Solution of nonlinear Riccati differential equation using Chebyshev wavelets

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Abstract: A generalized Chebyshev wavelet operational matrix (CWOM) is presented for the solution of nonlinear Riccati differential equations. The operational matrix together with suitable collocation points converts the fractional order Riccati differential equations into a system of algebraic equations. Accuracy and efficiency of the proposed method is verified through numerical examples and comparison with the recently developed approaches. The obtained results reveal that the performance of the proposed method is more accurate and reliable.

Key–Words: Riccati equation, Nonlinear ODE, Chebyshev wavelet, convergence, operational matrix.

1 Introduction

In this paper we consider the fractional-order Riccati differential equation of the form

$$D^{\alpha}y(t) = P(t)y^2 + Q(t)y + R(t),\ t > 0,$$

$\alpha \in (0, 1)$ with the initial condition

$$y(0) = k.$$

when $\alpha = 1$, the above equation is called classical Riccati differential equation and these equations will be investigated.

The Riccati differential equations (RDEs) has large variety of applications in engineering and applied science such as damping laws, rheology, diffusion processes, transmission line phenomena, optimal control theory problems etc., [1, 2, 3, 4, 5, 6]. It is to be noted that the RDEs are complicated in its structure and finding exact solutions for them cannot be simple. Applications and structure of RDEs attracted researchers to develop efficient methods to solve them in order to get more accurate solutions.

A substantial amount of research work has been done in the development of solution of RDEs. The most significant methods are Adomian decomposition method [7], homotopy perturbation method [8-11], homotopy analysis method [12, 13], Taylor matrix method [14] and Haar wavelet method [15], combination of Laplace, Adomian decomposition and Pad approximation [16] methods.

Several numerical methods for approximating the solution of nonlinear fractional-order Riccati differential equations equation are known. Raja et.al [17] developed a stochastic technique based on particle swarm optimization and simulated annealing. They were used as a tool for rapid global search method and simulated annealing for efficient local search method. A fractional variational iteration method described in the Riemann-Liouville derivative has been applied in [18], to give an analytical approximate solution to nonlinear fractional Riccati differential equation. A Combination of finite difference method and Pade? - variational iteration numerical scheme was proposed by Sweilam et. al [19]. Moreover an analytical scheme comprising the Laplace transform, the Adomian decomposition method (ADM), and the Pad approximation given in [16].

However, the above mentioned methods have some restrictions and disadvantages in their performance. For example, very complicated and toughest adomian polynomials are constructed in the Adomian decomposition method. In the variational iteration method identification Lagrange multiplier yields an underlying accuracy. The homotopy perturbation method needs a linear functional equation in each iteration to solve nonlinear equations, forming these functional equations are very difficult. The performance of the homotopy analysis method is very much depends on the chosen of the auxiliary parameter h of the zero-order deformation equation. Moreover, the convergence region and implementation of these results are very small.

In recent years, wavelets theory is one of the growing and predominantly a new method in the area of mathematical and engineering research. It has been applied in vast range of engineering sciences, particularly, they are used very successfully for waveform representation and segmentations in signal analysis, time-frequency analysis and in the mathematical sciences it is used in thriving manner for solving variety of linear and non linear differential and partial differential equations and fast algorithms for easy implementation [20]. Moreover wavelets build a connection with fast numerical algorithms [21, 29], this is due to wavelets admit the exact representation of a variety of function and operators. The application of Chebyshev wavelets are thoroughly considered in [22, 23].

In this work, the nonlinear Riccati differential equations of fractional-order approached analytically by using Chebyshev wavelets operational matrix (CWOM) method. The operational matrix of Chebyshev wavelet is generalized for fractional calculus in order to solve fractional and classical Riccati differential equations. The proposed CWOM method is illustrated by application, and obtained results are compared with recently proposed method for the fractional-order Riccati differential equation. We have adopted Chebyshev wavelet method to solve Riccati differential equations not only due to its emerging application of but also due to its greater convergence region.

The rest of the paper is as follows: In section 2 definitions and mathematical preliminaries of fractional calculus are presented. In section 3 Chebyshev wavelet, its properties, function approximations and generalized Chebyshev wavelet operational matrix fractional calculus are discussed. Section 4 establishes application of proposed method in the solution Riccati differential equations, existence and uniqueness solution of the proposed problem and convergence analysis of the proposed approach. Section 5 deals with the illustrative examples and their solutions by the proposed approach. Section 6 ends with our conclusion.

2 Preliminaries and notations

The notations, definitions and preliminary facts present in this section will be used in forthcoming sections of this work. Several definitions of fractional integrals and derivatives have been proposed after the logical definition given by Liouville. Important and few of these definitions include the Riemann-Liouville, the Caputo, the Weyl, the Hadamard, the Marchaud, the Riesz, the Grunwald-Letnikov and the Erdelyi-Kober. The Caputo fractional derivative uses initial and boundary conditions of integer order derivatives having some physical interpretations. Because of this specific reason, in this work we shall use the Caputo fractional derivative proposed by Caputo [24] in the theory of viscoelasticity.

