On Construction Third Order Approximation Using Values of Integrals

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Abstract: Sometimes results of experiments contain values of integrals of a function over sub-intervals however one needs to construct a continuous approximation of the function on the interval. This problem may be solved by the polynomial or by the trigonometrical integro-differential splines that are constructed in this paper. The integro-differential splines are useful for construction the approximation of the function on each sub-interval separately. We use the values of the integrals of the function over sub-intervals and construct a continuous approximation of the function on the interval by the polynomial integro-differential spline, and by the trigonometric integro-differential spline in two steps. First we obtain discontinuous third order approximation of the function in form of the polynomial integro-differential spline, or the trigonometrical integro-differential spline. Approximation on sub-interval uses only the values of the integrals of the function to be approximated over three intervals and basic functions which we obtain here. Then we construct continuous approximation of the function on the interval by solving the system of equations. After that we compare the properties of solutions in form of the polynomial and in form of the trigonometric integro-differential splines. One can see that for trigonometric function sometimes the trigonometric integro-differential spline gives better approximation then the polynomial integro-differential spline. Finally we construct approximation of the function in case we use quadrature formula of the third order instead of the value of the integral over sub-interval.

Key–Words: interpolation, splines, polynomial splines, trigonometric splines, polynomial integro-differential splines, trigonometric integro-differential splines, approximation, error of approximation

1 Introduction

Nowadays it is impossible to solve many different problems without help of splines [1–5].

Sometimes the results of experiments contain values of function by intervals and it is necessary to have approximation of the function. In this case we can construct approximation with integro-differential splines. Integro-differential smooth polynomial splines were invented by Kireev [6].

Nonpolynomial integro-differential splines were regarded in [7]. They may use the values of a function and its derivatives. We construct approximation by basic integro-differential splines on each interval separately. Integro-differential polynomial splines of the fifth order and approximations by them with different properties were presented in [8, 9]. Nonpolynomial splines without values of integrals were constructed in [10–14].

In this paper we consider approximation of functions by polynomial and trigonometric integro-differential splines. The approximations use only the values of the integrals of the function to be approximated. First we obtain discontinuous approximation, then we show how to construct continuous approximation, and construct approximation in case we approximate integral by quadrature formula.

2 Construction of polynomial splines

Let a, b be real numbers, n positive integer, $n \geq 2$. Suppose that $X : a = x_0 < \ldots < x_{k-1} < x_k < x_{k+1} < \ldots < x_n = b$ is a uniform grid of nodes in the interval $[a, b]$ with step $h = \frac{b-a}{n}$. We assume that $\varphi_i(x)$, $i = 1, 2, 3$ is Chebyshev system on $[x_0, x_n]$, $\varphi_i \in C^3[x_0, x_n]$. Let function u be such that $u \in C^3[a, b]$ and we have the values of $\int_{x_{k-1}}^{x_k} u(\tau)d\tau$, $k = 1, \ldots, n$. We construct an approximation of function $u(x)$ on each interval $(x_k,\, x_{k+1})$ in the form:

$$\tilde{u}_k(x) = \left(\int\limits_{x_{k-1}}^{x_k} u(t)dt\right)\omega_k^{<-1>}(x) + \left(\int\limits_{x_k}^{x_{k+1}} u(t)dt\right)\omega_k^{<0>}(x) + \left(\int\limits_{x_{k+1}}^{x_{k+2}} u(t)dt\right)\omega_k^{<1>}(x), \quad (1)$$

where $\omega_k^{<-1>}(x),\ \omega_k^{<0>}(x),\ \omega_k^{<1>}(x)$ are determined from conditions:

$$\tilde{u}_k(x) = u(x) \text{ for } u(x) = \varphi_i(x),\ i = 1, 2, 3.$$

Let $\varphi_i = x^{i-1},\ i = 1, 2, 3.$ On interval $(x_k,\, x_{k+1})$ the basis splines $\omega_k^{<s>}(x)$, $s = -1, 0, 1$, are obtained from the system of equations:

$$h\omega_k^{<-1>}(x) + h\omega_k^{<0>}(x) + h\omega_k^{<1>}(x) = 1,$$

$$-\frac{1}{2}h^2\omega_k^{<-1>}(x) + \frac{1}{2}h^2\omega_k^{<0>}(x) + \frac{3}{2}h^2\omega_k^{<1>}(x) = x,$$

$$\frac{1}{3}h^3\omega_k^{<-1>}(x) + \frac{1}{3}h^3\omega_k^{<0>}(x) + \frac{7}{3}h^3\omega_k^{<1>}(x) = x^2.$$

