The Riccati Sub-ODE Method For Fractional Differential-difference Equations

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Abstract: In this paper, we are concerned with seeking exact solutions for fractional differential-difference equations by an extended Riccati sub-ODE method. The fractional derivative is defined in the sense of the modified Riemann-liouville derivative. By a combination of this method and a fractional complex transformation, the iterative relations from indices n to $n \pm 1$ are established. As for applications, we apply this method to solve the two-component fractional Volterra lattice equations and the fractional m-KdV lattice equation. Some new exact solutions for the two fractional differential-difference equations are obtained.

Key–Words: Fractional differential-difference equations; Exact solutions; Riccati sub-ODE method; Fractional complex transformations; Traveling wave solutions; Nonlinear evolution equations
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1 Introduction

Nonlinear differential equations (NLDEs) and nonlinear differential-difference equations (NLDDEs) can find their applications in many aspects of mathematical physics. In the last decades, research on seeking exact solutions for NLDEs and NLDDEs has been a hot topic, and many effective methods have been presented so far (see [1-28] and the references therein). Among these investigations, we notice that little attention is paid to fractional differential-difference equations (FDDEs).

In this paper, we extend the Riccati sub-ODE method to seek exact solutions for FDDEs. The fractional derivative is defined in the sense of modified Riemann-liouville derivative [29-33] as follows.

$$D_t^\gamma f(t) = \begin{cases} \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t (t-\xi)^{-\gamma}(f(\xi)-f(0))d\xi, \\ \qquad 0<\gamma<1, \\ (f^{(n)}(t))^{(\gamma-n)}, \ n \le \gamma < n+1, \ n \ge 1. \end{cases}$$

Based on a fractional complex transformation, a given fractional differential-difference equation can be turned into another differential-difference equation of integer order, and the iterative relations of which from indices n to $n \pm 1$ are also established. By this approach, we will solve two fractional differential-difference equations: the two-component fractional Volterra lattice equations

$$\begin{cases} D_t^\gamma u_n = u_n(v_n - v_{n-1}), \\ D_t^\gamma v_n = v_n(u_{n+1} - u_n), \end{cases} \tag{1}$$

and the following fractional m-KdV lattice equation

$$D_t^\gamma u_n = (\alpha - u_n^2)(u_{n+1} - u_{n-1}), \tag{2}$$

where $0 < \gamma \le 1$, $u_n = u_n(t)$, $v_n = v_n(t)$, $n \in \mathbb{Z}$, and D_t^γ denotes the modified Riemann-liouville derivative of order γ with respect to the variable t.

When $\gamma = 1$, Eqs. (1) become the known two-component Volterra lattice equations [34], while Eq. (2) becomes the m-KdV lattice equation [34].

The following properties for the modified Riemann-Liouville are known to us (see [30-33]):

$$D_t^\gamma t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\gamma)} t^{r-\gamma}, \tag{3}$$

$$D_t^\gamma (f(t)g(t)) = g(t) D_t^\gamma f(t) + f(t) D_t^\gamma g(t), \tag{4}$$

$$D_t^\gamma f[g(t)] = f'_g[g(t)] D_t^\gamma g(t) = D_g^\gamma f[g(t)](g'(t))^\gamma. \tag{5}$$

2 Description of the extended Riccati sub-ODE method for fractional differential-difference equations

The main steps of the extended Riccati sub-ODE method for solving fractional differential-difference equations are summarized as follows:

Step 1. Consider a fractional differential difference equation in the form

$$P(u_{n+p_1}(x),...,u_{n+p_k}(x),...,D^{\gamma}u_{n+p_1}(x),...,$$
$$D^{\gamma}u_{n+p_k}(x),D^{2\gamma}u_{n+p_1}(x),...,D^{2\gamma}u_{n+p_k}(x),...)=0, \tag{6}$$

where the dependent variable u has M components u_i, the continuous variable x has N components x_j, the discrete variable n has Q components n_i, the k shift vectors $p_s \in \mathbb{Z}^{\mathbb{Q}}$ has Q components p_{sj}, $D^{k\gamma}$ denotes the collection of mixed derivative terms of order $k\gamma$, and the order of the derivatives with respect to every x_j are integer multiples of γ.

Step 2. Using a fractional complex transformation $X_j = \frac{x_j^{\gamma}}{\Gamma(1+\gamma)}$, and letting $u_{n+p_s}(x) = U_{n+p_s}(X)$, Eq. (6) can be turned into

$$\widetilde{P}(U_{n+p_1}(X),...,U_{n+p_k}(X),...,U'_{n+p_1}(X),...,$$
$$U'_{n+p_k}(X),U''_{n+p_1}(X),...,U''_{n+p_k}(X),...)=0, \tag{7}$$

Step 2. Using a wave transformation

$$U_{n+p_s}(X)=\widetilde{U}_{n+p_s}(\xi_n),\ \xi_n=\sum_{i=1}^{Q}d_in_i+\sum_{j=1}^{N}c_jX_j+\zeta,$$

where $d_i,\ c_j,\ \zeta$ are all constants, we can rewrite Eq. (7) in the following form

$$\widetilde{\widetilde{P}}(\widetilde{U}_{n+p_1}(\xi_n),...,\widetilde{U}_{n+p_k}(\xi_n),...,\widetilde{U}'_{n+p_1}(\xi_n),...,$$
$$\widetilde{U}'_{n+p_k}(\xi_n),...,\widetilde{U}^{(r)}_{n+p_1}(\xi_n),...,\widetilde{U}^{(r)}_{n+p_k}(\xi_n))=0. \tag{8}$$

Step 3: Suppose the solutions of Eq. (8) can be denoted by

$$\widetilde{U}_n(\xi_n)=\sum_{i=0}^{l}a_i\phi^i(\xi_n), \tag{9}$$

where a_i are constants to be determined later, l is a positive integer that can be determined by balancing the highest order linear term with the nonlinear terms in Eq. (8), $\phi(\xi_n)$ satisfies the known Riccati equation:

$$\phi'(\xi_n)=\sigma+\phi^2(\xi_n). \tag{10}$$

Step 4: We present some special solutions $\phi_1,...,\phi_6$ for Eq. (10):

When $\sigma<0$:

$$\begin{cases}\phi_1(\xi_n)=-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi_n+c_0),\\ \phi_2(\xi_n)=-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi_n+c_0),\\ \phi_{1,2}(\xi_{n+p_s})=\dfrac{\phi_{1,2}(\xi_n)-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\sum_{i=1}^{Q}d_ip_{si})}{1-\dfrac{\phi_{1,2}(\xi_n)}{\sqrt{-\sigma}}\tanh(\sqrt{-\sigma}\sum_{i=1}^{Q}d_ip_{si})},\end{cases} \tag{11}$$

where c_0 is an arbitrary constant.

