The Spectral Decomposition of Some Tridiagonal Matrices

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Abstract: Some properties of near-Toeplitz tridiagonal matrices with specific perturbations in the first and last main diagonal entries are considered. Applying the relation between the determinant and Chebyshev polynomial of the second kind, we first give the explicit expressions of determinant and characteristic polynomial, then eigenvalues are shown by finding the roots of the characteristic polynomial, which is due to the zeros of Chebyshev polynomial of the second kind, and the eigenvectors are obtained by solving symmetric tridiagonal linear systems in terms of Chebyshev polynomial of the third kind or the fourth kind. By constructing the inverse of the transformation matrices, we give the spectral decomposition of this kind of tridiagonal matrices. Furthermore, the inverse (if the matrix is invertible), powers and a square root are also determined.

Key–Words: Tridiagonal matrices, Spectral decomposition, Powers, Inverses, Chebyshev polynomials

1 Introduction

Tridiagonal matrices arise frequently in many areas of mathematics and engineering [1, 2, 3, 4]. In some problems in numerical analysis one is faced with solving a linear system of equations in which the matrix of the linear system is tridiagonal and Toeplitz, except for elements at the corners. For example, for the homogeneous difference system

$$\boldsymbol{u}(l+1) = \boldsymbol{A}\boldsymbol{u}(l), \qquad l \in \mathbb{Z},$$

where $\boldsymbol{A}$ is a nonsingular constant matrix and $\mathbb{Z}$ is the set of all integers including zero, the general solution can be written as $\boldsymbol{u}(l) = \boldsymbol{A}^l\boldsymbol{c}$, $l \in \mathbb{Z}$, where $\boldsymbol{c}$ is an arbitrary constant vector. Thus, to obtain the general solution of the above homogeneous difference system, we need to give the general expression for $\boldsymbol{A}^l$.

It is well known that if a matrix $\boldsymbol{A}$ has spectral decomposition $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$, then the lth ($l \in \mathbb{N}$) power of $\boldsymbol{A}$ can be obtained by $\boldsymbol{A}^l = \boldsymbol{S}\boldsymbol{\Lambda}^l\boldsymbol{S}^{-1}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix, the diagonal entries of which are the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{S}$ is the transforming matrix formed by eigenvectors of $\boldsymbol{A}$ with them as columns [5]. Therefore, the spectral decomposition plays an important role in computing powers of a matrix.

Rimas derived the general expression of the lth power ($l \in \mathbb{N}$) for one type of symmetric tridiagonal matrices with $0, 1, 0$ as lower diagonal entries, main diagonal entries and upper diagonal entries respectively in [6, 7, 8, 9]. And then he done some work about arbitrary positive integer powers for tridiagonal matrix

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \end{bmatrix}$$

in [10, 11] and presented that the expression of the lth power ($l \in \mathbb{N}$) of the matrix $\boldsymbol{B}$ is

$$\boldsymbol{B}^l = \frac{1}{n}\boldsymbol{Q}(l) = \frac{1}{n}(q_{ij}(l));$$

here

$$q_{ij}(l) = \sum_{k=1}^{n} \beta_k \lambda_k^l T_{\frac{2i-1}{2}}\left(\frac{\lambda_k}{2}\right) T_{\frac{2j-1}{2}}\left(\frac{\lambda_k}{2}\right), \qquad i, j = 1, 2, \ldots, n,$$

$$\beta_k = \begin{cases} 1, & \text{if } k = n, \\ 2, & \text{if } k \neq n, \end{cases}$$

$\lambda_k = -2\cos\frac{k\pi}{n}$, $k = 1, 2, \ldots, n$, are the eigenvalues of the matrix $\boldsymbol{B}$, n ($n \in \mathbb{N}$) is the order of the matrix $\boldsymbol{B}$. In addition, the odd order matrix $\boldsymbol{B}$ is nonsingular and the expression can be applied for computing negative integer powers of $\boldsymbol{B}$. Taking $l = -1$ he got the following expression for elements of the inverse matrix $\boldsymbol{B}^{-1}$:

$$\{\boldsymbol{B}^{-1}\}_{ij} = \frac{1}{n}\sum_{k=1}^{n}\frac{\beta_k}{\lambda_k}T_{\frac{2i-1}{2}}\left(\frac{\lambda_k}{2}\right)T_{\frac{2j-1}{2}}\left(\frac{\lambda_k}{2}\right), \quad i,j=1,2,\ldots,n.$$

The even order matrix $\boldsymbol{B}$ is singular and its inverse and negative powers do not exist.