If $t \in (0,\, 1)$, $x = x_k + th$, then:

$$\omega_k^{<-1>}(t) = \frac{1}{6h}(2 - 6t + 3t^2), \quad (2)$$

$$\omega_k^{<0>}(t) = -\frac{1}{6h}(-6t + 6t^2 - 5), \quad (3)$$

$$\omega_k^{<1>}(t) = \frac{1}{2h}t^2 - \frac{1}{6h}. \quad (4)$$

From (2)–(4) we find:

$$|\omega_k^{<-1>}(t)| \le \frac{1}{3h},\ |\omega_k^{<0>}(t)| \le \frac{13}{12h},\ |\omega_k^{<1>}(t)| \le \frac{1}{3h}.$$

Let us put:

$$\|f\| = \|f\|_{X(a,b)} = \max_k \sup_{x\in(x_{k-2},x_{k+2})} |f(x)|,$$

$$\|f\|_{(x_k,x_{k+1})} = \sup_{x\in(x_{k-2},x_{k+2})} |f(x)|.$$

Let us take $\widetilde{U}(x)$, $x \in (a, b)$, such that:

$$\widetilde{U}(x) = \widetilde{u}_k(x),\ x \in (x_k, x_{k+1}),\ k = 0,\, \ldots,\, n-1.$$

Theorem 1 *Let $u \in C^3[a, b]$, $\tilde{u}_k(x)$, defines by formulas (1), (2) – (4). Next relation is true:*

$$|\widetilde{u}_k(x) - u(x)| \le K_k h^3 \|u'''\|_{(x_{k-2},\, x_{k+2})}, \quad (5)$$

where $K_k \le 0.44,\ x \in (x_k,\, x_{k+1})$.

Proof: Using a Taylor formula to represent $u(t)$ at the x_k and taking into account formulas (2) – (4), one obtains for $x \in (x_k, x_{k+1})$:

$$|\widetilde{u}_k(x) - u(x)| \le 0.44h^3 \sup_{x\in(x_{k-2},x_{k+2})} |u'''(x)|. \quad (6)$$

Now we have (5) from (6) with $K_k \le 0.44$. □

Corollary 2 *Next relation is true:*

$$\|\widetilde{U} - u\|_{(a+2h,b-2h)} \le Kh^3\|u'''\|_{(a,b)},\ K \le 0.44.$$

Proof: It follows from (5). □

The maximums of absolute values of actual and theoretical errors of approximation by polynomial splines are defined by (1), (2)–(4) on interval $[-1,\, 1]$ with step $h = 0.1$ are presented in table 1.

Table 1: The maximums of actual and theoretical errors of approximation by polynomial integro-differential splines.

	$u(x)$	Actual err.	Theoret. err.
1	$1/(1+25x^2)$	$0.32 \cdot 10^{-1}$	0.257
2	$\sin(x)$	$0.83 \cdot 10^{-4}$	$0.44 \cdot 10^{-3}$
3	$x^3/3!$	$0.83 \cdot 10^{-4}$	$0.88 \cdot 10^{-3}$
4	$x^5/5!$	$0.38 \cdot 10^{-4}$	$0.53 \cdot 10^{-2}$

3 Construction of trigonometrical splines

Let $\varphi_1 = 1$, $\varphi_2 = \sin(x)$, $\varphi_2 = \cos(x)$. On interval $(x_k,\, x_{k+1})$ the basis splines $\widetilde{\omega}_k^{<s>}(x)$, $s = -1, 0, 1$, are obtained from system of equations:

$$h\widetilde{\omega}_k^{<-1>}(x) + h\widetilde{\omega}_k^{<0>}(x) + h\widetilde{\omega}_k^{<1>}(x) = 1, \quad (7)$$

$$\int\limits_{x_{k-1}}^{x_k} \sin(t)dt\, \widetilde{\omega}_k^{<-1>}(x) + \int\limits_{x_k}^{x_{k+1}} \sin(t)dt\, \widetilde{\omega}_k^{<0>}(x) + \int\limits_{x_k}^{x_{k+1}} \sin(t)dt\, \widetilde{\omega}_k^{<1>}(x) = \sin(x), \quad (8)$$

$$\int\limits_{x_{j-1}}^{x_j} \cos(t)dt\, \widetilde{\omega}_k^{<-1>}(x) + \int\limits_{x_k}^{x_{k+1}} \cos(t)dt\, \widetilde{\omega}_k^{<0>}(x) + \int\limits_{x_k}^{x_{k+1}} \cos(t)dt\, \widetilde{\omega}_k^{<1>}(x) = \cos(x). \quad (9)$$