When $\sigma>0$:

$$\begin{cases}\phi_3(\xi_n)=\sqrt{\sigma}\tan(\sqrt{\sigma}\xi_n+c_0),\\ \phi_4(\xi_n)=-\sqrt{\sigma}\cot(\sqrt{\sigma}\xi_n+c_0),\\ \phi_{3,4}(\xi_{n+p_s})=\dfrac{\phi_{3,4}(\xi_n)+\sqrt{\sigma}\tan(\sqrt{\sigma}\sum_{i=1}^{Q}d_ip_{si})}{1-\dfrac{\phi_{3,4}(\xi_n)}{\sqrt{\sigma}}\tan(\sqrt{\sigma}\sum_{i=1}^{Q}d_ip_{si})},\end{cases} \tag{12}$$

and

$$\begin{cases}\phi_5(\xi_n)=\sqrt{\sigma}[\tan(2\sqrt{\sigma}\xi_n+c_0)+|\sec(2\sqrt{\sigma}\xi_n+c_0)|],\\ \phi_5(\xi_{n+p_s})=\dfrac{\phi_5^{(1)}(\xi_n)+\sqrt{\sigma}\tan(2\sqrt{\sigma}\sum_{i=1}^{Q}d_ip_{si})}{1-\dfrac{\phi_5^{(1)}(\xi_n)}{\sqrt{\sigma}}\tan(2\sqrt{\sigma}\sum_{i=1}^{Q}d_ip_{si})}\\ \quad+\dfrac{\phi_5^{(2)}(\xi_n)\sec(2\sqrt{\sigma}\sum_{i=1}^{Q}d_ip_{si})}{1-\dfrac{\phi_5^{(1)}(\xi_n)}{\sqrt{\sigma}}\tan(2\sqrt{\sigma}\sum_{i=1}^{Q}d_ip_{si})},\end{cases} \tag{13}$$

where $\phi_5^{(1)}(\xi_n)=\sqrt{\sigma}tan(2\sqrt{\sigma}\xi_n+c_0),\ \phi_5^{(2)}(\xi_n)=\sqrt{\sigma}|sec(2\sqrt{\sigma}\xi_n+c_0)|$, and c_0 is an arbitrary constant.

When $\sigma=0$:

$$\begin{cases}\phi_6(\xi_n)=-\dfrac{1}{\xi_n+c_0},\\ \phi_6(\xi_{n+p_s})=\dfrac{\phi_6(\xi_n)}{1-\phi_6(\xi_n)\sum_{i=1}^{Q}d_ip_{si}},\end{cases} \tag{14}$$

where c_0 is an arbitrary constant.

Step 5: Substituting (9) into Eq. (8), by use of Eqs. (10)-(14), the left hand side of Eq. (8) can be converted into a polynomial in $\phi(\xi_n)$. Equating each coefficient of $\phi^i(\xi_n)$ to zero, yields a set of algebraic equations. Solving these equations, we can obtain the values of $a_i, d_i,\ c_j$.

Step 6: Substituting the values of a_i into (9), and combining with the various solutions of Eq. (10), we can obtain a variety of exact solutions for Eq. (6).

3 Application of the extended Riccati sub-ODE method to the two-component Volterra lattice equations

In this section, we apply the extended Riccati sub-ODE method described in Section 2 to solve the two-component Volterra lattice equations denoted by Eqs. (1).

Using a fractional complex transformation $T = \frac{t^\gamma}{\Gamma(1+\gamma)}$, and letting $u_{n+p_s}(t) = U_{n+p_s}(T),\ p_s = 0, \pm 1$, by (3) we have $D_t^\gamma T = 1$, and furthermore, by the first equality in (5), Eqs. (1) can be turned into

$$\begin{cases} U_n'(T) = U_n(T)[V_n(T) - V_{n-1}(T)], \\ V_n' = V_n(T)[U_{n+1}(T) - U_n(T)], \end{cases} \tag{15}$$

Letting

$$U_n(T) = \widetilde{U}_n(\xi_n),\ V_n(T) = \widetilde{V}_n(\xi_n),$$

$$\xi_n = d_1 n + c_1 T + \zeta, \tag{16}$$

where $d_1,\ c_1,\ \zeta$ are all constants, Eqs. (15) can be rewritten as the following form:

$$\begin{cases} c_1\widetilde{U}_n'(\xi_n) = \widetilde{U}_n(\xi_n)[\widetilde{V}_n(\xi_n) - \widetilde{V}_{n-1}(\xi_{n-1})], \\ c_1\widetilde{V}_n'(\xi_n) = \widetilde{V}_n(\xi_n)[\widetilde{U}_{n+1}(\xi_{n+1}) - \widetilde{U}_n(\xi_n)], \end{cases} \tag{17}$$

Suppose the solutions for (17) can be denoted by

$$\widetilde{U}_n(\xi_n) = \sum_{i=0}^{l_1} a_i\phi^i(\xi_n), \tag{18}$$

$$\widetilde{V}_n(\xi_n) = \sum_{i=0}^{l_2} b_i\phi^i(\xi_n), \tag{19}$$

where $\phi(\xi_n)$ satisfies Eq. (10). In Eqs. (17), by balancing the order of $\widetilde{U}_n'$ and $\widetilde{U}_n\widetilde{V}_n$, the order of $\widetilde{V}_n'$ and $\widetilde{V}_n\widetilde{U}_n$, we obtain $l_1 = l_2 = 1$. So we have

$$\widetilde{U}_n(\xi_n) = a_0 + a_1\phi(\xi_n). \tag{20}$$

$$\widetilde{V}_n(\xi_n) = b_0 + b_1\phi(\xi_n). \tag{21}$$

We will proceed to solve Eqs. (17) in several cases.

