Almost Periodic Solution of Schoener's Competition Model with Pure-Delays

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Abstract: In this paper, we consider an almost periodic Schoener's competition model with time-varying delays. By means of Mawhin's continuation theorem of coincidence degree theory, some new sufficient conditions are obtained for the existence of at least one positive almost periodic solution for a kind of Schoeners competition model with time-varying delays. The result of this paper complements previous results. Finally, two examples and numerical simulations are given to illustrate the feasibility and effectiveness of our main results.

Key–Words: Positive almost periodic solution; Coincidence degree; Schoener; Competition model; Time-varying delay.

1 Introduction

In recent years, the Schoener's competition system has been studied by many scholars. Topics such as existence, uniqueness and global attractivity of positive periodic solutions of the system were extensively investigated and many excellent results have been derived (see [1-9] and the references cited therein). In [6], Liu, Xu and Wang proposed and studied the global stability of the following Schoener's competition model with pure-delays:

$$\begin{cases} \dot{x}_1(t) = x_1(t)\Big[\frac{a_{10}(t)}{x_1(t-\tau_{10})+m_1(t)} \\ \qquad -a_{11}(t)x_1(t-\tau_{11}) \\ \qquad -a_{12}(t)x_2(t-\tau_{12}) - c_1(t)\Big], \\ \dot{x}_2(t) = x_2(t)\Big[\frac{a_{20}(t)}{x_2(t-\tau_{20})+m_2(t)} \\ \qquad -a_{21}(t)x_1(t-\tau_{21}) \\ \qquad -a_{22}(t)x_2(t-\tau_{22}) - c_2(t)\Big]. \end{cases} \tag{1}$$

In [9], we studied the existence and stability of a unique positive almost periodic solution of system (1) with impulsive effects by means of Lyapunov functional.

Time delays represent an additional level of complexity that can be incorporated in a more detailed analysis of a particular system. Specially, in the real world, the delays in differential equations of biological phenomena are usually time-varying. Thus, it is worthwhile continuing to consider the following Schoener's competition model with time-varying delays:

$$\begin{cases} \dot{x}_1(t) = x_1(t)\Big[\frac{a_{10}(t)}{x_1(t-\tau_{10}(t))+m_1(t)} \\ \qquad -a_{11}(t)x_1(t-\tau_{11}(t)) \\ \qquad -a_{12}(t)x_2(t-\tau_{12}(t)) - c_1(t)\Big], \\ \dot{x}_2(t) = x_2(t)\Big[\frac{a_{20}(t)}{x_2(t-\tau_{20}(t))+m_2(t)} \\ \qquad -a_{21}(t)x_1(t-\tau_{21}(t)) \\ \qquad -a_{22}(t)x_2(t-\tau_{22}(t)) - c_2(t)\Big], \end{cases} \tag{2}$$

where $x_1(t)$, $x_2(t)$ are population densities of species x_1, x_2 at time t, respectively.

In real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits and harvesting, etc. So it is usual to assume the periodicity of parameters in the systems. However, in applications, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples, see Example 1) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Hence, if we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity.

Example 1. *Let us consider the following simple population model:*

$$\dot{N}(t) = N(t)\bigg[\frac{|\sin(\sqrt{2}t)|}{N(t)+1} - |\sin(\sqrt{3}t)|N(t-1)\bigg]. \qquad (3)$$

In Eq. (3), $|\sin(\sqrt{2}t)|$ is $\frac{\sqrt{2}\pi}{2}$-periodic function and $|\sin(\sqrt{3}t)|$ is $\frac{\sqrt{3}\pi}{3}$-periodic function, which imply that Eq.(3) is with incommensurable periods. Then there is no a priori reason to expect the existence of periodic solutions of Eq.(3). Thus, it is significant to study the existence of almost periodic solutions of Eq. (3) .

It is well known that Mawhin's continuation theorem of coincidence degree theory is an important method to investigate the existence of positive periodic solutions of non-linear ecosystems (see [10-18]). However, it is difficult to be used to investigate the almost periodic solutions of non-linear ecosystems. Therefore, to the best of the author's knowledge, so far, there are scarcely any papers concerning with the existence of positive almost periodic solutions of system (2) by using Mawhin's continuation theorem. Motivated by the above reason, the main purpose of this paper is to establish some new sufficient conditions on the existence of positive almost periodic solutions of system (2) by using Mawhin's continuous theorem of coincidence degree theory.

Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}^+$ denote the sets of real numbers, integers and positive integers, respectively, $C(\mathbb{X},\mathbb{Y})$ and $C^1(\mathbb{X},\mathbb{Y})$ be the space of continuous functions and continuously differential functions which map $\mathbb{X}$ into $\mathbb{Y}$, respectively. Especially, $C(\mathbb{X}) := C(\mathbb{X},\mathbb{X})$, $C^1(\mathbb{X}) := C^1(\mathbb{X},\mathbb{X})$. Related to a continuous bounded function f, we use the following notations:

$$f^- = \inf_{s\in\mathbb{R}} f(s), \quad f^+ = \sup_{s\in\mathbb{R}} f(s), \quad |f|_\infty = \sup_{s\in\mathbb{R}} |f(s)|.$$

Throughout this paper, we always make the following assumption for system (2):

(H_1) a_{ij}, τ_{ij}, m_i and c_i are nonnegative almost periodic functions with $m_i^- > 0$, $i = 1,2$, $j = 0,1,2$.

The organization of this Letter is as follows. In Section 2, we make some preparations. In Section 3, by using Mawhin's continuation theorem of coincidence degree theory, we establish sufficient conditions for the existence of at least one positive almost periodic solution to system (2). Two illustrative examples are given in Section 4.