Let $t \in (0,\,1)$ $x = x_k + th$. From the system (7)–(9) we obtain:

$$\widetilde{\omega}_k^{<-1>}(t) = \frac{-h\cos(th-h)+\sin(h)}{-h\sin(2h)+2h\sin(h)}, \tag{10}$$

$$\widetilde{\omega}_k^{<0>}(t) = \frac{h\cos(th-h)-\sin(2h)+h\cos(th)}{-h\sin(2h)+2h\sin(h)}, \tag{11}$$

$$\widetilde{\omega}_k^{<1>}(t) = \frac{-\cos(th)h+\sin(h)}{-h\sin(2h)+2h\sin(h)}. \tag{12}$$

Taking into account that $t \in (0,\,1)$ we find:

$$|\widetilde{\omega}_k^{-1}(t)| \le \frac{h\cos(h)-\sin(h)}{2h(\cos(h)-1)\sin(h)}, \tag{13}$$

$$|\widetilde{\omega}_k^{<0>}(t)| \le \frac{h\sqrt{2}-s_1\sin(2h)}{-(s_1 h\sin(2h)+2s_1 h\sin(h))}, \tag{14}$$

where $s_1 = \sqrt{1/(\cos(h)+1)}$,

$$|\widetilde{\omega}_k^{<1>}(t)| \le \frac{-h\cos(h)+\sin(h)}{-h\sin(2h)+2h\sin(h)}. \tag{15}$$

We construct an approximation of function $u(x)$ on each interval $(x_k,\,x_{k+1})$ in form:

$$\widetilde{\widetilde{u}}_k(x) = \left(\int_{x_{k-1}}^{x_k} u(t)dt\right)\widetilde{\omega}_k^{<-1>}(x) + \left(\int_{x_k}^{x_{k+1}} u(t)dt\right)\widetilde{\omega}_k^{<0>}(x) + \left(\int_{x_{k+1}}^{x_{k+2}} u(t)dt\right)\widetilde{\omega}_k^{<1>}(x), \tag{16}$$

where $\widetilde{\omega}_k^{<-1>}(x)$, $\widetilde{\omega}_k^{<0>}(x)$, $\widetilde{\omega}_k^{<2>}(x)$ are determined by (10) – (12).

Theorem 3 *Let* $u \in C^3[a,b]$, $\widetilde{\widetilde{u}}_k(x)$, *defined by formulas (16), (10) – (12).*

Next relation is true:

$$|\widetilde{\widetilde{u}}_k(x) - u(x)| \le \widetilde{K}_k h^3 \|u' + u'''\|_{(x_{k-2},x_{k+2})}, \tag{17}$$

where $x \in (x_k,\,x_{k+1})$, $\widetilde{K}_k \le 1/8$.

Proof: The function $u(x)$ in trigonometric case (as was shown by the author at the conference in Gdansk, May 15–17, 2014) on interval (x_k, x_{k+1}) can be represented in the form:

$$u(x) = \frac{1}{2}\int_{x_k}^{x}((\sin(x/2-t/2))^2 dt + c_1 + c_2\sin(x) + c_3\cos(x),$$

where c_1, c_2, c_3 are arbitrary constants. We have

$$\int_{x_{k-1}}^{x_k} u(x)dx = (-1/2+(1/2)\cos^2(h/2)+h^2/8) + (\cos(x_{k-1})-\cos(x_k))c_2 + (-\sin(x_{k-1})+\sin(x_k))c_3 + c_1 h,$$

$$\int_{x_k}^{x_{k+1}} u(x)dx = (-1/2+(1/2)\cos^2(h/2)+h^2/8) + (\cos(x_k)-\cos(x_{k+1}))c_2 + (-\sin(x_k)+\sin(x_{k+1}))c_3 + c_1 h,$$

$$\int_{x_{k+1}}^{x_{k+2}} u(x)dx = (-(1/2)\cos^2(h/2)+3h^2/8 + (1/2)\cos^2(h)) + (\cos(x_{k+1})-\cos(x_{k+2}))c_2 + (-\sin(x_{k+1})+\sin(x_{k+2}))c_3 + c_1 h.$$

Taking into account formulas (13) – (15) we find (17) with constant $\widetilde{K}_k \le 1/8$. □

Let us define $\widetilde{\widetilde{U}}(x)$ $x \in (a,b)$, by relation:

$$\widetilde{\widetilde{U}}(x) = \widetilde{\widetilde{u}}_k(x),\ x \in (x_k, x_{k+1}),\ k = 0,\,\ldots,\,n-1.$$

Corollary 4 *Next relation is true:*

$$\|\widetilde{\widetilde{U}} - u\|_{(a+2h,b-2h)} \le Kh^3\|u''' + u'\|_{(a+2h,b-2h)},$$

$K \le 1/8$.