Case 1: If $\sigma < 0$, and assume (10) and (11) hold, then substituting (20), (21), (10) and (11) into Eqs. (17), collecting the coefficients of $\phi_{1,2}^i(\xi_n)$ and equating them to zero, we obtain a series of algebra equations. Solving these equations, yields

$$a_1 = -c_1, a_0 = -\frac{c_1\sqrt{-\sigma}}{\tanh(\sqrt{-\sigma}d_1)},$$

$$b_1 = c_1,\ b_0 = -\frac{c_1\sqrt{-\sigma}}{\tanh(\sqrt{-\sigma}d_1)},$$

$$d_1 = d_1,\ c_1 = c_1,$$

or

$$a_1 = \frac{a_0\tanh(\sqrt{-\sigma}d_1)}{\sqrt{-\sigma}},\ a_0 = a_0,$$

$$b_1 = -\frac{a_0\tanh(\sqrt{-\sigma}d_1)}{\sqrt{-\sigma}},\ b_0 = a_0,$$

$$c_1 = -\frac{a_0\tanh(\sqrt{-\sigma}d_1)}{\sqrt{-\sigma}},\ d_1 = d_1.$$

So we obtain the following four groups of solitary wave solutions:

$$\begin{cases} u_n(t) = -\frac{c_1\sqrt{-\sigma}}{\tanh(\sqrt{-\sigma}d_1)} \\ +c_1\sqrt{-\sigma}\tanh[\sqrt{-\sigma}(d_1 n + \frac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0], \\ v_n(t) = -\frac{c_1\sqrt{-\sigma}}{\tanh(\sqrt{-\sigma}d_1)} \\ -c_1\sqrt{-\sigma}\tanh[\sqrt{-\sigma}(d_1 n + \frac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0], \end{cases} \tag{22}$$

$$\begin{cases} u_n(t) = -\frac{c_1\sqrt{-\sigma}}{\tanh(\sqrt{-\sigma}d_1)} \\ +c_1\sqrt{-\sigma}\coth[\sqrt{-\sigma}(d_1 n + \frac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0], \\ v_n(t) = -\frac{c_1\sqrt{-\sigma}}{\tanh(\sqrt{-\sigma}d_1)} \\ -c_1\sqrt{-\sigma}\coth[\sqrt{-\sigma}(d_1 n + \frac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0], \end{cases} \tag{23}$$

where $d_1,\ c_1,\ c_0$ are arbitrary constants, and

$$\begin{cases} u_n(t) = a_0 - a_0\tanh(\sqrt{-\sigma}d_1)\times \\ \tanh[\sqrt{-\sigma}(d_1 n - \frac{a_0\tanh(\sqrt{-\sigma}d_1)t^\gamma}{\sqrt{-\sigma}\Gamma(1+\gamma)} + \zeta) + c_0], \\ v_n(t) = a_0 + a_0\tanh(\sqrt{-\sigma}d_1)\times \\ \tanh[\sqrt{-\sigma}(d_1 n - \frac{a_0\tanh(\sqrt{-\sigma}d_1)t^\gamma}{\sqrt{-\sigma}\Gamma(1+\gamma)} + \zeta) + c_0], \end{cases} \tag{24}$$

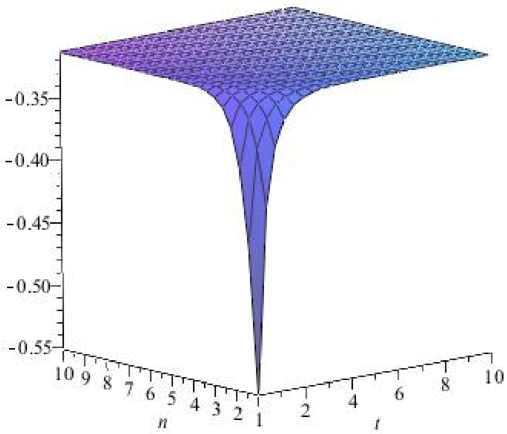

Fig. 1 The solitary wave solution $u_n(t)$ in (22) with $\gamma = 4/5$, $\sigma = -1$, $\zeta = 0$, $c_0 = 0$, $c_1 = 1$, $d_1 = 1$

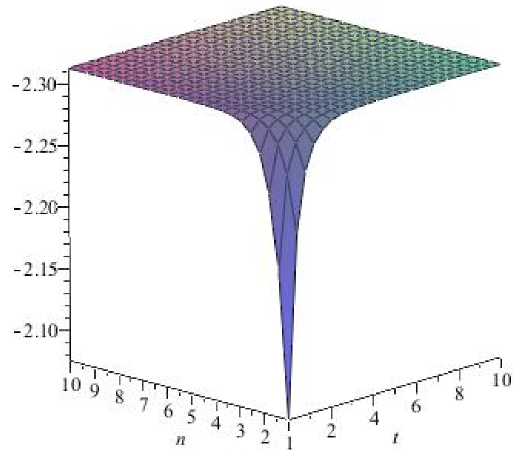

Fig. 2 The solitary wave solution $v_n(t)$ in (22) with $\gamma = 4/5$, $\sigma = -1$, $\zeta = 0$, $c_0 = 0$, $c_1 = 1$, $d_1 = 1$

$$\begin{cases} u_n(t) = a_0 - a_0 \tanh(\sqrt{-\sigma} d_1) \times \\ \coth[\sqrt{-\sigma}(d_1 n - \dfrac{a_0 \tanh(\sqrt{-\sigma} d_1) t^\gamma}{\sqrt{-\sigma}\Gamma(1+\gamma)} + \zeta) + c_0], \\ v_n(t) = a_0 + a_0 \tanh(\sqrt{-\sigma} d_1) \times \\ \coth[\sqrt{-\sigma}(d_1 n - \dfrac{a_0 \tanh(\sqrt{-\sigma} d_1) t^\gamma}{\sqrt{-\sigma}\Gamma(1+\gamma)} + \zeta) + c_0], \end{cases} \tag{25}$$

where $d_1,\ c_0,\ a_0$ are arbitrary constants.