2 Preliminaries

Definition 2. *([19]) $x \in C(\mathbb{R},\mathbb{R}^n)$ is called almost periodic, if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $\|x(t+\tau) - x(t)\| < \epsilon$, $\forall t \in \mathbb{R}$, where $\|\cdot\|$ is arbitrary norm of $\mathbb{R}^n$. τ is called to the ϵ-almost period of x, $T(x,\epsilon)$ denotes the set of ϵ-almost periods for x and $l(\epsilon)$ is called to the length of the inclusion interval for $T(x,\epsilon)$. The collection of those functions is denoted by $AP(\mathbb{R},\mathbb{R}^n)$. Let $AP(\mathbb{R}) := AP(\mathbb{R},\mathbb{R})$.*

Lemma 3. *([20]) Assume that $x \in AP(\mathbb{R}) \cap C^1(\mathbb{R})$ with $\dot{x} \in C(\mathbb{R})$. For arbitrary interval $I = [a,b]$ with $b - a = \omega > 0$, let $\xi \in [a,b]$ and*

$$I_1 = \big\{s \in [\xi,b] : \dot{x}(s) \geq 0\big\},$$

then ones have

$$x(t) \leq x(\xi) + \int_{I_1} \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [\xi,b].$$

Lemma 4. *([20]) If $x \in AP(\mathbb{R})$, then for arbitrary interval $I = [a,b]$ with $b - a = \omega > 0$, there exist $\xi \in [a,b]$, $\underline{\xi} \in (-\infty,a]$ and $\bar{\xi} \in [b,+\infty)$ such that*

$$x(\underline{\xi}) = x(\bar{\xi}) \quad \text{and} \quad x(\xi) \leq x(s), \ \forall s \in [\underline{\xi},\bar{\xi}].$$

Lemma 5. *([20]) If $x \in AP(\mathbb{R})$, then for arbitrary interval $[a,b]$ with $I = b - a = \omega > 0$, there exist $\eta \in [a,b]$, $\underline{\eta} \in (-\infty,a]$ and $\bar{\eta} \in [b,+\infty)$ such that*

$$x(\underline{\eta}) = x(\bar{\eta}) \quad \text{and} \quad x(\eta) \geq x(s), \ \forall s \in [\underline{\eta},\bar{\eta}].$$

Lemma 6. *([20]) If $x \in AP(\mathbb{R})$, then for $\forall n \in \mathbb{N}^+$, there exists $\alpha_n \in \mathbb{R}$ such that $x(\alpha_n) \in [x^* - \frac{1}{n}, x^*]$, where $x^* = \sup_{s\in\mathbb{R}} x(s)$.*

Lemma 7. *([20]) Assume that $x \in AP(\mathbb{R})$ and $\bar{x} > 0$, then for $\forall t_0 \in \mathbb{R}$ and $\epsilon_0 \in (0,\bar{x})$, there exists a positive constant $T_0 = T_0(\epsilon_0)$ independent of t_0 such that*

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(s)\,\mathrm{d}s \in [\bar{x} - \epsilon_0, \bar{x} + \epsilon_0], \quad \forall T \geq T_0.$$

Let $\epsilon_0 = \frac{\bar{x}}{2}$ in the above lemma, we obtain

Lemma 8. *Assume that $x \in AP(\mathbb{R})$ and $\bar{x} > 0$, then for $\forall t_0 \in \mathbb{R}$, there exists a positive constant T_0 independent of t_0 such that*

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(s)\,\mathrm{d}s \in \left[\frac{\bar{x}}{2}, \frac{3\bar{x}}{2}\right], \quad \forall T \geq T_0.$$

3 Main results

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. This requires us to introduce a few concepts and results from Gaines and Mawhin [21].

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}$ be a linear mapping and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\mathrm{Im}L$ is closed in $\mathbb{Y}$ and $\mathrm{dimKer}L = \mathrm{codimIm}L < +\infty$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$. It follows that $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P} : (I - P)\mathbb{X} \to \mathrm{Im}L$ is invertible and its inverse is denoted by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to \mathbb{X}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Lemma 9. *([21]) Let $\Omega \subseteq \mathbb{X}$ be an open bounded set, L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. If all the following conditions hold:*

(a) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap \mathrm{Dom}L, \lambda \in (0, 1)$;

(b) $QNx \neq 0, \forall x \in \partial\Omega \cap \mathrm{Ker}L$;

(c) $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$, *where $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is an isomorphism.*

Then $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}L$.

Under the invariant transformation $(N_1, N_2)^T = (e^u, e^v)^T$, system (2) reduces to

$$\begin{cases} \dot{u}(t) = b_1(t) - a_1(t)e^{u(t-\mu_1(t))} \\ \qquad\quad -\frac{\alpha_1(t)e^{u(t)}}{1+me^{2u(t)}}e^{v(t-\nu(t))}, \\ \dot{v}(t) = -b_2(t) - a_2(t)e^{v(t)} \\ \qquad\quad +\frac{\alpha_2(t)e^{2u(t-\mu_2(t))}}{1+me^{2u(t-\mu_2(t))}}. \end{cases} \tag{4}$$

For $f \in AP(\mathbb{R})$, we denote by

$$\bar{f} = m(f) = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(s)\,\mathrm{d}s,$$

$$\Lambda(f) = \left\{\varpi \in \mathbb{R} : \lim_{T\to\infty}\frac{1}{T}\int_0^T f(s)e^{-\mathrm{i}\varpi s}\mathrm{d}s \neq 0\right\},$$

$$\mathrm{mod}(f) = \left\{\sum_{j=1}^m n_j\varpi_j : n_j \in \mathbb{Z}, m \in \mathbb{N}, \right.$$
$$\left. \varpi_j \in \Lambda(f), j = 1, 2\ldots, m\right\}$$

the mean value, the set of Fourier exponents and the module of f, respectively.

Set $\mathbb{X} = \mathbb{Y} = \mathbb{V}_1 \bigoplus \mathbb{V}_2$, where

$$\mathbb{V}_1 = \Big\{z = (u, v)^T \in AP(\mathbb{R}, \mathbb{R}^2) :$$
$$\mathrm{mod}(u) \subseteq \mathrm{mod}(L_u),$$
$$\mathrm{mod}(v) \subseteq \mathrm{mod}(L_v),$$
$$\forall\varpi \in \Lambda(u) \cup \Lambda(v), |\varpi| \geq \theta_0\Big\},$$

$$\mathbb{V}_2 = \big\{z = (u, v)^T \equiv (k_1, k_2)^T, k_1, k_2 \in \mathbb{R}\big\},$$

where

$$L_u = L_u(t, \varphi) = \frac{a_{10}(t)}{e^{\varphi_1(-\tau_{10}(0))} + m_1(t)}$$
$$-a_{11}(t)e^{\varphi_2(-\tau_{11}(0))}$$
$$-a_{12}(t)e^{\varphi_2(-\tau_{12}(0))} - c_1(t),$$
$$L_v = L_v(t, \varphi) = \frac{a_{20}(t)}{e^{\varphi_2(-\tau_{20}(0))} + m_2(t)}$$
$$-a_{21}(t)e^{\varphi_1(-\tau_{21}(0))}$$
$$-a_{22}(t)e^{\varphi_2(-\tau_{22}(0))} - c_2(t),$$

$\varphi = (\varphi_1, \varphi_2)^T \in C([-\tau, 0], \mathbb{R}^2)$, $\tau = \max_{i=1,2}\{\mu_i^M, \nu^M\}$, θ_0 is a given positive constant. Define the norm

$$\|z\|_{\mathbb{X}} = \max\left\{\sup_{s\in\mathbb{R}}|u(s)|, \sup_{s\in\mathbb{R}}|v(s)|\right\},$$

where $z = (u, v)^T \in \mathbb{X} = \mathbb{Y}$.