Proof: It follows from (17). □

The maximums of absolute values of actual and theoretical errors of approximation by trigonometrical integro-differential splines are defined by (16), (10) – (12) on interval $[-1,\,1]$ with step $h = 0.1$ are presented in table 2.

Table 2: Actual and theoretical errors of approximation by trigonometrical integro-differential splines.

	$u(x)$	Actual err.	Theoret.err.
1	$1/(1+25x^2)$	$0.33 \cdot 10^{-1}$	$0.73 \cdot 10^{-1}$
2	$\sin(x)$	0.	0
3	$x^3/3!$	$0.12 \cdot 10^{-3}$	$0.38 \cdot 10^{-3}$
4	$x^5/5!$	$0.42 \cdot 10^{-4}$	$0.16 \cdot 10^{-2}$

4 Construction of continuous polynomial approximations

Let us take $\varphi_i = x^{i-1}$, $i = 1, 2, 3$. Let C_{k+1}, C_k be real numbers, $N = n - 2$. Let us take $\omega_k^{<i>}(x)$ in form (2) – (4).

We construct an approximation for $u(x)$ on (x_k, x_{k+1}) in form:

$$\tilde{u}_k^C(x) = \left(\int_{x_{k-1}}^{x_k} u(t)dt\right) \omega_k^{<-1>}(x) + \left(\int_{x_k}^{x_{k+1}} u(t)dt\right)\omega_k^{<0>}(x) + C_{k+1}\,\omega_k^{<1>}(x). \quad (18)$$

On (x_{k-1}, x_k) we take another approximation for $u(x)$ in form:

$$\tilde{u}_{k-1}^C(x) = \left(\int_{x_{k-2}}^{x_{k-1}} u(t)dt\right) \omega_{k-1}^{<-1>}(x) + \left(\int_{x_{k-1}}^{x_k} u(t)dt\right)\omega_{k-1}^{<0>}(x) + C_k\,\omega_{k-1}^{<1>}(x),$$

From the condition $\tilde{u}_{k-1}^C(x_k-) = \tilde{u}_k^C(x_k+)$ we obtain the system of equations:

$$C_{k+1} + 2C_k = f_k, \quad k = 2, \dots, N-1, \quad (19)$$

$$f_k = \int\limits_{x_{k-2}}^{x_{k-1}} u(t)dt - 3\int\limits_{x_{k-1}}^{x_k} u(t)dt + 5\int\limits_{x_k}^{x_{k+1}} u(t)dt,$$

$$2C_N = f_N, \quad (20)$$

$$f_N = \int\limits_{x_{N-2}}^{x_{N-1}} u(t)dt - 3\int\limits_{x_{N-1}}^{x_N} u(t)dt + 5\int\limits_{x_N}^{x_{N+1}} u(t)dt - \int\limits_{x_{N+1}}^{x_{N+2}} u(t)dt.$$

Solving (19)–(20) we obtain:

$$C_N = f_N/2,$$
$$C_{N-i-1} = -(C_{N-i} - f_{N-i-1})/2, \quad i = 1, \dots, N-1.$$

5 Construction of continuous trigonometrical approximations

Let $\varphi_1 = 1$, $\varphi_2 = \sin(x)$, $\varphi_2 = \cos(x)$, $\widetilde{C_i}$ be real numbers. Let us find $\widetilde{\omega}_k^{<i>}(x)$ from (10)–(12). On (x_k, x_{k+1}) approximation for $u(x)$ we take in form:

$$\tilde{\tilde{u}}_k^C(x) = \left(\int_{x_{k-1}}^{x_k} u(t)dt\right) \widetilde{\omega}_k^{<-1>}(x) + \left(\int_{x_k}^{x_{k+1}} u(t)dt\right)\widetilde{\omega}_k^{<0>}(x) + \widetilde{C}_{k+1}\,\widetilde{\omega}_k^{<1>}(x). \quad (21)$$