In Figs 1-2, the solitary wave solutions (22) with some special parameters are demonstrated.

Case 2: If $\sigma > 0$, and assume (10) and (12) hold, then substituting (20), (21), (10) and (12) into Eqs. (17), collecting the coefficients of $\phi_{3,4}^i(\xi_n)$ and equating them to zero, we obtain a series of algebra equations. Solving these equations, yields

$$a_1 = -c_1, a_0 = -\frac{c_1\sqrt{\sigma}}{\tan(\sqrt{\sigma} d_1)},\ b_1 = c_1,$$

$$b_0 = -\frac{c_1\sqrt{\sigma}}{\tan(\sqrt{\sigma} d_1)},\ d_1 = d_1,\ c_1 = c_1,$$

or

$$a_1 = \frac{a_0 \tan(\sqrt{\sigma} d_1)}{\sqrt{\sigma}},\ a_0 = a_0,\ b_1 = -\frac{a_0 \tan(\sqrt{\sigma} d_1)}{\sqrt{\sigma}},$$

$$b_0 = a_0,\ c_1 = -\frac{a_0 \tan(\sqrt{\sigma} d_1)}{\sqrt{\sigma}},\ d_1 = d_1.$$

So we obtain the following periodic wave solutions:

$$\begin{cases} u_n(t) = -\dfrac{c_1\sqrt{\sigma}}{\tan(\sqrt{\sigma} d_1)} \\ -c_1\sqrt{\sigma}\tan[\sqrt{\sigma}(d_1 n + \dfrac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0], \\ v_n(t) = -\dfrac{c_1\sqrt{\sigma}}{\tan(\sqrt{\sigma} d_1)} \\ +c_1\sqrt{\sigma}\tan[\sqrt{\sigma}(d_1 n + \dfrac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0], \end{cases} \tag{26}$$

$$\begin{cases} u_n(t) = -\dfrac{c_1\sqrt{\sigma}}{\tan(\sqrt{\sigma} d_1)} \\ +c_1\sqrt{\sigma}\cot[\sqrt{\sigma}(d_1 n + \dfrac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0], \\ v_n(t) = -\dfrac{c_1\sqrt{\sigma}}{\tan(\sqrt{\sigma} d_1)} \\ -c_1\sqrt{\sigma}\cot[\sqrt{\sigma}(d_1 n + \dfrac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0], \end{cases} \tag{27}$$

where $d_1,\ c_1,\ c_0$ are arbitrary constants, and

$$\begin{cases} u_n(t) = a_0 + a_0 \tan(\sqrt{\sigma} d_1) \times \\ \tan[\sqrt{\sigma}(d_1 n - \dfrac{a_0 \tan(\sqrt{\sigma} d_1) t^\gamma}{\sqrt{\sigma}\Gamma(1+\gamma)} + \zeta) + c_0], \\ v_n(t) = a_0 - a_0 \tan(\sqrt{\sigma} d_1) \times \\ \tan[\sqrt{\sigma}(d_1 n - \dfrac{a_0 \tan(\sqrt{\sigma} d_1) t^\gamma}{\sqrt{\sigma}\Gamma(1+\gamma)} + \zeta) + c_0], \end{cases} \tag{28}$$

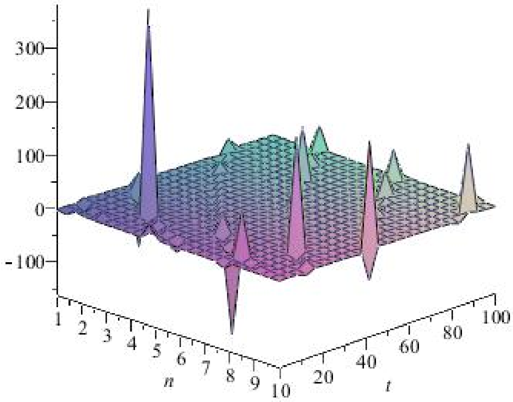

Fig. 3 The periodic wave solution $u_n(t)$ in (26) with γ=1/2,σ=1, ζ=0, $c_0 = 0$, $c_1 = 1$, $d_1 = 1$

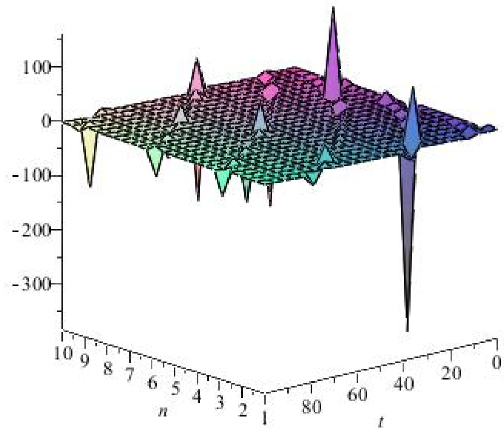

Fig. 4 The periodic wave solution $v_n(t)$ in (26) with γ=1/2,σ=1, ζ=0, $c_0 = 0$, $c_1 = 1$, $d_1 = 1$

$$\begin{cases} u_n(t) = a_0 - a_0 \tan(\sqrt{\sigma}d_1)\times \\ \cot[\sqrt{\sigma}(d_1 n - \frac{a_0 \tan(\sqrt{\sigma}d_1)t^\gamma}{\sqrt{\sigma}\Gamma(1+\gamma)} + \zeta) + c_0], \\ v_n(t) = a_0 + a_0 \tan(\sqrt{\sigma}d_1)\times \\ \cot[\sqrt{\sigma}(d_1 n - \frac{a_0 \tan(\sqrt{\sigma}d_1)t^\gamma}{\sqrt{\sigma}\Gamma(1+\gamma)} + \zeta) + c_0], \end{cases} \tag{29}$$

where $d_1,\ c_0,\ a_0$ are arbitrary constants.