The Caputo fractional derivative of order $\alpha > 0, (\alpha \in R,\ n-1 < \alpha \leq n,\ n \in N)$ and $h : (0,\infty) \longrightarrow R$ is continuous is defined by

$$D^{\alpha} f(t) = I^{n-\alpha}\left(\frac{d^n}{dt^n} f(t)\right) \tag{1}$$

where

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds \tag{2}$$

is the Riemann-Liouville fractional integral operator of order $\alpha > 0$ and Γ is the gamma function. The fractional integral of t^{β}, $\beta > -1$is given as

$$I^{\alpha}(t-\alpha)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha},\ a \geq 0 \tag{3}$$

Properties of fractional integrals and derivatives are as follows [25], for $\alpha, \beta > 0$:

The fractional order integral satisfies the semi group property

$$I^{\alpha}\left(I^{\beta} f(t)\right) = I^{\beta}\left(I^{\alpha} f(t)\right) = I^{\alpha+\beta} f(t) \tag{4}$$

The integer order derivative D^n and fractional order derivative D^{α} commute with each other,

$$D^n\left(D^{\alpha} f(t)\right) = D^{\alpha}\left(D^n f(t)\right) = D^{\alpha+n} f(t) \tag{5}$$

The fractional integral operator and fractional derivative operator do not satisfy the commutative property. In general,

$$I^{\alpha}\left(D^{\alpha} f(t)\right) = f(t) - \sum_{k=0}^{n-1} f^k(0)\frac{t^k}{k!} \tag{6}$$

but in the reverse way we have

$$D^{\alpha}\left(I^{\beta} f(t)\right) = D^{\alpha-\beta} f(t) \tag{7}$$

3 Properties of Chebyshev wavelets and its operational matrix of fractional integration

A family of functions constituted by Wavelets, constructed from dilation and translation of a single function called mother wavelet. When the parameters a of dilation and b of translation vary continuously, following are the family of continuous wavelets [26]

$$\psi_{a,b}(t) = |a|^{-1/2}\, \psi\left(\frac{t-b}{a}\right),\ a, b \in R,\ a \neq 0.$$

If the parameters a and b are restricted to discrete values as $a = a_0^{-k},\ b = nb_0a_0^{-k},\ a_0 > 1,\ b_0 > 0$ and n, and k are positive integers, following are the family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{k/2}\,\psi\left(a_0^k t - nb_0\right)$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$, and $b_0 = 1,\ \psi_{k,n}(t)$ forms an orthonormal basis [26].

Chebyshev wavelets $\psi_{k,n}(t) = \psi(k, n, m, t)$ have four arguments; $n = 1, 2, 3, \ldots 2^{k-1},\ k$ can assume any positive integer, m is the degree of Chebyshev polynomials of first kind and t denotes the normalized time. They are defined on the interval [0, 1) as [27, 28]

$$\psi_{nm}(t) = \begin{cases} \frac{\alpha_m 2^{(k+1)/2}}{\sqrt{\pi}} T_m(2^k t - 2n + 1) \\ \quad \text{for } \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0, \ \text{otherwise} \end{cases}$$

where

$$\alpha_m = \begin{cases} \sqrt{2} & \text{for } m = 0 \\ 2 & \text{for } m = 1, 2, \ldots \end{cases}$$

and $m = 0, 1, \ldots, M - 1$ and $n = 1, 2, 3, \ldots, 2^{k-1}$. $T_m(t)$ are the well-known Chebyshev polynomials of order m defined on the interval [-1, 1], which are orthogonal with respect to the weight function $\omega(t) = 1 \wedge \sqrt{1 - x^2}$ and they can be determined with the aid of the following recurrence formulae:

$$T_0(t) = 1, \quad T_1(t) = t,$$
$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1, 2, 3, \ldots$$

The Chebyshev wavelet series representation of the function $f(t)$ defined over [0, 1) is given by

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t) \tag{8}$$

where $c_{nm} = \langle f(t),\ \psi_{nm}(t)\rangle$, in which $\langle ., .\rangle$ denotes the inner product. If the infinite series in Eq. (8) is truncated, then Eq. (8) can be written as

$$f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t) = \hat{f}(t) \tag{9}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C = \begin{bmatrix} c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, c_{21}, \ldots, c_{2M-1}, \\ \cdots, c_{2^{k-1}0}, c_{2^{k-1}1}, \ldots, c_{2^{k-1}M-1} \end{bmatrix}^T$$

and

$$\Psi(t) = \begin{bmatrix} \psi_{10}(t) & \psi_{11}(t) & \ldots & \psi_{1M-1}(t), \\ \psi_{20}(t) & \psi_{21}(t) & \ldots & \psi_{2M-1}(t), \\ \vdots & \ldots & \ldots & \vdots \\ \psi_{2^{k-1}0}(t) & \psi_{2^{k-1}1}(t) & \ldots & \psi_{2^{k-1}M-1}(t) \end{bmatrix}^T \tag{10}$$

Taking suitable collocation points as following

$$t_i = \frac{(2i-1)}{2^k M}, \quad i = 1, 2, \ldots, 2^{k-1}M,$$

we defined the $\dot{m}$-square Chebyshev matrix

$$\phi_{\dot{m}\times\dot{m}} = \left[\Psi\left(\frac{1}{2^k M}\right), \Psi\left(\frac{3}{2^k M}\right), \ldots, \Psi\left(\frac{2^k M-1}{2^k M}\right)\right]$$

where $\dot{m} = 2^{k-1}M$, correspondingly we have

$$\hat{f} = \left[\hat{f}\left(\tfrac{1}{2^k M}\right), \hat{f}\left(\tfrac{3}{2^k M}\right), \ldots, \hat{f}\left(\tfrac{2^k M-1}{2^k M}\right)\right]$$
$$= C^T\phi_{\dot{m}\times\dot{m}}$$

The Chebyshev matrix $\phi_{\dot{m}\times\dot{m}}$ is an invertible matrix, the coefficient vector C^T is obtained by $C^T = \hat{f}\phi^{-1}_{\dot{m}\times\dot{m}}$.

3.1 Operational matrix of the fractional integration

The integration of the $\Psi(t)$ defined in Eq.(10) can be approximated by Chebyshev wavelet series with Chebyshev wavelet coefficient matrix P

$$\int_0^t \Psi(t)dt = P_{\dot{m}\times\dot{m}}\Psi(t)$$

where P is called Chebyshev wavelet operational matrix of integration.