Similar to the proof as that in articles [20], it follows that

Lemma 10. *$\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces endowed with $\|\cdot\|_{\mathbb{X}}$.*

Lemma 11. *Let $L : \mathbb{X} \to \mathbb{Y}$, $Lz = L(u, v)^T = (\dot{u}, \dot{v})^T$, then L is a Fredholm mapping of index zero.*

Lemma 12. *Define* $N : \mathbb{X} \to \mathbb{Y}$, $P : \mathbb{X} \to \mathbb{X}$ *and* $Q : \mathbb{Y} \to \mathbb{Y}$ *by*

$$Nz = \begin{pmatrix} \frac{a_{10}(t)}{e^{u(t-\tau_{10}(t))}+m_1(t)} \\ -a_{11}(t)e^{u(t-\tau_{11}(t))} \\ -a_{12}(t)e^{v(t-\tau_{12}(t))} - c_1(t), \\ \frac{a_{20}(t)}{e^{v(t-\tau_{20}(t))}+m_2(t)} \\ -a_{21}(t)e^{u(t-\tau_{21}(t))} \\ -a_{22}(t)e^{v(t-\tau_{22}(t))} - c_2(t) \end{pmatrix},$$

$$Pz = P\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} m(u) \\ m(v) \end{pmatrix} = Qz.$$

Then N *is L-compact on* $\bar{\Omega}(\Omega$ *is an open and bounded subset of* $\mathbb{X})$.

Now we are in the position to present and prove our result on the existence of at least two positive almost periodic solutions of system (2).

Let

$$\tau := \max_{1\le i\le 2, 0\le j\le 2}\{\tau_{ij}^+\}, \quad \mu_i(s) = \frac{a_{i0}(s)}{m_i(s)},$$

$$\nu_1(s) = \frac{a_{10}(s)}{e^{\rho_1} + m_1^+} - a_{12}(s)e^{\varrho_1} - c_1(s),$$

$$\nu_2(s) = \frac{a_{20}(s)}{e^{\varrho_1} + m_2^+} - a_{21}(s)e^{\rho_1} - c_2(s),$$

$$\rho_1 = \ln\frac{6\bar{\mu}_1}{\bar{a}_{11}} + \frac{a_{10}^+\omega}{m_1^-},$$

$$\varrho_1 = \ln\frac{6\bar{\mu}_2}{\bar{a}_{22}} + \frac{a_{20}^+\omega}{m_2^-}, \quad \forall s\in\mathbb{R}, \quad i = 1, 2,$$

where ω is defined as that in (7).

Theorem 13. *Assume that* (H_1) *holds. Suppose further that*

(H_2) $\bar{a}_{i0} > 0$ *and* $\bar{a}_{ii} > 0$, $i = 1, 2$.

(H_3) $\bar{\nu}_1 > 0$.

(H_4) $\bar{\nu}_2 > 0$.

Then system (2) admits at least one positive almost periodic solution.

Proof. It is easy to see that if system (4) has one almost periodic solution $(u, v)^T$, then $(x_1, x_2)^T = (e^u, e^v)^T$ is a positive almost periodic solution of system (2). Therefore, to completes the proof it suffices to show that system (4) has one almost periodic solution.

In order to use Lemma 9, we set the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ as those in Lemma 10 and L, N, P, Q the same as those defined in Lemmas 11 and 12, respectively. It remains to search for an appropriate open and bounded subset $\Omega \subseteq \mathbb{X}$.

Corresponding to the operator equation $Lz = \lambda z$, $\lambda \in (0, 1)$, we have

$$\begin{cases} \dot{u}(t) = \lambda\Big[\frac{a_{10}(t)}{e^{u(t-\tau_{10}(t))}+m_1(t)} \\ \qquad -a_{11}(t)e^{u(t-\tau_{11}(t))} \\ \qquad -a_{12}(t)e^{v(t-\tau_{12}(t))} - c_1(t)\Big], \\ \dot{v}(t) = \lambda\Big[\frac{a_{20}(t)}{e^{v(t-\tau_{20}(t))}+m_2(t)} \\ \qquad -a_{21}(t)e^{u(t-\tau_{21}(t))} \\ \qquad -a_{22}(t)e^{v(t-\tau_{22}(t))} - c_2(t)\Big]. \end{cases} \tag{5}$$

Suppose that $z = (u, v)^T \in \mathrm{Dom}L \subseteq \mathbb{X}$ is a solution of system (3.1) for some $\lambda \in (0, 1)$, where $\mathrm{Dom}L = \{z = (u, v)^T \in \mathbb{X} : u, v \in C^1(\mathbb{R}), \dot{u}, \dot{v} \in C(\mathbb{R})\}$. By Lemma 6, there exist two sequences $\{\alpha_n : n \in \mathbb{N}^+\}$ and $\{\beta_n : n \in \mathbb{N}^+\}$ such that

$$u(\alpha_n) \in [u^* - \frac{1}{n}, u^*], \quad v(\beta_n) \in [v^* - \frac{1}{n}, v^*], \tag{6}$$

where $u^* = \sup_{s\in\mathbb{R}} u(s)$, $v^* = \sup_{s\in\mathbb{R}} v(s)$.

From (H_1)-(H_2) and Lemma 8, for $\forall t_0 \in \mathbb{R}$, there exists a constant $\omega \in (2\tau, +\infty)$ independent of t_0 such that

$$\begin{cases} \frac{1}{T}\int_{t_0}^{t_0+T} a_{i0}(s)\,\mathrm{d}s \in \left[\frac{\bar{a}_{i0}}{2}, \frac{3\bar{a}_{i0}}{2}\right], \\ \frac{1}{T}\int_{t_0}^{t_0+T} a_{ii}(s)\,\mathrm{d}s \in \left[\frac{\bar{a}_{ii}}{2}, \frac{3\bar{a}_{ii}}{2}\right], \\ \frac{1}{T}\int_{t_0}^{t_0+T} \mu_i(s)\,\mathrm{d}s \in \left[\frac{\bar{\mu}_i}{2}, \frac{3\bar{\mu}_i}{2}\right], \\ \forall T \ge \frac{\omega}{2}, \quad i = 1, 2. \end{cases} \tag{7}$$