In Figs 3-4, the periodic wave solutions (26) with some special parameters are demonstrated.

Case 3: If $\sigma > 0$, and assume (10) and (13) hold, then substituting (20), (21), (10) and (13) into Eqs. (17), using $[\phi_5^{(2)}(\xi_n)]^2 = \sigma + [\phi_5^{(1)}(\xi_n)]^2$, collecting the coefficients of $[\phi_5^{(1)}(\xi_n)]^i[\phi_5^{(2)}(\xi_n)]^j$ and equating them to zero, we obtain a series of algebra equations. Solving these equations, we get that

$$a_1 = -c_1,\ a_0 = 0,\ b_1 = c_1,$$

$$b_0 = 0,\ d_1 = \frac{\pi}{2\sqrt{\sigma}},\ c_1 = c_1,$$

or

$$a_1 = -c_1,\ a_0 = b_0,\ b_1 = c_1,$$

$$b_0 = b_0,\ d_1 = \frac{1}{2\sqrt{\sigma}}\arcsin(-\frac{2c_1 b_0\sqrt{\sigma}}{b_0^2 + c_1^2\sigma}),\ c_1 = c_1.$$

So we obtain the following trigonometric function solutions:

$$\begin{cases} u_n(t) = -c_1\sqrt{\sigma}\{\tan[2\sqrt{\sigma}(\frac{\pi}{2\sqrt{\sigma}}n + \frac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) \\ \quad +c_0] + |\sec[2\sqrt{\sigma}(\frac{\pi}{2\sqrt{\sigma}}n + \frac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0]|\}, \\ v_n(t) = c_1\sqrt{\sigma}\{\tan[2\sqrt{\sigma}(\frac{\pi}{2\sqrt{\sigma}}n + \frac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) \\ \quad +c_0] + |\sec[2\sqrt{\sigma}(\frac{\pi}{2\sqrt{\sigma}}n + \frac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0]|\}, \end{cases} \tag{30}$$

where $c_1,\ c_0$ are an arbitrary constants, and

$$\begin{cases} u_n(t) = -c_1\sqrt{\sigma}\{\tan[2\sqrt{\sigma}(\frac{1}{2\sqrt{\sigma}}\arcsin(-\frac{2c_1 b_0\sqrt{\sigma}}{b_0^2 + c_1^2\sigma})n \\ \quad +\frac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0] \\ \quad +|\sec[2\sqrt{\sigma}(\frac{1}{2\sqrt{\sigma}}\arcsin(-\frac{2c_1 b_0\sqrt{\sigma}}{b_0^2 + c_1^2\sigma})n \\ \quad +\frac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0]|\} + b_0, \\ v_n(t) = c_1\sqrt{\sigma}\{\tan[2\sqrt{\sigma}(\frac{1}{2\sqrt{\sigma}}\arcsin(-\frac{2c_1 b_0\sqrt{\sigma}}{b_0^2 + c_1^2\sigma})n \\ \quad +\frac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0] \\ \quad +|\sec[2\sqrt{\sigma}(\frac{1}{2\sqrt{\sigma}}\arcsin(-\frac{2c_1 b_0\sqrt{\sigma}}{b_0^2 + c_1^2\sigma})n \\ \quad +\frac{c_1 t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0]|\} + b_0, \end{cases} \tag{31}$$

where $c_1,\ b_0,\ c_0$ are an arbitrary constants.

Case 4: If $\sigma = 0$, and assume (10) and (14) hold, then substituting (20), (21), (10) and (14) into Eqs. (17), collecting the coefficients of $\phi_6^i(\xi_n)$ and equating them to zero, we obtain a series of algebra equations. Solving these equations, yields

$$a_1 = d_1 b_0,\ a_0 = b_0,\ b_1 = -d_1 b_0,$$

$$b_0 = b_0,\ d_1 = d_1,\ c_1 = -d_1 b_0.$$

Then we obtain the following rational solutions:

$$\begin{cases} u_n(t) = \dfrac{-d_1 b_0}{d_1 n - \dfrac{d_1 b_0 t^\gamma}{\Gamma(1+\gamma)} + \zeta + c_0} + b_0, \\ v_n(t) = \dfrac{d_1 b_0}{d_1 n - \dfrac{d_1 b_0 t^\gamma}{\Gamma(1+\gamma)} + \zeta + c_0} + b_0, \end{cases} \quad (32)$$

where $d_1,\ b_0,\ c_0$ are an arbitrary constants.

Remark 1 *In [34, Eqs. (46), (47), (51), (52)], Ayhan and Bekir presented some exact solutions for the two-component Volterra lattice equations by the (G'/G)-expansion method. We note that our results (22), (23) are generalizations of [34, Eqs. (46), (47)], while (26), (27) are generalizations of [34, Eqs. (51), (52)]. In fact, if we let*

$$\gamma = 1,\ c_0 = arth(\frac{C_2}{C_1}),\ \sigma = \frac{4\mu - \lambda^2}{4}$$

or

$$\gamma = 1,\ c_0 = arcoth(\frac{C_1}{C_2}),\ \sigma = \frac{4\mu - \lambda^2}{4},$$

then our results (22), (23) reduce to [34, Eq. (46), (47)]. If we let

$$\gamma = 1,\ c_0 = \arctan(-\frac{C_2}{C_1}),\ \sigma = \frac{4\mu - \lambda^2}{4}$$

or

$$\gamma = 1,\ c_0 = arccot(-\frac{C_1}{C_2}),\ \sigma = \frac{4\mu - \lambda^2}{4},$$

then our results (26), (27) reduce to [34, Eq. (51), (52)].

Remark 2 *The established results by (30-32) are new exact solutions for the two-component Volterra lattice equations so far to our best knowledge.*

4 Application of the extended Riccati sub-ODE method to the fractional m-KdV lattice equation

In this section, we apply the extended Riccati sub-ODE method to solve the fractional m-KdV lattice equation denoted by Eq. (2).