The m-set of Block Pulse Functions is defined on $[0, l)$ as follows

$$b_i(t) = \begin{cases} 1, & \frac{i}{m} \le t < \frac{(i+1)}{m} \\ 0, & \text{otherwise} \end{cases}$$

where $i = 0, 1, 2, \ldots, m$
The functions b_i are disjoint and orthogonal. That is,

$$b_i(t)b_j(t) = \begin{cases} b_i(t), & i = j \\ 0, & i \neq j \end{cases}$$

and

$$\int_0^1 b_i(t)b_j(t)dt = \begin{cases} \frac{1}{m}, & i = j \\ 0, & i \neq j \end{cases}$$

The orthogonality property of block-pulse function is obtained from the disjointness property. An arbitrary function $f \in L_2[0,1)$, can be expanded into block-pulse functions as

$$f(t) \approx \sum_{i=0}^{m-1} f_i b_i(t) = f^T B(t)$$

where f_i are the coefficients of the block-pulse function, given by

$$f_i = \frac{m}{l} \int_0^l f(t) b_i(t) dt$$

The Chebyshev wavelets can be expanded into m-set of block-pulse functions as

$$\Psi(t) = \phi_{\dot{m}\times\dot{m}} B(t) \tag{11}$$

where $B(t) = [b_0(t), b_1(t), \cdots, b_i(t), \cdots, b_{m-1}(t)]^T$.

The fractional integral of block-pulse function vector can be written as

$$\left(I^\alpha B\right)(t) = F^\alpha_{\dot{m}\times\dot{m}} B(t) \tag{12}$$

where $F^\alpha_{\dot{m}\times\dot{m}}$ is given in [27].

Now we derive the Chebyshev wavelet operational matrix of the fractional integration

$$\left(I^\alpha \Psi\right)(t) \approx P^\alpha_{\dot{m}\times\dot{m}} \Psi(t) \tag{13}$$

where the $\dot{m}$ - square matrix $P_{\dot{m}\times\dot{m}}$ is called Chebyshev wavelet operational matrix of the fractional integration.

Using Eqs. (11) and (13) we have

$$\begin{aligned} &\left(I^\alpha \Psi\right)(t) \approx \left(I^\alpha \phi_{\dot{m}\times\dot{m}} B\right)(t) \\ &= \phi_{\dot{m}\times\dot{m}} \left(I^\alpha B\right)(t) \approx \phi_{\dot{m}\times\dot{m}} \left(F^\alpha B\right)(t) \end{aligned} \tag{14}$$

From Eqs. (13) and (14) we get

$$P^\alpha_{\dot{m}\times\dot{m}} \Psi(t) = \phi_{\dot{m}\times\dot{m}} F^\alpha B(t) \tag{15}$$

and by the Eq. (11), the Eq. (15) becomes

$$P^\alpha_{\dot{m}\times\dot{m}} \phi_{\dot{m}\times\dot{m}} B(t) = \phi_{\dot{m}\times\dot{m}} F^\alpha B(t) \tag{16}$$

Then the Chebyshev wavelet operational matrix $P^\alpha_{\dot{m}\times\dot{m}}$ of fractional integration is given by

$$P^\alpha_{\dot{m}\times\dot{m}} = \phi_{\dot{m}\times\dot{m}} F^\alpha \phi^{-1}_{\dot{m}\times\dot{m}} \tag{17}$$

Following is the Chebyshev wavelet operational matrix $P^\alpha_{\dot{m}\times\dot{m}}$ of fractional order integration, for the particular values of $k = 2, M = 3, \alpha = 0.5$

$$P^{0.5}_{6\times6} = \begin{pmatrix} 0.5415 & 0.4324 & 0.1819 & -0.0871 & -0.0179 & 0.0154 \\ 0 & 0.5415 & 0 & 0.1819 & 0 & -0.0179 \\ -0.2046 & 0.071 & 0.2243 & -0.0449 & 0.0798 & 0.0119 \\ 0 & -0.2046 & 0 & 0.2243 & 0 & 0.0798 \\ 0.1781 & 0.2506 & -0.0252 & -0.0652 & 0.1555 & 0.0143 \\ 0 & 0.1781 & 0 & -0.0252 & 0 & 0.1555 \end{pmatrix}$$

4 Application to fractional Riccati differential Equation

In this section, we will use the generalized Chebyshev wavelet operational matrix to solve nonlinear Riccati differential equation and we discuss the existence and uniqueness of solutions with initial conditions and convergence criteria of the proposed CWOM approach.

Consider the fractional-order Riccati differential equation of the form

$$D^\alpha y(t) = P(t)y^2 + Q(t)y + R(t),\ t > 0, \tag{18}$$

$0 < \alpha \leq 1$ with the initial condition

$$y(0) = k. \tag{19}$$

Let us suppose that the functions $D^\alpha y(t)$, $P(t)$, $Q(t)$, $R(t)$ are approximated using Chebyshev wavelet as follows:

$$\begin{aligned} D^\alpha y(t) = U^T\Psi(t), \quad & P(t) = V^T\Psi(t) \\ Q(t) = W^T\Psi(t), \quad & R(t) = X^T\Psi(t) \end{aligned} \tag{20}$$

where U, V, W, X and $\Psi(t)$ are given in Eqs. (10). Using the Eq.(6), we can write

$$y(t) = I^\alpha\left(D^\alpha y(t)\right) - y(0) \tag{21}$$

By the Eqs. (13) and (19), the Eq. (21) leads to

$$y(t) \approx U^T P^\alpha_{\dot{m}\times\dot{m}} \Psi(t) + Y_0^T(t)\Psi(t) = C^T\Psi(t) \tag{22}$$

where

$$y(0) = k \approx Y_0^T(t)\Psi(t),\ C = \left(U^T P^\alpha_{\dot{m}\times\dot{m}} + Y_0^T\right)^T$$