On (x_{k-1}, x_k) we take an approximation for $u(x)$ in form:

$$\tilde{\tilde{u}}_{k-1}^C(x) = \left(\int_{x_{k-2}}^{x_{k-1}} u(t)dt\right) \widetilde{\omega}_{k-1}^{<-1>}(x) + \left(\int_{x_{k-1}}^{x_k} u(t)dt\right)\widetilde{\omega}_{k-1}^{<0>}(x) + \widetilde{C}_k\,\widetilde{\omega}_{k-1}^{<1>}(x),$$

From relation $\tilde{\tilde{u}}_{k-1}^C(x_k-) = \tilde{\tilde{u}}_k^C(x_k+)$ we find:

$$d_{k+1}\widetilde{C}_{k+1} + d_k\widetilde{C}_k = 6hf_k, \ \ k = 2, \dots, N-1,$$
$$d_N\widetilde{C}_N = 6hf_N,$$

where

$$d_k = \frac{6h(\sin h - h\cos h)}{2h\sin h(1-\cos h)},$$
$$d_{k+1} = -\frac{6h(\sin h - h)}{2h\sin h(1-\cos h)},$$

$$f_k = A(h)\left(\int_{x_{k-2}}^{x_{k-1}} u(t)dt\right) + B(h)\left(\int_{x_{k-1}}^{x_k} u(t)dt\right) + C(h)\left(\int_{x_k}^{x_{k+1}} u(t)dt\right);$$

$$d_N = \frac{6h(\sin h - h\cos h)}{2h\sin h(1-\cos h)},$$

$$f_N = A(h)\left(\int_{x_{N-2}}^{x_{N-1}} u(t)dt\right) + B(h)\left(\int_{x_{N-1}}^{x_N} u(t)dt\right) + C(h)\left(\int_{x_N}^{x_{N+1}} u(t)dt\right) - A(h)\left(\int_{x_{N+1}}^{x_{N+2}} u(t)dt\right),$$

$$A(h) = \frac{\sin(h) - h}{2h\sin h(\cos h - 1)},$$
$$B(h) = \frac{2h\cos h - \sin h - \sin 2h + h}{2h\sin h(\cos h - 1)},$$
$$C(h) = \frac{\sin 2h - h\cos h - h}{2h\sin h(\cos h - 1)}.$$

6 Some results

Lemma 5 *1) In polynomial case next relation is true:*

$$|C_k - \int_{x_k}^{x_{k+1}} u(t)dt| \le K_1 h^4 \|u'''\|_{(x_{k-2},x_{k+2})},$$

$$K_1 = \frac{3}{2}. \quad (22)$$

2) In trigonometrical case next relation is true:

$$|\widetilde{C}_k - \int_{x_k}^{x_{k+1}} u(t)dt| \le K_2 h^4 \|u' + u'''\|_{(x_{k-2},x_{k+2})},$$

$$K_2 = 9/4. \quad (23)$$

Proof: In the system of equations: (19) – (20) we make a change of variables. We put $S_k = C_k - \int_{x_k}^{x_{k+1}} u(t)dt$. Now we have the system of equations:

$$S_{k+1} + 2S_k = F_k, \ \ k = 2, \dots, N-1,$$

$$F_k = f_k - \int_{x_{k+1}}^{x_{k+2}} u(t)dt - 2\int_{x_k}^{x_{k+1}} u(t)dt,$$

$$F_N = \int_{x_{N-2}}^{x_{N-1}} u(t)dt - 3\int_{x_{N-1}}^{x_N} u(t)dt + 3\int_{x_N}^{x_{N+1}} u(t)dt - \int_{x_{N+1}}^{x_{N+2}} u(t)dt.$$

In polynomial case using Taylor formula and theorem of the mean for integrals one obtains:

$$|F_k| \le h^4 K_1 \max_{[x_{k-2},x_{k+2}]} |u^{(3)}|, \ \text{ where } K_1 = \frac{3}{2}.$$

As known (see [13]) $|S_k| \le \max_i |F_i|$. Thus, the inequality (22) is proved with $K_1 = 3/2$.