Using a fractional complex transformation $T = \frac{t^\gamma}{\Gamma(1+\gamma)}$, and letting $u_{n+p_s}(t) = U_{n+p_s}(T),\ p_s = 0, \pm 1$, by (3) we have $D_t^\gamma T = 1$, and furthermore, by the first equality in (5), Eq. (2) can be turned into

$$U_n'(T) = [\alpha - U_n^2(T)][U_{n+1}(T) - U_{n-1}(T)]. \quad (33)$$

Letting

$$U_n(T) = \widetilde{U}_n(\xi_n),\ \xi_n = d_1 n + c_1 T + \zeta, \quad (34)$$

where $d_1,\ c_1,\ \zeta$ are all constants, Eq. (33) can be rewritten in the following form:

$$c_1 \widetilde{U}_n'(\xi_n) - (\alpha - \widetilde{U}_n^2(\xi_n))(\widetilde{U}_{n+1}(\xi_{n+1}) - \widetilde{U}_{n-1}(\xi_{n-1})) = 0. \quad (35)$$

Suppose the solutions $\widetilde{U}_n(\xi_n)$ for Eq. (3.3) can be denoted by

$$\widetilde{U}_n(\xi_n) = \sum_{i=0}^{l} a_i \phi^i(\xi_n), \quad (36)$$

where $\phi(\xi_n)$ satisfies Eq. (10). By balancing the highest order linear term with the nonlinear terms in Eq. (35) we obtain $l + 1 = 2l$, and then $l = 1$. So we have

$$\widetilde{U}_n(\xi_n) = a_0 + a_1 \phi(\xi_n). \quad (37)$$

We will proceed to solve Eq. (35) in several cases.

Case 1: If $\sigma < 0$, and assume (10) and (11) hold, then substituting (37), (10) and (11) into Eq. (35), collecting the coefficients of $\phi_{1,2}^i(\xi_n)$ and equating them to zero, we obtain a series of algebraic equations. Solving these equations, yields

$$a_1 = \pm\sqrt{-\frac{\alpha}{\sigma}} \tanh(\sqrt{-\sigma} d_1),\ a_0 = 0,$$

$$d_1 = d_1,\ c_1 = \frac{2\alpha}{\sqrt{-\sigma}} \tanh(\sqrt{-\sigma} d_1).$$

So we obtain the following solitary wave solutions:

$$u_n(t) = \pm\sqrt{\alpha} \tanh(\sqrt{-\sigma} d_1)$$

$$\tanh[\sqrt{-\sigma}(d_1 n + \frac{2\alpha}{\sqrt{-\sigma}} \tanh(\sqrt{-\sigma} d_1) \frac{t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0], \quad (38)$$

and

$$u_n(t) = \pm\sqrt{\alpha} \tanh(\sqrt{-\sigma} d_1)$$

$$\coth[\sqrt{-\sigma}(d_1 n + \frac{2\alpha}{\sqrt{-\sigma}} \tanh(\sqrt{-\sigma} d_1) \frac{t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0], \quad (39)$$

where $d_1,\ c_0$ are arbitrary constants.

$Case$ 2: If $\sigma > 0$, and assume (10) and (12) hold, then substituting (37), (10) and (12) into Eq. (35), collecting the coefficients of $\phi^i_{3,4}(\xi_n)$ and equating them to zero, we obtain a series of algebraic equations. Solving these equations, yields

$$a_1 = \pm\sqrt{\frac{\alpha}{\sigma}}\tan(\sqrt{\sigma}d_1),\ a_0 = 0,$$

$$d_1 = d_1,\ c_1 = \frac{2\alpha}{\sqrt{\sigma}}\tan(\sqrt{\sigma}d_1).$$

Then we have the following periodic wave solutions:

$$u_n(t) = \pm\sqrt{\alpha}\tan(\sqrt{\sigma}d_1)$$

$$\tan[\sqrt{\sigma}(d_1 n + \frac{2\alpha}{\sqrt{\sigma}}\tan(\sqrt{\sigma}d_1)\frac{t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0], \quad (40)$$

and

$$u_n(t) = \pm\sqrt{\alpha}\tan(\sqrt{\sigma}d_1)$$

$$\cot[\sqrt{\sigma}(d_1 n + \frac{2\alpha}{\sqrt{\sigma}}\tan(\sqrt{\sigma}d_1)\frac{t^\gamma}{\Gamma(1+\gamma)} + \zeta) + c_0], \quad (41)$$

where $d_1,\ c_0$ are arbitrary constants.

In [34, Eqs. (32) and (36)], Ayhan and Bekir presented some exact solutions for m-KdV lattice equation by the (G'/G)-expansion method as follows:

$$u_n = \pm\sqrt{\alpha}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)\times$$

$$(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n) + C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n) + C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n))}), \quad (42)$$

where $\xi_n = d_1 n + \frac{4\alpha}{\sqrt{\lambda^2-4\mu}}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)t + \zeta$, and

$$u_n = \pm\sqrt{\alpha}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)\times$$

$$(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n))}), \quad (43)$$

where $\xi_n = d_1 n + \frac{4\alpha}{\sqrt{4\mu-\lambda^2}}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)t + \zeta$.

We note that our results (38) and (40) are solutions of more general forms than Eqs. (42) and (43). In fact, if we let $\gamma = 1,\ c_0 = arth(\frac{C_2}{C_1}),\ \sigma = \frac{4\mu-\lambda^2}{4}$ or $\gamma = 1,\ c_0 = arcoth(\frac{C_1}{C_2}),\ \sigma = \frac{4\mu-\lambda^2}{4}$, then our result (38) reduces to (42). If we let $\gamma = 1,\ c_0 = \arctan(-\frac{C_2}{C_1}),\ \sigma = \frac{4\mu-\lambda^2}{4}$ or $\gamma = 1,\ c_0 = arccot(-\frac{C_1}{C_2}),\ \sigma = \frac{4\mu-\lambda^2}{4}$, then our result (40) reduces to (43).