For $\forall n_0 \in \mathbb{N}^+$, we consider $[\alpha_{n_0} - \omega, \alpha_{n_0}]$ and $[\beta_{n_0} - \omega, \beta_{n_0}]$, where ω is defined as that in (7). By Lemma 4, there exist $\xi \in [\alpha_{n_0} - \omega, \alpha_{n_0}]$, $\underline{\xi} \in (-\infty, \alpha_{n_0} - \omega]$ and $\bar{\xi} \in [\alpha_{n_0}, +\infty)$ such that

$$u(\underline{\xi}) = u(\bar{\xi}) \quad \text{and} \quad u(\xi) \le u(s), \quad \forall s \in [\underline{\xi}, \bar{\xi}]. \tag{8}$$

Integrating the first equation of system (5) from $\underline{\xi}$ to $\bar{\xi}$ leads to

$$\int_{\underline{\xi}}^{\bar{\xi}} \left[\frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))} + m_1(s)} - a_{11}(s)e^{u(s-\tau_{11}(s))} \right. \\ \left. -a_{12}(s)e^{v(s-\tau_{12}(s))} - c_1(s)\right]\mathrm{d}s = 0,$$

which yields that

$$\begin{aligned}&\int_{\underline{\xi}+\tau_{11}^+}^{\bar{\xi}} a_{11}(s)e^{u(s-\tau_{11}(s))}\,\mathrm{d}s\\ &\le \int_{\underline{\xi}}^{\bar{\xi}} a_{11}(s)e^{u(s-\tau_{11}(s))}\,\mathrm{d}s\\ &\le \int_{\underline{\xi}}^{\bar{\xi}} \frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))}+m_1(s)}\,\mathrm{d}s. \qquad (9)\end{aligned}$$

By the integral mean value theorem and (7), there exists $s_0 \in [\underline{\xi}+\tau_{11}^+, \bar{\xi}]$ $(s_0-\tau_{11}(s_0) \in [\underline{\xi}, \bar{\xi}])$ such that

$$\begin{aligned}&\frac{1}{\bar{\xi}-\underline{\xi}}\int_{\underline{\xi}+\tau_{11}^+}^{\bar{\xi}} a_{11}(s)e^{u(s-\tau_{11}(s))}\,\mathrm{d}s\\ = & \frac{\bar{\xi}-\underline{\xi}-\tau_{11}^+}{\bar{\xi}-\underline{\xi}}\frac{1}{\bar{\xi}-\underline{\xi}-\tau_{11}^+}\\ & \int_{\underline{\xi}+\tau_{11}^+}^{\bar{\xi}} a_{11}(s)\,\mathrm{d}s e^{u(s_0-\tau_{11}(s_0))}\\ \ge & \frac{\bar{\xi}-\underline{\xi}-\tau_{11}^+}{\bar{\xi}-\underline{\xi}}\frac{\bar{a}_{11}}{2}e^{u(s_0-\tau_{11}(s_0))}\\ \ge & \left[1-\frac{\tau_{11}^+}{\omega}\right]\frac{\bar{a}_{11}}{2}e^{u(s_0-\tau_{11}(s_0))}\\ \ge & \frac{\bar{a}_{11}}{4}e^{u(s_0-\tau_{11}(s_0))}. \qquad (10)\end{aligned}$$

By (8)-(10), we have

$$\begin{aligned}&\frac{\bar{a}_{11}}{4}e^{u(\xi)} \le \frac{\bar{a}_{11}}{4}e^{u(s_0-\tau_{11}(s_0))} \le \frac{1}{\bar{\xi}-\underline{\xi}}\\ &\int_{\underline{\xi}}^{\bar{\xi}} \frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))}+m_1(s)}\,\mathrm{d}s \le \frac{3}{2}\bar{\mu}_1,\end{aligned}$$

which implies that

$$u(\xi) \le \ln\frac{6\bar{\mu}_1}{\bar{a}_{11}}. \qquad (11)$$

Let $I=[\xi,\alpha_{n_0}]$ and $I_1=\{s\in I : \dot{u}(s)\ge 0\}$. It follows from system (5) that

$$\begin{aligned}\int_{I_1}\dot{u}(s)\,\mathrm{d}s = & \int_{I_1}\lambda\bigg[\frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))}+m_1(s)}\\ & -a_{11}(s)e^{u(s-\tau_{11}(s))}\\ & -a_{12}(s)e^{v(s-\tau_{12}(s))}-c_1(s)\bigg]\,\mathrm{d}s\\ \le & \int_{I_1}\frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))}+m_1(s)}\,\mathrm{d}s\\ \le & \int_{\alpha_{n_0}-\omega}^{\alpha_{n_0}}\frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))}+m_1(s)}\,\mathrm{d}s\\ \le & \frac{a_{10}^+\omega}{m_1^-}. \qquad (12)\end{aligned}$$

By Lemma 3, it follows from (11)-(12) that

$$\begin{aligned}u(t) \le & u(\xi)+\int_{I_1}\dot{u}(s)\,\mathrm{d}s\\ \le & \ln\frac{6\bar{\mu}_1}{\bar{a}_{11}}+\frac{a_{10}^+\omega}{m_1^-} := \rho_1, \quad \forall t\in[\xi,\alpha_{n_0}],\end{aligned}$$

which implies that

$$u(\alpha_{n_0}) \le \rho_1.$$

In view of (6), letting $n_0 \to +\infty$ in the above inequality leads to

$$u^* = \lim_{n_0\to+\infty} u(\alpha_{n_0}) \le \rho_1. \qquad (13)$$

Similar to the argument as that in (13), we can obtain that

$$v^* \le \ln\frac{6\bar{\mu}_2}{\bar{a}_{22}}+\frac{a_{20}^+\omega}{m_2^-} := \varrho_1. \qquad (14)$$

From (H_3)-(H_4) and Lemma 8, for $\forall t_0\in\mathbb{R}$, there exists a constant $\omega_0\in[\omega,+\infty)$ independent of t_0 such that

$$\frac{1}{T}\int_{t_0}^{t_0+T}\nu_i(s)\,\mathrm{d}s \in \left[\frac{\bar{\nu}_i}{2},\frac{3\bar{\nu}_i}{2}\right],\ \forall T\ge\omega_0,\ i=1,2.$$

Let

$$l=\max\left\{\omega_0, \frac{4a_{11}^+e^{\rho_1}\tau_{11}^+}{\bar{\nu}_1}, \frac{4a_{22}^+e^{\varrho_1}\tau_{22}^+}{\bar{\nu}_2}\right\}.$$