In trigonometrical case, by using $u(t) = \frac{1}{2}\int_{x_k}^{t}(u' + u''')\sin^2\frac{t-\tau}{2}d\tau + c_1 + c_2\sin(t) + c_3\cos(t)$, we obtain $|F_k| \le h^4 K_2 \|u' + u'''\|_{(x_{k-2},x_{k+2})}$, where $K_2 = \frac{9}{4}$. Thus, the inequality (23) is proved with $K_2 = 9/4$. □

Theorem 6 *Let $u \in C^3[a,b]$.*

1) Let $\tilde{u}_k^C(x)$, $x \in (x_k, x_{k+1})$ defined by formula (18) and polynomial splines (2)–(4), then the next relation is true:

$$|\widetilde{u}_k^C(x) - u(x)| \le K h^3 \|u'''\|_{(x_{k-2},x_{k+2})}, \ K \le 0.94.$$

2) Let $\tilde{\tilde{u}}_k^C(x)$, $x \in (x_k, x_{k+1})$ defined by formula (21) and trigonometrical splines (10)–(12), then the next relation is true:

$$|\widetilde{\widetilde{u}}_k^C(x) - u(x)| \le K h^3 \|u' + u'''\|_{(x_{k-2},x_{k+2})}, \ K \le 0.87.$$

Proof: For polynomial splines $|\widetilde{u}^C(x) - u(x)| \le |\widetilde{u}^C(x) - \widetilde{u}(x)| + |\widetilde{u}(x) - u(x)|$, we have the same for trigonometric splines and then we use (5), (17), (22), (23). □

In tables 3 and 4, the maximums of the absolute values of actual and theoretical errors of approximation by continuous polynomial splines are defined by (18), (2)–(4) and continuous trigonometrical splines are defined by (21), (10)–(12) on interval $[-1,\ 1]$ with step $h = 0.1$ are presented.

Table 3: Actual errors of approximation by continuous polynomial and trigonometric splines (in Maple, Digits=15).

	$u(x)$	Actual err. pol. spl.	Actual err. trig.spl.
1	$1/(1+25x^2)$	$0.21 \cdot 10^{-1}$	$0.21 \cdot 10^{-1}$
2	$\sin(x)$	$0.40 \cdot 10^{-4}$	$0.16 \cdot 10^{-14}$
3	$x^3/3!$	$0.80 \cdot 10^{-4}$	$0.12 \cdot 10^{-3}$
4	$x^5/5!$	$0.43 \cdot 10^{-4}$	$0.47 \cdot 10^{-4}$

Table 4: Theoretical errors of approximation by continuous polynomial and trigonometric splines.

	$u(x)$	Theor.err. pol.spl	Theor. err. trig.spl.
1	$1/(1+25x^2)$	0.60	0.52
2	$\sin(x)$	$0.51 \cdot 10^{-3}$	0.
3	$x^3/3!$	$0.94 \cdot 10^{-3}$	$0.13 \cdot 10^{-2}$
4	$x^5/5!$	$0.47 \cdot 10^{-3}$	$0.47 \cdot 10^{-3}$

In table 5, the maximums of the absolute values of actual errors of approximation by continuous polynomial splines are defined by (18), (2)–(4) and by continuous trigonometrical splines defined by (21), (10)–(12) on interval $[-1,\ 1]$ with step $h = 0.01$ (in Maple, Digits=30) are presented.

Table 5: Actual errors of approximation by continuous polynomial and trigonometric splines.

	$u(x)$	Actual err. pol. spl.	Actual err. trig.spl.
1	$1/(1+25x^2)$	$0.16 \cdot 10^{-4}$	$0.16 \cdot 10^{-4}$
2	$\sin(x)$	$0.44 \cdot 10^{-7}$	$0.16 \cdot 10^{-11}$
3	$x^3/3!$	$0.79 \cdot 10^{-7}$	$0.12 \cdot 10^{-6}$
4	$x^5/5!$	$0.40 \cdot 10^{-7}$	$0.43 \cdot 10^{-7}$

The approximation of the function $1/(1+25x^2)$ on $[-1,1]$ with $h = 0.1$ by continuous trigonometric splines (21), (10)–(12) is represented in figure 1. The approximation of the function $1/(1+25x^2)$ on $[-1,1]$ with $h = 0.1$ by continuous polynomial splines (18), (2)–(4) is represented in figure 2.

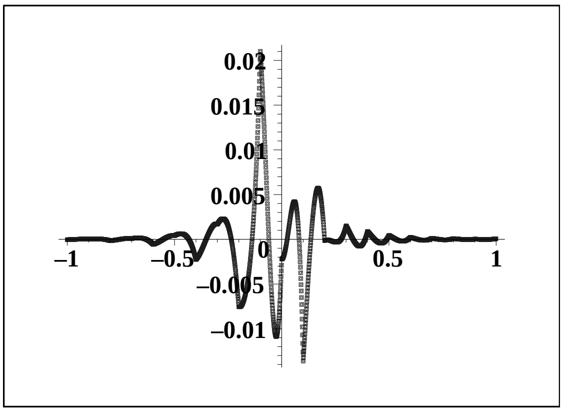

Fig. 1: Graph of error of approximation of Runge function $1/(1+25x^2)$ on $[-1,1]$ with $h = 0.1$ by continuous trigonometric splines (21), (10)–(12).