$Case$ 3: If $\sigma > 0$, and assume (10) and (13) hold, then substituting (37), (10) and (13) into Eq. (35), using $[\phi_5^{(2)}(\xi_n)]^2 = \sigma + [\phi_5^{(1)}(\xi_n)]^2$, collecting the coefficients of $[\phi_5^{(1)}(\xi_n)]^i[\phi_5^{(2)}(\xi_n)]^j$ and equating them to zero, we obtain a series of algebraic equations. Solving these equations, we get three families of values as follows:

$$a_1 = \pm\sqrt{\frac{\alpha}{\sigma}},\ a_0 = 0,$$

$$d_1 = -\frac{\pi}{4\sqrt{\sigma}},\ c_1 = -\frac{2\alpha}{\sqrt{\sigma}},$$

$$a_1 = \pm\sqrt{\frac{\alpha}{\sigma}},\ a_0 = 0,$$

$$d_1 = \frac{\pi}{4\sqrt{\sigma}},\ c_1 = \frac{2\alpha}{\sqrt{\sigma}},$$

or

$$a_1 = \pm\frac{\sqrt{2\alpha - \alpha\sin^2(2\sqrt{\sigma}d_1) - 2\alpha\cos(2\sqrt{\sigma}d_1)}}{\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)},$$

$$d_1 = d_1,\ a_0 = 0,\ c_1 = -\frac{2\alpha(\cos(2\sqrt{\sigma}d_1) - 1)}{\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)}$$

So we obtain the following trigonometric function solutions:

$$u_n(t) = \pm\sqrt{\alpha}\{\tan[2\sqrt{\sigma}(\frac{\pi}{4\sqrt{\sigma}}n + \frac{2\alpha t^\gamma}{\sqrt{\sigma}\Gamma(1+\gamma)} + \zeta)$$

$$+c_0] + |\sec[2\sqrt{\sigma}(\frac{\pi}{4\sqrt{\sigma}}n + \frac{2\alpha t^\gamma}{\sqrt{\sigma}\Gamma(1+\gamma)} + \zeta) + c_0]|\}, \quad (44)$$

$$u_n(t) = \pm\sqrt{\alpha}\{\tan[2\sqrt{\sigma}(-\frac{\pi}{4\sqrt{\sigma}}n - \frac{2\alpha t^\gamma}{\sqrt{\sigma}\Gamma(1+\gamma)} + \zeta)$$

$$+c_0] + |\sec[2\sqrt{\sigma}(-\frac{\pi}{4\sqrt{\sigma}}n - \frac{2\alpha t^\gamma}{\sqrt{\sigma}\Gamma(1+\gamma)} + \zeta) + c_0]|\}, \quad (45)$$

where c_0 is an arbitrary constant, and

$$u_n(t) = \pm \frac{\sqrt{2\alpha - \alpha\sin^2(2\sqrt{\sigma}d_1) - 2\alpha\cos(2\sqrt{\sigma}d_1)}}{\sin(2\sqrt{\sigma}d_1)}$$
$$\{\tan[2\sqrt{\sigma}(d_1 n - \frac{2\alpha(\cos(2\sqrt{\sigma}d_1)-1)t^\gamma}{\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)\Gamma(1+\gamma)} + \zeta) + c_0]$$
$$+|\sec[2\sqrt{\sigma}(d_1 n - \frac{2\alpha(\cos(2\sqrt{\sigma}d_1)-1)t^\gamma}{\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)\Gamma(1+\gamma)} + \zeta) + c_0]|\}, \quad (46)$$

where $d_1,\ c_0$ are arbitrary constants.

Case 4: If $\sigma = 0$, and assume (10) and (14) hold, then substituting (37), (10) and (14) into Eq. (35), collecting the coefficients of $\phi_6^i(\xi_n)$ and equating them to zero, we obtain a series of algebraic equations. Solving these equations, yields

$$a_1 = \pm\sqrt{\alpha}d_1,\ a_0 = 0,\ d_1 = d_1,\ c_1 = 2d_1\alpha.$$

Then we obtain the following rational solution:

$$u_n(t) = \frac{\pm\sqrt{\alpha}d_1}{d_1 n + \frac{2d_1\alpha t^\gamma}{\Gamma(1+\gamma)} + \zeta + c_0}, \quad (47)$$

where $d_1,\ c_0$ are arbitrary constants.

Remark 3 *Our results (44)-(47) have not been reported by other authors so far to our best knowledge.*

5 Conclusions

The Riccati sub-ODE method is extended to seek exact solutions for fractional differential-difference equations. By this method, we solved the two-component fractional Volterra lattice equations and the fractional m-KdV lattice equation successfully, and as a result, some generalized exact solutions including solitary wave solutions, periodic wave solutions and rational function solutions for them have been found with the aid of mathematical software. Being concise and effective, we note that this approach can be applied to solve other fractional differential-difference equations.

References:

[1] M. L. Wang and X. Z. Li, Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation, *Chaos, Solitons and Fractals*, 24, 2005, pp. 1257-1268.

[2] J. H. He and X. H. Wu, Exp-function method for nonlinear wave equations, *Chaos, Solitons and Fractals*, 30, 2006, pp. 700-708.

[3] M. L. Wang, X. Z. Li and J. L. Zhang, The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, *Phys. Lett. A*, 372, 2008, pp. 417-423.

[4] E. M. E. Zayed and K. A. Gepreel, The modified (G'/G)-expansion method and its applications to construct exact solutions for nonlinear PDEs, *WSEAS Transactions on Mathematics*, 10(8), 2011, pp. 270-278.

[5] E. M. E. Zayed, A further improved (G'/G)-expansion method and the extended tanh-method for finding exact solutions of nonlinear PDEs, *WSEAS Transactions on Mathematics*, 10(2), 2011, pp. 56-64.

[6] E. M. E. Zayed and M. Abdelaziz, Exact traveling wave solutions of nonlinear variable coefficients evolution equations with forced terms using the generalized (G'/G)-expansion method, *WSEAS Transactions on Mathematics*, 10(3), 2011, pp. 115-124.