Substituting Eqs. (20) and (22) into Eq.(18), we have

$$\begin{aligned} U^T\Psi(t) = V^T\Psi(t)\left[C^T\Psi(t)\right]^2 \\ + W^T\Psi(t)C^T\Psi(t) + X^T\Psi(t) \end{aligned} \tag{23}$$

Substituting Eq. (11) into the Eq. (23), we have

$$\begin{aligned} U^T\phi_{\dot{m}\times\dot{m}} = V^T\left[C^T\phi_{\dot{m}\times\dot{m}}\right]^2 \\ + W^T C^T \phi_{\dot{m}\times\dot{m}} + X^T \end{aligned} \tag{24}$$

where C, V, W and $\phi_{\dot{m}\times\dot{m}}$ are known. Eq. (24) represents a system of nonlinear equations with unknown vector U. This system of nonlinear equations can be solved by Newton method for the unknown vector U and we can get the approximation solution by including U into Eq.(22).

4.1 Existence and uniqueness of solutions

Consider the fractional-order Riccati differential equation of the form of Eqs.(18) and (19). The non-linear term in Eq. (18) is y^2 and $P(t), Q(t), R(t)$ are known functions. For $\alpha = 1$, the fractional-order Riccati converts into the classical Riccati differential equation.

Definition 1 *Let $I = [0, l], l < \infty$ and $C(I)$ be the class of all continuous function defined on I, with the norm*

$$\|y\| = \sup_{t \in I} \left| e^{-ht} y(t) \right|, h > 0$$

which is equivalent to the sup-norm of y.

Remark 2 *Assume that solution $y(t)$ of fractional-order Riccati differential Eqs. (18) and (19) belongs to the space $S = \{y \in R : |y| \leq c,\ c\, is\, any\, constant\}$, in order to study the existence and uniqueness of the initial value problem.*

Definition 3 *The Space of integrable functions $L_1[0, l]$ in the interval $[0, l]$ is defined as*

$$L_1[0, l] = \left\{ u(t) : \int_0^l |u(t)| dt < \infty \right\}.$$

Theorem 4 *The initial value problem given by Eqs. (18) and (19) has a unique solution*

$$y \in C(I), y' \in \left\{ y \in L_1[0, l], \|y\| = \left| e^{-ht} y(t) \right|_{L_1} \right\}$$

Proof: By Eq. (1), the fractional differential Eq. (18) can be written as

$$I^{1-\alpha} \frac{dy(t)}{dt} = P(t)y^2 + Q(t)y + R(t) \qquad (25)$$

becomes

$$y(t) = I^\alpha \left(P(t)y^2 + Q(t)y + R(t) \right) \qquad (26)$$

Now we define the operator $\Theta : C(I) \longrightarrow C(I)$ by

$$\Theta y(t) = I^\alpha \left(P(t)y^2 + Q(t)y + R(t) \right) \qquad (27)$$

then,

$$\begin{aligned} & e^{-ht} (\Theta y - \Theta w) \\ & = e^{-ht} I^\alpha \begin{bmatrix} (P(t)y^2 + Q(t)y + R(t)) \\ -(P(t)w^2 + Q(t)w + R(t)) \end{bmatrix} \\ & \leq \tfrac{1}{\Gamma(\alpha)} \int_0^t \begin{pmatrix} (t-s)^{\alpha-1} e^{-h(t-s)} \\ ((y-w)(y+w) - k(y-w)) e^{-hs} \end{pmatrix} ds \\ & \leq \|y - w\| \tfrac{1}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} e^{-hs} ds \end{aligned}$$

hence, we have

$$\|\Theta y - \Theta w\| < \|y - w\|$$

which implies the operator given by Eq. (27), has a unique fixed point and consequently the given integral equation has a unique solution $y \in C(I)$. Also we can see that

$$I^\alpha P(t)y^2 + Q(t)y + R(t)|_{t=0} = k \qquad (28)$$

Now from Eq. (26), we have

$$y(t) = \begin{bmatrix} \frac{t^\alpha}{\Gamma(\alpha+1)} (P y_0^2 + Q y_0 + R) + \\ I^{\alpha+1} (P' y^2 + 2y' P + Q' y + Q y' + R') \end{bmatrix}$$

and

$$\frac{dy}{dt} = \begin{bmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} (P y_0^2 + Q y_0 + R) + \\ I^\alpha (P' y^2 + 2y' P + Q' y + Q y' + R') \end{bmatrix}$$

$$e^{-ht} \frac{dy}{dt} = e^{-ht} \begin{bmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} (P y_0^2 + Q y_0 + R) + \\ I^\alpha (P' y^2 + 2y' P + Q' y + Q y' + R') \end{bmatrix}$$

from which we can deduce that $y' \in C(I)$ and $y' \in S$ Now again from Eqs. (26), (27) and (28) we get

$$\frac{dy}{dt} = \frac{d}{dt} I^\alpha [P y^2 + Q y + R]$$

$$\begin{aligned} & I^{1-\alpha} \tfrac{dy}{dt} = I^{1-\alpha} \tfrac{d}{dt} I^\alpha [P y^2 + Q y + R] \\ & = \tfrac{d}{dt} I^{1-\alpha} I^\alpha [P y^2 + Q y + R] \end{aligned}$$

$$\begin{aligned} & D^\alpha y(t) = \tfrac{d}{dt} I [P y^2 + Q y + R] \\ & = P y^2 + Q y + R \end{aligned}$$

and

$$y(0) = I^\alpha P y^2 + Q y + R|_{t=0} = k$$

which implies that the integral equation (28) is equivalent to the initial value problem (19) and the theorem is proved.