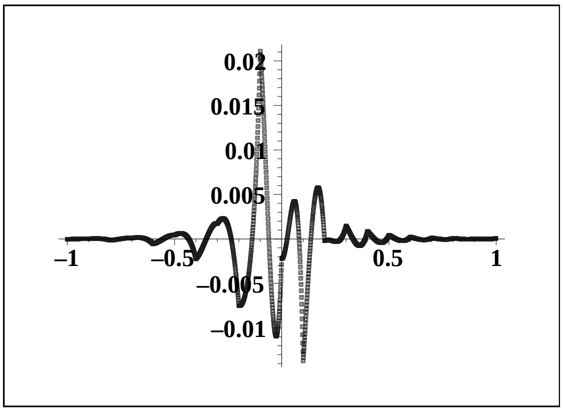

Fig. 2: Graph of error of approximation of Runge function $1/(1+25x^2)$ on $[-1,1]$ with $h = 0.1$ by continuous polynomial splines (18), (2)–(4).

7 Continuous approximation

If we know the values of the function in points x_k then we can take continuous trigonometric approximation of the function in form:

$$\widetilde{V}(x) = u(x_j)\widetilde{w}_j(x) + u(x_{j+1})\widetilde{w}_{j+1}(x) + u(x_{j+2})\widetilde{w}_{j+2}(x), \ \ x \in [x_j, x_{j+1}], \quad (24)$$

where for $x = x_j + th$, $t \in [0,1]$:

$$\widetilde{w}_j(x_j+th) = \frac{\sin(h) - \sin(th-h) + \sin(th-2h)}{2\sin(h) - \sin(2h)}, \quad (25)$$

$$\widetilde{w}_{j+1}(x_j+th) = \frac{-\sin(th-2h) + \sin(th) - \sin(2h)}{2\sin(h) - \sin(2h)}, \quad (26)$$

$$\widetilde{w}_{j+2}(x_j+th) = \frac{\sin(th-h) - \sin(th) + \sin(h)}{2\sin(h) - \sin(2h)}. \quad (27)$$

It is easy to show that:

$$\widetilde{V}(x) = u(x), \ \ u = 1, \sin(x), \cos(x).$$

Therefore

$$\int_{x_j}^{x_{j+1}} \widetilde{V}(x)dx = u(x_j)W_0 + u(x_{j+1})W_1 + u(x_{j+2})W_2, \quad (28)$$

where

$$W_0 = \frac{\sin(2h) - h\cos(h) - h}{-1 + \cos(2h)},$$

$$W_1 = \frac{h\cos(h) - \sin(h)}{-1 + \cos(h)},$$

$$W_2 = \frac{-h\cos(h) + 2\sin(h) - h}{-1 + \cos(2h)}.$$

It can be shown that:

$$\int_{x_j}^{x_{j+1}} \widetilde{V}(x)dx - \int_{x_j}^{x_{j+1}} u(x)dx = O(h^3).$$

Let us compare continuous trigonometrical approximations (24)–(27) and (21), (10)–(12) where integrals were calculated by (28).

The errors of approximation of function $1/(1+25x^2)$, $x \in [-1,1]$, $h = 0.1$, by splines (24)–(27) are represented in figure 3 and errors of approximation of function $1/(1+25x^2)$, $x \in [-1,1]$, $h = 0.1$ by splines (21), (10)–(12), (28) are represented in figure 4.

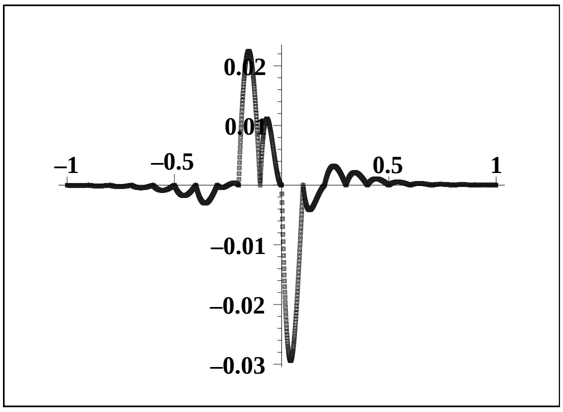

Fig. 3: Graph of error of approximation of Runge function $1/(1+25x^2)$ on $[-1,1]$ with $h=0.1$ by continuous trigonometric splines (24)–(27).