[7] Q. H. Feng and B. Zheng, Traveling Wave Solutions for the Fifth-Order Sawada-Kotera Equation and the General Gardner Equation by (G'/G)-Expansion Method, *WSEAS Transactions on Mathematics*, 9(3), 2010, pp. 171-180.

[8] Y. Wang, Variable-coefficient Simplest Equation Method For Solving Nonlinear Evolution Equations In Mathematical Physics, *WSEAS Transactions on Mathematics*, 12(5), 2013, pp. 512-520.

[9] E. M. E. Zayed and S. Al-Joudi, An Improved (G'/G)-expansion Method for Solving Nonlinear PDEs in Mathematical Physics, *ICNAAM 2010, AIP. Conf. Proc.*, Vol. 1281, 2010, pp. 2220-2224.

[10] Q. H. Feng and B. Zheng, Traveling Wave Solutions for the Fifth-Order Kdv Equation and the BBM Equation by (G'/G)-Expansion Method, *WSEAS Transactions on Mathematics*, 9(3), 2010, pp. 201-210.

[11] B. Zheng, Application Of A Generalized Bernoulli Sub-ODE Method For Finding Traveling Solutions Of Some Nonlinear Equations, *WSEAS Transactions on Mathematics*, 11(7), 2012, pp. 634-642.

[12] C. C. Kong, D. Wang, L. N. Song and H. Q. Zhang, New exact solutions to MKDV-Burgers equation and (2+1)-dimensional dispersive long wave equation via extended Riccati equation method, *Chaos, Solitons and Fractals*, 39, 2009, pp. 697-706.

[13] M. Eslami, A. Neyrame and M. Ebrahimi, Explicit solutions of nonlinear (2+1)-dimensional dispersive long wave equation, *J. King Saud Univer.-Sci.* , 24, 2012, pp. 69-71.

[14] I. Aslan, Discrete exact solutions to some nonlinear differential-difference equations via the (G'/G)-expansion method, *Appl. Math. Comput.*, 215, 2009, pp. 3140-3147.

[15] X. B. Hu and W. X. Ma, Application of Hirota's bilinear formalism to the Toeplitz lattice some special soliton-like solutions, *Phys. Lett. A*, 293, 2002, pp. 161-165.

[16] B. Tang, Y. N. He, L. L. Wei and S. L. Wang, Variable-coefficient discrete (G'/G)-expansion method for nonlinear differential-difference equations, *Phys. Lett. A* , 375, 2011, pp. 3355-3361.

[17] W. Zhen, Discrete tanh method for nonlinear difference-differential equations, *Comput. Phys. Commun.*, 180, 2009, pp. 1104-1108.

[18] I. Aslan, A discrete generalization of the extended simplest equation method, *Commun. Nonlinear Sci. Numer. Simul.*, 15, 2010, pp. 1967-1973.

[19] C. Q. Dai, X. Cen and S. S. Wu, Exact travelling wave solutions of the discrete sine-Gordon equation obtained via the exp-function method, *Nonlinear Anal.*, 70, 2009, pp. 58-63.

[20] C. S. Liu, Exponential function rational expansion method for nonlinear differential-difference equations, *Chaos, Solitons and Fractals*, 40, 2009, pp. 708-716.

[21] S. Zhang, L. Dong, J. M. Ba and Y. N. Sun, The (G'/G)-expansion method for nonlinear differential-difference equations, *Phys. Lett. A*, 373, 2009, pp. 905-910.

[22] B. Ayhan and A. Bekir, The (G'/G)-expansion method for the nonlinear lattice equations, *Commun. Nonlinear Sci. Numer. Simulat.*, 17, 2012, pp. 3490-3498.

[23] H. Xin, The exponential function rational expansion method and exact solutions to nonlinear lattice equations system, *Appl. Math. Comput.*, 217, 2010, pp. 1561-1565.

[24] K. A. Gepreel and A. R. Shehata, Rational Jacobi elliptic solutions for nonlinear differential-difference lattice equations, *Appl. Math. Lett.*, 25, 2012, pp. 1173-1178.

[25] W. H. Huang and Y. L. Liu, Jacobi elliptic function solutions of the Ablowitz-Ladik discrete nonlinear Schrödinger system, *Chaos, Solitons and Fractals*, 40, 2009, pp. 786-792.

[26] C. B. Wen, Traveling Wave Solutions For Two Nonlinear Lattice Equations By An Extended Riccati Sub-equation Method, *WSEAS Transactions on Mathematics*, 11(12), 2012, pp. 1085-1093.

[27] C. B. Wen and B. Zheng, A New Fractional Sub-equation Method For Fractional Partial Differential Equations, *WSEAS Transactions on Mathematics*, 12(5), 2013, pp. 564-571.

[28] Q. H. Feng and B. Zheng, Traveling Wave Solutions For The Variant Boussinseq Equation And The (2+1)-dimensional Nizhnik-Novikov-Veselov (NNV) System By (G'/G)-expansion method, *WSEAS Transactions on Mathematics*, 9(3), 2010, pp. 191-200.

[29] G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results, *Comput. Math. Appl.*, 51, 2006, pp. 1367-1376.

[30] B. Lu, Bäcklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations, *Phys. Lett. A*, 376, 2012, pp. 2045-2048.

[31] S. M. Guo, L. Q. Mei, Y. Li and Y. F. Sun, The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics, *Phys. Lett. A*, 376, 2012, pp. 407-411.

[32] S. Zhang and H. Q. Zhang, Fractional sub-equation method and its applications to nonlinear fractional PDEs, *Phys. Lett. A*, 375, 2011, pp. 1069-1073.

[33] B. Zheng, (G'/G)-Expansion Method for Solving Fractional Partial Differential Equations in the Theory of Mathematical Physics, *Commun. Theor. Phys. (Beijing, China)*, 58, 2012, pp. 623-630.

[34] B. Ayhan and A. Bekir, The (G'/G)-expansion method for the nonlinear lattice equations, *Commun. Nonlinear Sci. Numer. Simulat.*, 17, 2012, pp. 3490-3498.