4.2 Convergence analyses

Let

$$\psi_{k,n}(t) = |a_0|^{k/2} \psi \left(a_0^k t - n b_0 \right)$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$, and $b_0 = 1$, $\psi_{k,n}(t)$ forms an orthonormal basis [26].

Let

$$y(t) = \sum_{i=1}^{M-1} c_{1i} \psi_{1i}(t)$$

be the solution of the Eq. (18) where $c_{1i} = \langle y(t), \psi_{1i}(t) \rangle$ for $k = 1$ in which $\langle ., . \rangle$ denotes the inner product.

$$y(t) = \sum_{i=1}^{n} \langle y(t), \psi_{1i}(t) \rangle \, \psi_{1i}(t) \tag{29}$$

Let $\beta_j = \langle y(t), \psi(t) \rangle$ where

$$\psi(t) = \psi_{1i}(t) \tag{30}$$

Let $x_n = \sum_{j=1}^{n} \beta_j \psi(t_j)$ be a sequence of partial sums. Then,

$$\begin{aligned} \langle y(t), x_n \rangle &= \left\langle y(t), \sum_{j=1}^{n} \beta_j \psi(t_j) \right\rangle \\ &= \sum_{j=1}^{n} \bar{\beta}_j \langle y(t), \psi(t_j) \rangle \\ &= \sum_{j=1}^{n} \bar{\beta}_j \beta_j = \sum_{j=1}^{n} |\beta_j|^2 \end{aligned}$$

Further

$$\begin{aligned} \|x_n - x_m\|^2 &= \left\| \textstyle\sum_{j=m+1}^{n} \beta_j \psi(t_j) \right\|^2 \\ &= \left\langle \sum_{i=m+1}^{n} \beta_i \psi(t_i), \sum_{j=m+1}^{n} \beta_j \psi(t_j) \right\rangle \\ &= \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \beta_i \bar{\beta}_j \langle \psi(t_i), \psi(t_j) \rangle \\ &= \sum_{j=m+1}^{n} |\beta_j|^2 \end{aligned}$$

As $n \to \infty$, from Bessel's inequality, we have $\sum_{j=1}^{\infty} |\beta_j|^2$ is convergent. It implies that x_n is a Cauchy sequence and it converges to x (say).

Also

$$\begin{aligned} &\langle x - y(t), \psi(t_j) \rangle = \langle x, \psi(t_j) \rangle - \langle y(t), \psi(t_j) \rangle \\ &= \langle Lt_{n\to\infty} x_n, \psi(t_j) \rangle - \beta_j \\ &= Lt_{n\to\infty} \langle x_n, \psi(t_j) \rangle - \beta_j \\ &= Lt_{n\to\infty} \left\langle \sum_{j=1}^{n} \beta_j \psi(t_j), \psi(t_j) \right\rangle - \beta_j \\ &= \beta_j - \beta_j \\ &= 0. \end{aligned}$$

which is possible only if $y(t) = x$. i.e., both $y(t)$ and x_n converges to the same value, which indeed give the guarantee of convergence of CWOM.

5 Numerical Examples

In order to show the effectiveness of the Chebyshev wavelets operational matrix method (CWOM), we implement CWOM to the nonlinear fractional Riccati differential equations. All the numerical experiments carried out on a personal computer with some MATLAB codes. The specification of PC is intel core i5 processor and with Turbo boost up to 3.1GHz and 4 GB of DDR3 memory. The following problems of nonlinear Riccati differential equations are solved with real coefficients.

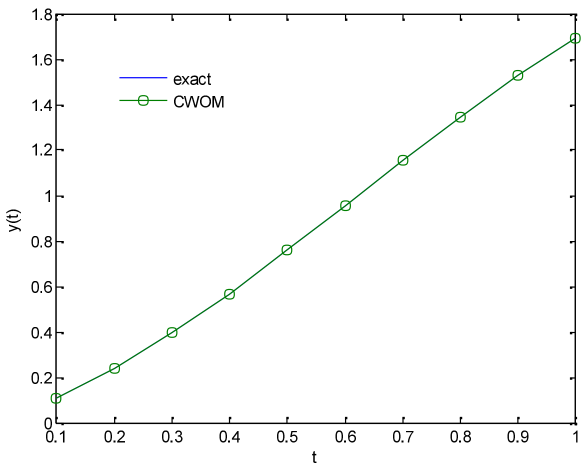

Figure 1: Numerical Results of Example 5 by CWOM for α = 1

Example 5 *Consider the following nonlinear fractional Riccati differential equation*

$$D^{\alpha} y(t) = 1 + 2y(t) - y^2(t), \; 0 < \alpha \leq 1 \tag{31}$$

with initial condition

$$y(0) = 0 \tag{32}$$

Exact solution for $\alpha = 1$ was found to be

$$y(t) = 1 + \sqrt{2} \tanh \left(\sqrt{2} t + \frac{1}{2} \log \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right)$$

The integral representation of the Eqs.(31) and (32) is given by

$$I^{\alpha} \left(D^{\alpha} y(t) \right) = I^{\alpha} \left(1 + 2y(t) - y^2(t) \right)$$

$$y(t) = y(0) + \frac{t^{\alpha}}{\Gamma(\alpha + 1)} + 2I^{\alpha} y(t) - I^{\alpha} y^2(t) \tag{33}$$

Let

$$y(t) = C^T \Psi(t) \tag{34}$$

then

$$\begin{aligned} I^{\alpha} y(t) &= C^T I^{\alpha} \Psi(t) \\ &= C^T P^{\alpha}_{2^{k-1}M \times 2^{k-1}M} \Psi(t) \end{aligned} \tag{35}$$