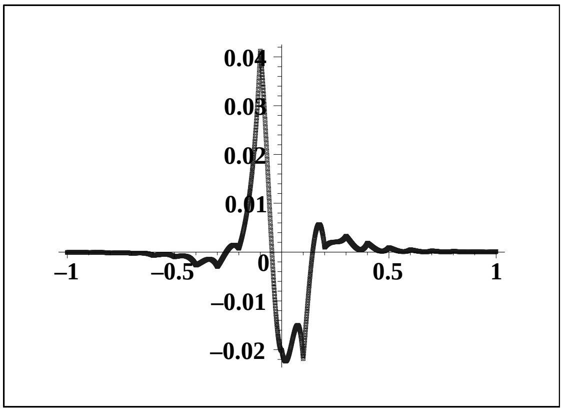

Fig. 4: Graph of error of approximation of Runge function $1/(1+25x^2)$ on $[-1,1]$ with $h=0.1$ by continuous trigonometric splines (21), (10)–(12), (28).

If we know the values of the function in points x_k then we can take continuous polynomial approximation of the function in form:

$$V(x) = u(x_j)w_j(x) + u(x_{j+1})w_{j+1}(x) + u(x_{j+2})w_{j+2}(x), \quad x \in [x_j, x_{j+1}], \quad (29)$$

where for $x = x_j + th$, $t \in [0,1]$

$$w_j(x_j+th) = 1 - (3/2)t + (1/2)t^2, \quad (30)$$

$$w_{j+1}(x_j+th) = -(t-2)t, \quad (31)$$

$$w_{j+2}(x_j+th) = (t-1)t/2. \quad (32)$$

It is easy to show that $V(x) = u(x),\ u = 1,\ x,\ x^2$.

In (18), (2)–(4) we can use:

$$\int_{x_k}^{x_{k+1}} u(t)dt = (u(x_k) + u(x_{k+1}))\, h/2 + O(h^3). \quad (33)$$

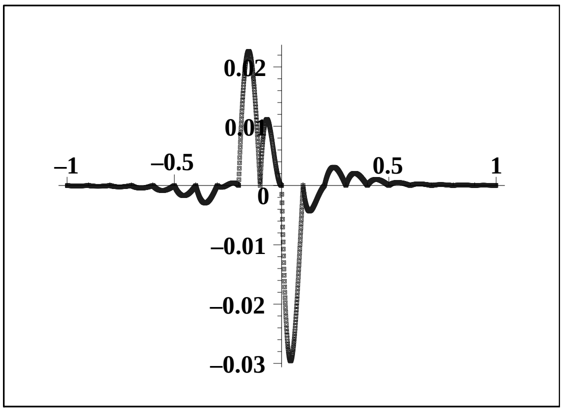

Fig. 5: Graph of error of approximation of Runge function $1/(1+25x^2)$ on $[-1,1]$ with $h=0.1$ by continuous polynomial splines (29)–(32).

Errors of approximation of Runge function $1/(1+25x^2)$ on $[-1,1]$ by $h=0.1$ by polynomial splines (29)–(32) are represented in figure 5. Errors of approximation of Runge function on $[-1,1]$ by $h=0.1$ by polynomial splines (18), (2)–(4), (33) are represented in graphics 6.

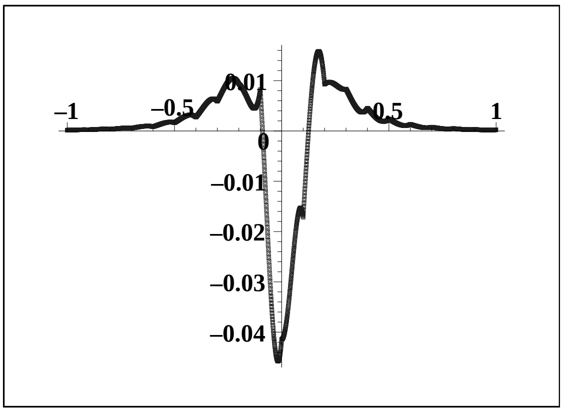

Fig. 6: Graphs of error approximation of Runge function $1/(1+25x^2)$ on $[-1,1]$ with $h=0.1$) by continuous polynomial splines (18), (2)–(4), (33).

The results show that the polynomial and the trigonometric integro-differential splines of the third order approximation may be used in practice but Lebesque constant increase when h is tending to zero.

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