On the other hand, for $\forall n_0\in\mathbb{Z}$, by Lemma 5, we can conclude that there exist $\eta\in[n_0l, n_0l+l]$, $\underline{\eta}\in(-\infty, n_0l]$ and $\bar{\eta}\in[n_0l+l,+\infty)$ such that

$$u(\underline{\eta})=u(\bar{\eta}), \quad u(\eta)\ge u(s), \quad \forall s\in[\underline{\eta},\bar{\eta}]. \qquad (15)$$

Integrating the first equation of system ((5)) from $\underline{\eta}$ to $\bar{\eta}$ leads to

$$\begin{aligned}&\int_{\underline{\eta}}^{\bar{\eta}}\bigg[\frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))}+m_1(s)}-a_{11}(s)e^{u(s-\tau_{11}(s))}\\ &\qquad -a_{12}(s)e^{v(s-\tau_{12}(s))}-c_1(s)\bigg]\,\mathrm{d}s=0,\end{aligned}$$

which yields that

$$\begin{aligned}&\frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}} a_{11}(s)e^{u(s-\tau_{11}(s))}\,\mathrm{d}s\\ = & \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}}\bigg[\frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))}+m_1(s)}\\ & -a_{12}(s)e^{v(s-\tau_{12}(s))}-c_1(s)\bigg]\,\mathrm{d}s\\ \ge & \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}}\left[\frac{a_{10}(s)}{e^{\rho_1}+m_1^+}-a_{12}(s)e^{\varrho_1}-c_1(s)\right]\mathrm{d}s\\ = & \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}}\nu_1(s)\,\mathrm{d}s \ge \frac{\bar{\nu}_1}{2}. \qquad (16)\end{aligned}$$

By (**??**), we have that

$$\begin{aligned}
&\frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}} a_{11}(s)e^{u(s-\tau_{11}(s))}\,\mathrm{d}s \\
\leq\ & \frac{a_{11}^{+}}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}} e^{u(s-\tau_{11}(s))}\,\mathrm{d}s \\
=\ & \frac{a_{11}^{+}}{\bar{\eta}-\underline{\eta}}\bigg[\int_{\underline{\eta}+\tau_{11}^{+}}^{\bar{\eta}} e^{u(s-\tau_{11}(s))}\,\mathrm{d}s \\
& +\int_{\underline{\eta}}^{\underline{\eta}+\tau_{11}^{+}} e^{u(s-\tau_{11}(s))}\,\mathrm{d}s\bigg] \\
\leq\ & \frac{a_{11}^{+}}{\bar{\eta}-\underline{\eta}}\Big[e^{u(\eta)}(\bar{\eta}-\underline{\eta}-\tau_{11}^{+})+e^{\rho_1}\tau_{11}^{+}\Big] \\
\leq\ & a_{11}^{+}e^{u(\eta)}+\frac{a_{11}^{+}e^{\rho_1}\tau_{11}^{+}}{l} \\
\leq\ & a_{11}^{+}e^{u(\eta)}+\frac{\bar{\nu}_1}{4}. \qquad (17)
\end{aligned}$$

From (16)-(17), it follows that

$$u(\eta)\geq \ln\frac{\bar{\nu}_1}{4a_{11}^{+}}. \qquad (18)$$

Further, we obtain from system (5) that

$$\begin{aligned}
&\int_{n_0 l}^{n_0 l+l}|\dot{u}(s)|\,\mathrm{d}s \\
=\ & \int_{n_0 l}^{n_0 l+l}\lambda\bigg|\frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))}+m_1(s)} \\
& -a_{11}(s)e^{u(s-\tau_{11}(s))} \\
& -a_{12}(s)e^{v(s-\tau_{12}(s))}-c_1(s)\bigg|\,\mathrm{d}s \\
\leq\ & \bigg[\frac{3\bar{\mu}_1}{2}+(a_{11}^{+}e^{\rho_1}+a_{12}^{+}e^{\varrho_1}+c_1^{+})\bigg]l. \qquad (19)
\end{aligned}$$

It follows from (18)-(19) that

$$\begin{aligned}
u(t)\ \geq\ & u(\eta)-\int_{n_0 l}^{n_0 l+l}|\dot{u}(s)|\,\mathrm{d}s \\
\geq\ & \ln\frac{\bar{\nu}_1}{4a_{11}^{+}}-\bigg[\frac{3\bar{\mu}_1}{2}+(a_{11}^{+}e^{\rho_1} \\
& +a_{12}^{+}e^{\varrho_1}+c_1^{+})\bigg]l \\
:=\ & \rho_2,\quad \forall t\in[n_0 l, n_0 l+l]. \qquad (20)
\end{aligned}$$

Obviously, ρ_2 is a constant independent of n_0. So it follows from (20) that

$$\begin{aligned}
u_*\ =\ & \inf_{s\in\mathbb{R}} u(s)=\inf_{n_0\in\mathbb{Z}}\Big\{\min_{s\in[n_0 l, n_0 l+l]} u(s)\Big\} \\
\geq\ & \inf_{n_0\in\mathbb{Z}}\{\rho_2\}=\rho_2. \qquad (21)
\end{aligned}$$

Similar to the argument as that in (21), we can obtain that

$$\begin{aligned}
v_*\ \geq\ & \ln\frac{\bar{\nu}_2}{4a_{22}^{+}}-\bigg[\frac{3\bar{\mu}_2}{2}+(a_{22}^{+}e^{\varrho_1}+a_{21}^{+}e^{\rho_1}+c_2^{+})\bigg]l \\
:=\ & \varrho_2. \qquad (22)
\end{aligned}$$

Set $C=|\rho_1|+|\rho_2|+|\varrho_1|+|\varrho_2|+1$. Clearly, C is independent of $\lambda\in(0,1)$. Let $\Omega=\{z\in\mathbb{X}:\|z\|_{\mathbb{X}}<C\}$. Therefore, Ω satisfies condition (a) of Lemma 9.

Now we show that condition (b) of Lemma 9 holds, i.e., we prove that $QNz\neq 0$ for all $z=(u,v)^T\in\partial\Omega\cap\mathrm{Ker}L=\partial\Omega\cap\mathbb{R}^2$. If it is not true, then there exists at least one constant vector $z_0=(u_0,v_0)^T\in\partial\Omega$ such that

$$\begin{cases} 0=m\Big[\frac{a_{10}}{e^{u_0}+m_1}\Big]-\bar{a}_{11}e^{u_0}-\bar{a}_{12}e^{v_0}-\bar{c}_1, \\ 0=m\Big[\frac{a_{20}}{e^{v_0}+m_2}\Big]-\bar{a}_{21}e^{u_0}-\bar{a}_{22}e^{v_0}-\bar{c}_2. \end{cases}$$

Similar to the argument as that in (13), (14), (21) and (22), it follows that

$$\rho_2<u_0<\rho_1,\quad \varrho_2<v_0<\varrho_1.$$

Then $z_0\in\Omega\cap\mathbb{R}^2$. This contradicts the fact that $z_0\in\partial\Omega$. This proves that condition (b) of Lemma 9 holds.