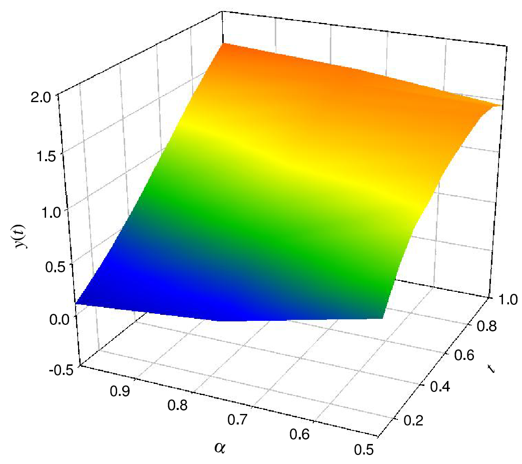

Figure 2: Numerical results of Example 5 for different values of α

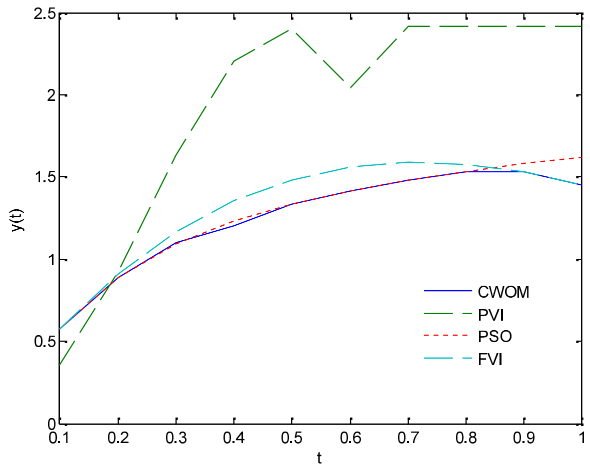

Figure 3: Comparative results of Example 5 when α = 1/2

By substituting Eqs.(34) and (35) in (33), we get the following system of algebraic equations

$$C^T\Psi(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} + 2C^T P^{\alpha}_{2^{k-1}M\times 2^{k-1}M}\Psi(t) - C^T P^{2\alpha}_{2^{k-1}M\times 2^{k-1}M}\Psi(t)$$

By solving the above system of linear equations, we can find the vector C. Numerical results are obtained for different values of k, M, α. Solution obtained by the proposed CWOM approach for $\alpha = 1, k = 1$ and $M = 3$ is given in Figure 1 and for different values of α = 0.6, 0.7, 0.8 and 0.9 and for $k = 2$ and $M = 5$ are graphically given in Figure 2. It can be seen from Figure 1 that the solution obtained by the proposed CWOM approach is more close to the exact solution.

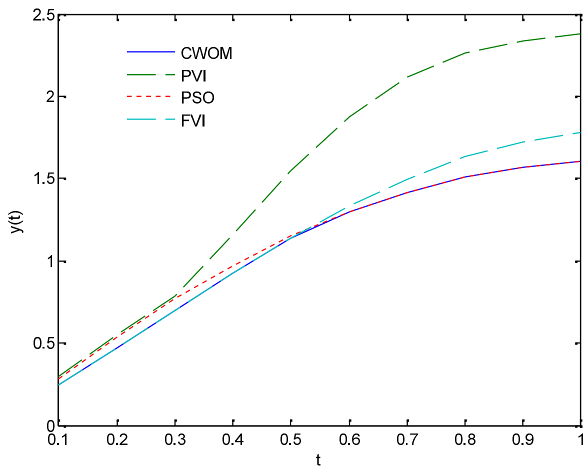

Figure 4: Comparative results of Example 5 when α = 3/4

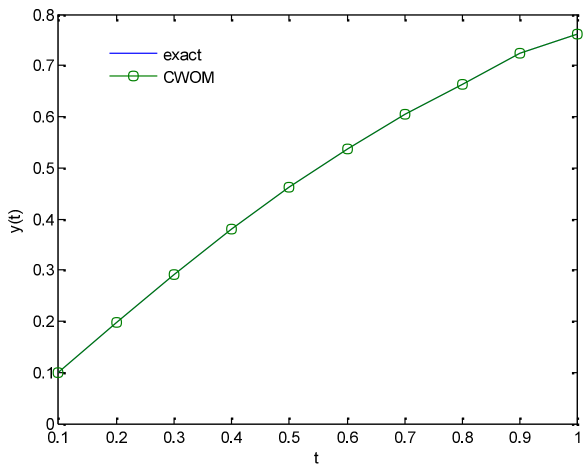

Figure 5: Numerical Results of Example 6 by CWOM for α = 1

In order to analyze the effectiveness of the proposed approach further, the obtained results of Example 5.1 for α = 0.5, 0.75 and for $k = 1$ and $M = 3$ are compared with reported results of other numerical, analytical and stochastic solver such as solution by PSO [17] based on swarm intelligence, a finite difference numerical iteration scheme by Pade-variational iteration method (PVI) [19] and analytical approximation solution obtained by fractional variational iteration method (FVI) [18] based on Riemann-Liouville derivative. The compared results are provided in the Figure 3 and 4, it indicates that the results obtained by proposed CWOM approach has good convergence than the other approaches in the given applicable domain.