Finally, we will show that condition (c) of Lemma 9 is satisfied. Let us consider the homotopy

$$H(\iota,z)=\iota QNz+(1-\iota)\Phi z,\ \ (\iota,z)\in[0,1]\times\mathbb{R}^2,$$

where

$$\Phi z=\Phi\begin{pmatrix} u \\ v\end{pmatrix}=\begin{pmatrix} \frac{\bar{a}_{10}}{e^{u}+m_1^{+}}-\bar{a}_{11}e^{u} \\ \frac{\bar{a}_{20}}{e^{v}+m_2^{+}}-\bar{a}_{22}e^{v}\end{pmatrix}.$$

From the above discussion it is easy to verify that $H(\iota,z)\neq 0$ on $\partial\Omega\cap\mathrm{Ker}L$, $\forall\iota\in[0,1]$. Further, $\Phi z=0$ has a solution:

$$(u^*,v^*)^T=(\ln k,\ln l)^T,$$

where

$$k=\frac{\sqrt{(m_1^{+}\bar{a}_{11})^2+4\bar{a}_{10}\bar{a}_{11}}-m_1^{+}\bar{a}_{11}}{2\bar{a}_{11}},$$

$$l=\frac{\sqrt{(m_2^{+}\bar{a}_{22})^2+4\bar{a}_{20}\bar{a}_{22}}-m_2^{+}\bar{a}_{22}}{2\bar{a}_{22}}.$$

It is easy to verify that

$$\rho_2<\ln k<\rho_1,\quad \varrho_2<\ln l<\varrho_1.$$

Therefore $(u^*, v^*)^T \in \Omega$. A direct computation yields

$$\begin{aligned}
&\deg\big(\Phi, \Omega \cap \mathrm{Ker}L, 0\big)\\
&= \mathrm{sign}\begin{vmatrix} -\frac{\bar{a}_{10}e^{u^*}}{e^{u^*}+m_1^+} - \bar{a}_{11}e^{u^*} & 0 \\ 0 & -\frac{\bar{a}_{20}e^{v^*}}{e^{v^*}+m_2^+} - \bar{a}_{22}e^{v^*} \end{vmatrix}\\
&= \mathrm{sign}\bigg[\bigg(\frac{\bar{a}_{10}e^{u^*}}{e^{u^*}+m_1^+} + \bar{a}_{11}e^{u^*}\bigg)\\
&\quad \bigg(\frac{\bar{a}_{20}e^{v^*}}{e^{v^*}+m_2^+} + \bar{a}_{22}e^{v^*}\bigg)\bigg]\\
&> 0.
\end{aligned}$$

By the invariance property of homotopy, we have

$$\deg\big(JQN, \Omega \cap \mathrm{Ker}L, 0\big) = \deg\big(QN, \Omega \cap \mathrm{Ker}L, 0\big)$$
$$= \deg\big(\Phi, \Omega \cap \mathrm{Ker}L, 0\big) \neq 0,$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree and J is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$. Obviously, all the conditions of Lemma 9 are satisfied. Therefore, system (4) has at least one almost periodic solution, that is, system (2) has at least one positive almost periodic solution. This completes the proof. □

Corollary 14. *Assume that* (H_1)*-*(H_4) *hold. Suppose further that* a_{ij}, τ_{ij}, m_i *and* c_i *in system* (1.2) *are continuous nonnegative periodic functions with periods* α_{ij}, β_i, σ_i *and* δ_i, *respectively,* $i = 1, 2$, $j = 0, 1, 2$, *then system* (1.2) *has at least one positive almost periodic solution.*

Remark 15. *By Corollary 14, it is easy to obtain the existence of at least one positive almost periodic solution of Eq. (3) in Example 1, although there is no a priori reason to expect the existence of positive periodic solutions of Eq.(3).*

In Corollary 14, let $\alpha_{ij} = \beta_i = \sigma_i = \delta_i = \omega$, $i = 1, 2$, $j = 0, 1, 2$, then we obtain that

Corollary 16. *Assume that* (H_1)*-*(H_4) *hold. Suppose further that* a_{ij}, τ_{ij}, m_i *and* c_i *in system (2) are continuous nonnegative* ω*-periodic functions,* $i = 1, 2$, $j = 0, 1, 2$, *then system (2) has at least one positive* ω*-periodic solution.*

Let

$$\phi_1(s) := a_{11}(s)e^{\rho_1} + a_{12}(s)e^{\varrho_1} + c_1(s),$$

$$\phi_2(s) := a_{21}(s)e^{\rho_1} + a_{22}(s)e^{\varrho_1} + c_2(s), \quad \forall s \in \mathbb{R},$$

where ρ_1 and ϱ_1 are defined as that in Theorem 13.

Theorem 17. *Assume that* (H_1)*-*(H_2) *hold. Suppose further that*

(H_5) $\bar{a}_{10} > \bar{\phi}_1 m_1^+ > 0$,

(H_6) $\bar{a}_{20} > \bar{\phi}_2 m_2^+ > 0$,

then system (2) admits at least one positive almost periodic solution.

Proof. Proceeding as in the proof of Theorem 13, it remains to search for an appropriate open and bounded subset $\Omega \subseteq \mathbb{X}$.