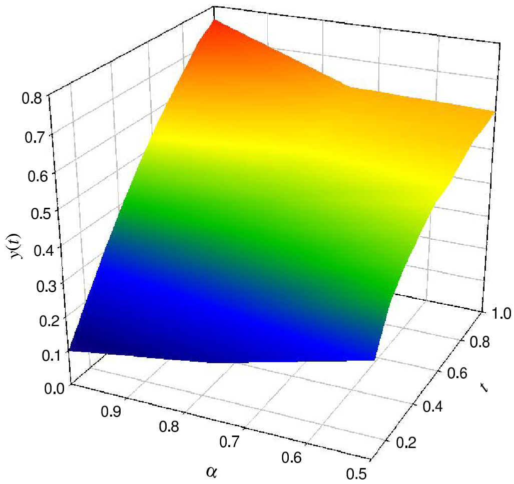

Figure 6: Numerical results of Example 6 for different values of α

Example 6 *Consider another fractional order Riccati differential equation*

$$D^{\alpha}y(t) = 1 - y^2(t),\ 0 < \alpha \leq 1 \tag{36}$$

with initial condition

$$y(0) = 0 \tag{37}$$

Exact solution for $\alpha = 1$ was found to be

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}$$

The integral representation of the Eqs.(36) and (37) is given by

$$y(t) = y(0) + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - I^{\alpha}y^2(t)$$

Let

$$y(t) = C^T\Psi(t) \tag{38}$$

then

$$\begin{aligned} I^{\alpha}y(t) &= C^T I^{\alpha}\Psi(t) \\ &= C^T P^{\alpha}_{2^{k-1}M\times 2^{k-1}M}\Psi(t) \end{aligned} \tag{39}$$

By substituting Eqs.(38) and (39) in (36), we get the following system of algebraic equations

$$C^T\Psi(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} - C^T P^{2\alpha}_{2^{k-1}M\times 2^{k-1}M}\Psi(t)$$

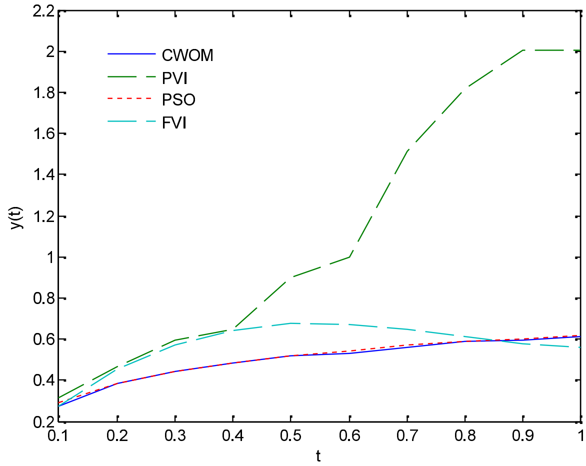

Figure 7: Comparative results of Example 6 when α = 1/2

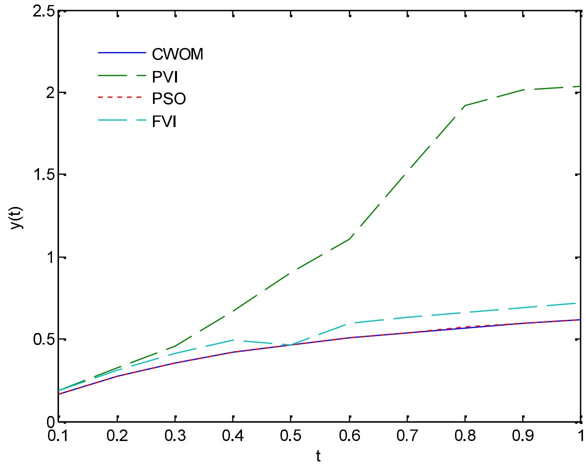

Figure 8: Comparative results of Example 6 when α = 3/4

By solving the above system of linear equations, we can find the vector C. Numerical results are obtained for different values of $k, M\alpha$. Results obtained by CWOM for $\alpha = 1, k = 2, M = 3$ shown in the Figure 5 and it can be seen from the figure that solution given by the CWOM merely coincide with the exact solution. Figure 6, show that the obtained results of Eqs. (36) and (37) by CWOM for different values of α and for $k = 2$ and $M = 5$. Since exact solution for fractional order case is not available, like example 5.1, for the Eqs. (36) and (37) comparisons made with the approximate solution given by the proposed approach and reported approximate results of other approaches PSO [17], FVI [18], PVI [19]. Obtained results are provided in the Figures 7 and 8, from these figures we can identify that guarantee of convergence of the proposed CWOM approach is very high.

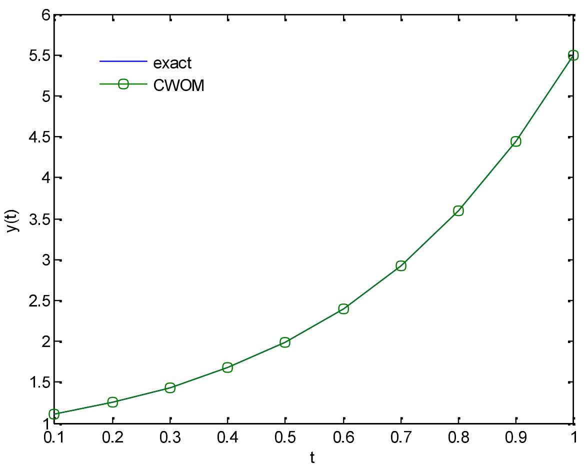

Figure 9: Numerical Results of Example 7 by CWOM for $\alpha = 1$

Example 7 *Let us consider another problem of nonlinear Riccati differential equation*

$$D^\alpha y(t) = t^2 + y^2(t), \ 0 < \alpha \leq 1 \tag{40}$$

with initial condition

$$y(0) = 0 \tag{41}$$

Exact solution for $\alpha = 1$ was found to be

$$y(t) = \frac{t\left(J_{-3/4}(t^2/2)\Gamma(1/4) + 2J_{3/4}(t^2/2)\Gamma(3/4)\right)}{J_{1/4}(t^2/2)\Gamma(1/4) - J_{-1/4}(t^2/2)\Gamma(3/4)}$$

where $J_n(t)$ is the Bessel function of first kind.