Consider the operator equations (5). From the proof of Theorem 13, (13)-(14) are valid. In view of (H_5)-(H_6), there must exist small enough $\epsilon_0 > 0$ such that

$$\frac{(1-\epsilon_0)(\bar{a}_{i0}-\epsilon_0)}{\bar{\phi}_i + \epsilon_0} > m_i^+, \quad i = 1, 2.$$

By Lemma 7, for $\forall t_0 \in \mathbb{R}$, there must exist large enough $T_0 = T_0(\epsilon_0)$ such that $\frac{\tau_{i0}^+}{T_0} \leq \epsilon_0$ and

$$\frac{1}{T}\int_{t_0}^{t_0+T} a_{i0}(s)\,\mathrm{d}s \in \left[\bar{a}_{i0} - \epsilon_0, \bar{a}_{i0} + \epsilon_0\right],$$

$$\frac{1}{T}\int_{t_0}^{t_0+T} \phi_i(s)\,\mathrm{d}s \in \left[\bar{\phi}_i - \epsilon_0, \bar{\phi}_i + \epsilon_0\right], \ \forall T \geq T_0,$$

where $i = 1, 2$. From Lemma 7, there also exist $\eta \in [n_0T_0, n_0T_0 + T_0]$, $\underline{\eta} \in (-\infty, n_0T_0]$ and $\bar{\eta} \in [n_0T_0 + T_0, +\infty)(n_0 \in \mathbb{Z})$ such that

$$u(\underline{\eta}) = u(\bar{\eta}), \quad u(\eta) \geq u(s), \quad \forall s \in [\underline{\eta}, \bar{\eta}]. \tag{23}$$

Integrating the first equation of system(5) from $\underline{\eta}$ to $\bar{\eta}$ leads to

$$\begin{aligned}
\int_{\underline{\eta}}^{\bar{\eta}} \bigg[&\frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))} + m_1(s)} - a_{11}(s)e^{u(s-\tau_{11}(s))}\\
&-a_{12}(s)e^{v(s-\tau_{12}(s))} - c_1(s)\bigg]\,\mathrm{d}s = 0,
\end{aligned}$$

which yields that

$$\begin{aligned}
&\frac{(1-\epsilon_0)(\bar{a}_{10}-\epsilon_0)}{e^{u(\eta)}+m_1^+}\\
\le\ & \frac{1}{e^{u(\eta)}+m_1^+}\\
&\frac{\bar{\eta}-\underline{\eta}-\tau_{10}^+}{\bar{\eta}-\underline{\eta}}\frac{1}{\bar{\eta}-\underline{\eta}-\tau_{10}^+}\int_{\underline{\eta}+\tau_{10}^+}^{\bar{\eta}} a_{10}(s)\,\mathrm{d}s\\
\le\ & \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}+\tau_{10}^+}^{\bar{\eta}}\frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))}+m_1(s)}\,\mathrm{d}s\\
\le\ & \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}}\frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))}+m_1(s)}\,\mathrm{d}s\\
=\ & \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}}\bigg[a_{11}(s)e^{u(s-\tau_{11}(s))}\\
&+a_{12}(s)e^{v(s-\tau_{12}(s))}+c_1(s)\bigg]\,\mathrm{d}s\\
\le\ & \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}}\left[a_{11}(s)e^{\rho_1}+a_{12}(s)e^{\varrho_1}+c_1(s)\right]\mathrm{d}s\\
\le\ & (\bar{\phi}_1+\epsilon_0). \qquad (24)
\end{aligned}$$

From (24), it follows that

$$u(\eta)\ge \ln\left[\frac{(1-\epsilon_0)(\bar{a}_{10}-\epsilon_0)}{\bar{\phi}_1+\epsilon_0}-m_1^+\right]. \qquad (25)$$

Further, we obtain from the first equation of system (5) that

$$\begin{aligned}
&\int_{n_0T_0}^{n_0T_0+T_0}|\dot{u}(s)|\,\mathrm{d}s\\
=\ & \int_{n_0T_0}^{n_0T_0+T_0}\lambda\bigg|\frac{a_{10}(s)}{e^{u(s-\tau_{10}(s))}+m_1(s)}\\
&-a_{11}(s)e^{u(s-\tau_{11}(s))}-a_{12}(s)e^{v(s-\tau_{12}(s))}\\
&-c_1(s)\bigg|\,\mathrm{d}s\\
\le\ & \left(\frac{a_{10}^+}{m_1^-}+\phi_1^+\right)T_0. \qquad (26)
\end{aligned}$$

It follows from (25)-(26) that

$$\begin{aligned}
u(t)\ &\ge\ u(\eta)-\int_{n_0T_0}^{n_0T_0+T_0}|\dot{u}(s)|\,\mathrm{d}s\\
&\ge\ \ln\left[\frac{(1-\epsilon_0)(\bar{a}_{10}-\epsilon_0)}{\bar{\phi}_1+\epsilon_0}-m_1^+\right]\\
&\quad -\left(\frac{a_{10}^+}{m_1^-}+\phi_1^+\right)T_0\\
&:=\ \rho_3,\quad \forall t\in[n_0T_0,n_0T_0+T_0]. \qquad (27)
\end{aligned}$$

Obviously, ρ_3 is a constant independent of n_0. So it follows from (27) that

$$\begin{aligned}
u_* &= \inf_{s\in\mathbb{R}} u(s)=\inf_{n_0\in\mathbb{Z}}\Big\{\min_{s\in[n_0T_0,n_0T_0+T_0]}u(s)\Big\}\\
&\ge \inf_{n_0\in\mathbb{Z}}\{\rho_3\}=\rho_3. \qquad (28)
\end{aligned}$$

Similar to the argument as that in (28) , we can obtain that

$$\begin{aligned}
v_* &\ge \ln\left[\frac{(1-\epsilon_0)(\bar{a}_{20}-\epsilon_0)}{\bar{\phi}_2+\epsilon_0}-m_2^+\right]\\
&\quad -\left(\frac{a_{20}^+}{m_2^-}+\phi_2^+\right)T_0 := \varrho_3. \qquad (29)
\end{aligned}$$

Set $C_0=|\rho_1|+|\rho_3|+|\varrho_1|+|\varrho_3|+1$. Clearly, C_0 is independent of $\lambda\in(0,1)$. Let $\Omega=\{z\in\mathbb{X}:\|z\|_{\mathbb{X}}<C_0\}$. From the proof in Theorem 13, it is easy to verify that Ω satisfies conditions (a)-(c) of Lemma 9. Obviously, all the conditions of Lemma 9 are satisfied. Therefore, system (4) has one almost periodic solution, that is, system (2) has at least one positive almost periodic solution. This completes the proof. □

Together with Theorem 13 and Theorem 18, we can easily show that

Theorem 18. *Assume that (H_1)-(H_3) and (H_6) hold, then system (2) admits at least one positive almost periodic solution.*

Theorem 19. *Assume that (H_1)-(H_2) and (H_4)-(H_5) hold, then system (2) admits at least one positive almost periodic solution.*

Together with Corollaries 14-16, we obtain that

Corollary 20. *Assume that (H_1)-(H_3) and (H_6) hold. Suppose further that a_{ij}, τ_{ij}, m_i and c_i in system (2) are continuous nonnegative periodic functions with periods α_{ij}, β_i, σ_i and δ_i, respectively, $i=1,2$, $j=0,1,2$, then system (2) has at least one positive almost periodic solution.*