$$y(t) = 1 + \frac{2}{\Gamma(\alpha+1) + 2\Gamma(\alpha)} t^{\alpha+2} - I^\alpha y^2(t)$$

Let

$$y(t) = C^T \Psi(t) \tag{42}$$

then

$$\begin{aligned} I^\alpha y(t) &= C^T I^\alpha \Psi(t) \\ &= C^T P^\alpha_{2^{k-1}M \times 2^{k-1}M} \Psi(t) \end{aligned} \tag{43}$$

The integral representation of the Eqs.(40) and (41) is given by By substituting Eqs.(42) and (43) in (40), we get the following system of algebraic equations

$$\begin{aligned} C^T\Psi(t) &= 1 + \tfrac{2}{\Gamma(\alpha+1)+2\Gamma(\alpha)} (C^T\Psi(t))^{\alpha+2} \\ &-C^T P^{2\alpha}_{2^{k-1}M \times 2^{k-1}M} \Psi(t) \end{aligned}$$

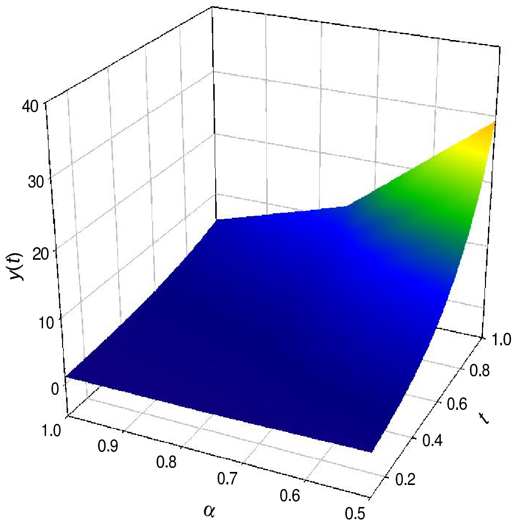

Figure 10: Numerical results of Example 7 for different values of α

By solving the above system of linear equations, we can find the vector C. Numerical results are obtained for different values of $k, M\alpha$. Results obtained by CWOM for $\alpha = 1, k = 2, M = 3$ shown in the Figure 9 and it can be seen from the figure that solution given by the CWOM merely coincide with the exact solution. Figure 10, show that the obtained results of Eqs. (40) and (41) by CWOM for different values of α and for $k = 2$ and $M = 5$. Since exact solution for fractional order case is not available, like example 5.1, for the Eqs. (40) and (41) comparisons made with the approximate solution given by the proposed approach and reported approximate results of other approaches PSO [17], FVI [18], PVI [19]. From these results we can see that the proposed CWOM approach gives the solution which are very close to the exact solution and outperformed recently developed approaches for the nonlinear fractional order Riccati differential equations in terms of solution quality and convergence criteria.

6 Conclusion

Nonlinear fractional order Riccati differential equations play an important role in the modeling of many biological, physical, chemical and real life problems. Therefore it is necessary to develop a method which would give more accurate solutions to such type of problems with greater convergence criteria.

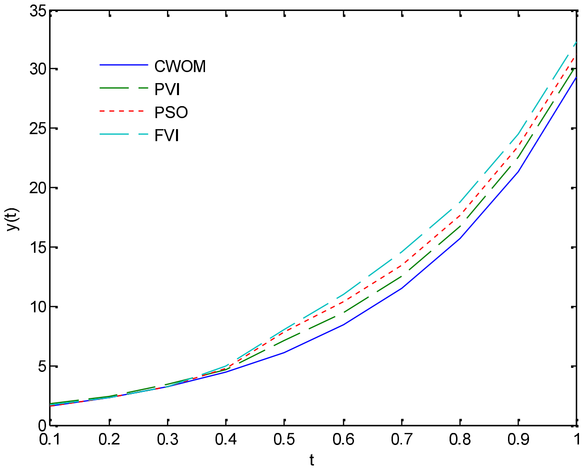

Figure 11: Comparative results of Example 6 when α = 1/2

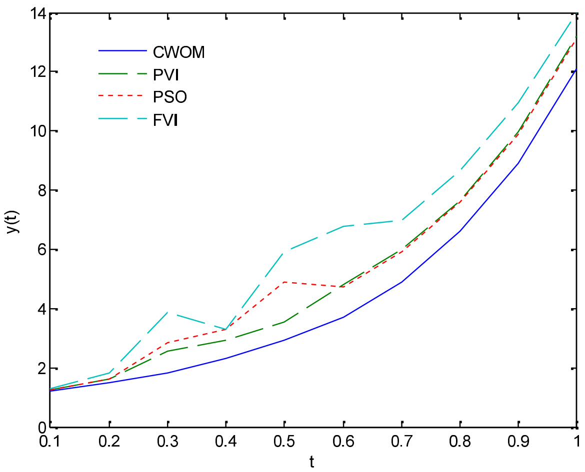

Figure 12: Comparative results of Example 7 when α = 3/4

In this work, a Chebyshev wavelet operational matrix method called CWOM, proposed for solving nonlinear fractional order Riccati differential equations. Comparison made for the solutions obtained by the proposed method and with the other recent approaches developed for same problem; obtained results show that the proposed CWOM yields more accurate and reliable solutions with less computational effort. Further we have discussed the convergence criteria of proposed scheme, which indeed provides the guarantee of consistency and stability of the proposed CWOM scheme for the solutions of nonlinear fractional Riccati differential equations.

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