Corollary 21. *Assume that (H_1)-(H_2) and (H_4)-(H_5) hold. Suppose further that a_{ij}, τ_{ij}, m_i and c_i in system (2) are continuous nonnegative periodic functions with periods α_{ij}, β_i, σ_i and δ_i, respectively, $i=1,2$, $j=0,1,2$, then system (2) has at least one positive almost periodic solution.*

In Corollaries 20-21, let $\alpha_{ij}=\beta_i=\sigma_i=\delta_i=\omega$, $i=1,2$, $j=0,1,2$, then we have

Corollary 22. *Assume that (H_1)-(H_3) and (H_6) hold. Suppose further that a_{ij}, τ_{ij}, m_i and c_i in system (2) are continuous nonnegative ω-periodic functions, $i=1,2$, $j=0,1,2$, then system (2) has at least one positive ω-periodic solution.*

Corollary 23. *Assume that (H_1)-(H_2) and (H_4)-(H_5) hold. Suppose further that a_{ij}, τ_{ij}, m_i and c_i in system (2) are continuous nonnegative ω-periodic functions, $i = 1, 2$, $j = 0, 1, 2$, then system (2) has at least one positive ω-periodic solution.*

4 Two examples and numerical simulations

Example 24. *Consider the following Schoener's competition model with time-varying delays:*

$$\begin{cases} \dot{x}_1(t) = x_1(t)\Bigg[\frac{|\sin(\sqrt{3}t)|}{x_1(t-\sin^2(\sqrt{2}t))+1} \\ \qquad - \sin^2(\sqrt{3}t)x_1(t-1) \\ \qquad -\frac{10^{-4}}{3000}x_2(t) - 10^{-5}\Bigg], \\ \dot{x}_2(t) = x_2(t)\Bigg[\frac{|\cos(\sqrt{2}t)|}{x_2(t)+1} \\ \qquad -\frac{10^{-4}}{3000}x_1(t) \\ \qquad -\cos^2(\sqrt{2}t)x_2(t-\cos^2(t)) - 10^{-5}\Bigg]. \end{cases} \tag{30}$$

Then system (30) has at least one positive almost periodic solution.

Proof. Corresponding to system (2), we have $\bar{a}_{10} = \bar{a}_{20} = \frac{2}{\pi}$, $\bar{a}_{11} = \bar{a}_{22} = \frac{1}{2}$, $m_i \equiv 1$, $a_{12} = a_{21} \equiv \frac{10^{-4}}{3000}$, $c_1 = c_2 \equiv 10^{-5}$, $i = 1, 2$. Obviously, (H_1)-(H_2) hold. Further, for $\forall t_0 \in \mathbb{R}$, we can choose $\omega = 8$ so that (7) holds, that is,

$$\frac{1}{T}\int_a^{a+T} a_{i0}(s)\,\mathrm{d}s \in \left[\frac{1}{\pi}, \frac{3}{\pi}\right],$$

$$\frac{1}{T}\int_a^{a+T} a_{ii}(s)\,\mathrm{d}s \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

$$\forall T \geq \frac{\omega}{2} = 4, \ i = 1, 2.$$

By a easy calculation, we obtain that

$$\rho_1 = \varrho_1 \approx 6, \quad \bar{\nu}_1 = \bar{\nu}_2 > 0.00005.$$

So (H_3)-(H_4) in Theorem 13 hold. Therefore, all the conditions of Theorem 13 are satisfied. By Theorem 13, system (30) admits at least one positive almost periodic solution (see Figure 1). This completes the proof. □

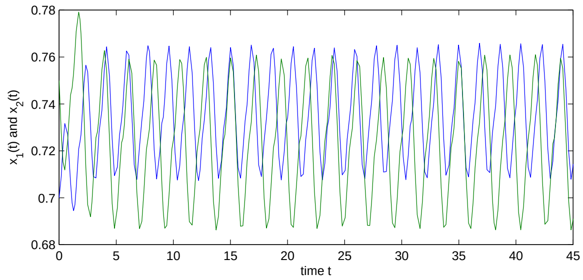

Figure 1 State variables x_1 and x_2 of system (30)

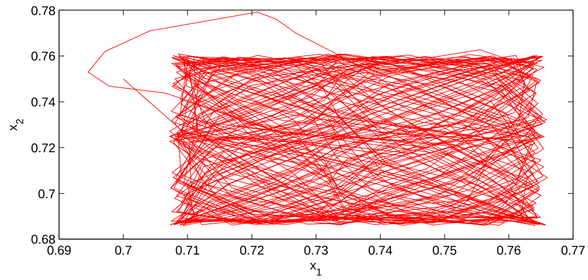

Figure 2 Phase responses of states x_1 and x_2 of system (30)

Remark 25. *In system (30), $|\cos\sqrt{2}t|$ is $\frac{\sqrt{2}\pi}{2}$-periodic function and $|\sin\sqrt{3}t|$ is $\frac{\sqrt{3}\pi}{3}$-periodic function. So system (30) is with incommensurable periods. Through all the coefficients of system (30) are periodic functions, the positive periodic solutions of system (30) could not possibly exist. However, by Theorem 13, the positive almost periodic solutions of system (30) exactly exist.*

Example 26. *Consider the following almost periodic Schoener's competition model with time-varying delays:*

$$\begin{cases} \dot{x}_1(t) = x_1(t)\Bigg[\frac{|\sin\sqrt{2}t|+|\sin\sqrt{3}t|}{2x_1(t-\sin^2(\sqrt{2}t))+2} \\ \qquad - \sin^2(\sqrt{3}t)x_1(t-1) \\ \qquad -\frac{10^{-4}}{3000}x_2(t) - 10^{-5}\Bigg], \\ \dot{x}_2(t) = x_2(t)\Bigg[\frac{|\cos(\sqrt{2}t)|}{x_2(t)+1} - \frac{10^{-4}}{3000}x_1(t) \\ \qquad -0.5\left(\cos^2(\sqrt{2}t) + \cos^2(\sqrt{3}t)\right) \\ \qquad x_2(t-\cos^2(t)) - 10^{-5}\Bigg]. \end{cases} \tag{31}$$

In system (31), $|\sin\sqrt{2}t| + |\sin\sqrt{3}t|$ and $\cos^2(\sqrt{2}t) + \cos^2(\sqrt{3}t)$ are almost periodic functions, which are not periodic functions. Similar to the argument as that in Example 24, it is easy to obtain that system (30) admits at least one positive almost periodic solution.

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