J. Gutirrez-Gutirrez derived the entries of positive integer powers of an $n\times n$ Hermitian tridiagonal matrix in [12]. And in [13] he studied the entries of positive integer powers of an $n\times n$ complex tridiagonal Toeplitz (constant diagonals) matrix of the form

$$\boldsymbol{A}_n = \mathrm{tridiag}_n(a_1, a_0, a_{-1}) = \begin{bmatrix} a_0 & a_{-1} & & & & \\ a_1 & a_0 & a_{-1} & & & \\ & a_1 & \ddots & \ddots & & \\ & & \ddots & \ddots & a_{-1} & \\ & & & a_1 & a_0 & a_{-1} \\ & & & & a_1 & a_0 \end{bmatrix},$$

where $a_1a_{-1}\neq 0$. He gave the following result:

Consider $a_1, a_0, a_{-1}\in\mathbb{C}$, $a_1a_{-1}\neq 0$ and $n\in\mathbb{N}$. Let $\boldsymbol{A}_n = \mathrm{tridiag}_n(a_1,a_0,a_{-1})$, $\beta=\sqrt{\frac{a_1}{a_{-1}}}$ and $\lambda_h = -2\cos\frac{h\pi}{n+1}$ for every $1\le h\le n$. Then

$$\begin{aligned}[\boldsymbol{A}_n^q]_{j,k} = &\frac{\beta^{j-k}}{2n+2}\Bigg[2(1+(-1)^{n+1})a_0^qU_{j-1}(0)U_{k-1}(0) \\ &+\sum_{h=1}^{\lfloor\frac{n}{2}\rfloor}(4-\lambda_{n-h+1}^2)U_{j-1}\left(\frac{\lambda_{n-h+1}}{2}\right) \\ &\times U_{k-1}\left(\frac{\lambda_{n-h+1}}{2}\right)[(a_0+a_{-1}\beta\lambda_{n-h+1})^q \\ &+(-1)^{j+k}(a_0-a_{-1}\beta\lambda_{n-h+1})^q]\Bigg]\end{aligned}$$

for all $q\in\mathbb{N}$ and $1\le j,k\le n$, where $\lfloor x\rfloor$ denotes the largest integer less than or equal to x.

In this paper, we consider the near-Toeplitz tridiagonal matrices of order n ($n\in\mathbb{N}, n\ge 2$) with the same specific perturbations in the first and last main diagonal entries as follows:

$$\boldsymbol{A} = \begin{bmatrix} \alpha+b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & b & c \\ & & & a & \alpha+b \end{bmatrix}, \qquad (1)$$

where $\alpha, a, b, c\in\mathbb{C}$, and $\alpha=\pm\sqrt{ac}$, $ac\neq 0$.

If $a=c$, then $\boldsymbol{A}$ is symmetric. For a general real symmetric matrix is orthogonally equivalent to a symmetric tridiagonal matrix, so solving the spectral decomposition problem of the symmetric tridiagonal matrices makes a contribution to that of the general real symmetric matrices.

The outline of the paper is as follows. In next section, we review some basic definition and facts about the Chebyshev polynomials and an equality on the sum of trigonometric function without proof. In section 3, we first compute trace, determinant, the characteristic polynomial, the eigenvalues and eigenvectors. The eigenvalues and eigenvectors are calculated by using root-finding scheme and solving symmetric tridiagonal linear system of equations respectively, which are different from the techniques used in [14]. As we all know, the powers are easily determined if we know the spectral decomposition. Therefore, we present the spectral decomposition by constructing the inverse of the similarity matrix of which column vectors are eigenvectors of $\boldsymbol{A}$. On the grounds of the spectral decomposition, we discuss the conditions under which $\boldsymbol{A}$ can be unitarily diagonalizable. In addition, we give some conclusions when $\boldsymbol{A}$ is a symmetric tridiagonal matrix. In section 4, using the results in section 3, we present the powers, inverse (if invertible) and a square root of $\boldsymbol{A}$. In the end, to make the application of the obtained results clear, we solve a difference system as example and verify the result obtained by J. Rimas is a special case of our conclusion. Moreover, the algorithms of Maple 13 are given.

2 Preliminaries

There are several kinds of Chebyshev polynomials. In particular we shall introduce the first and second kind polynomials $T_n(x)$ and $U_n(x)$, as well as a pair of related (Jacobi) polynomials $V_n(x)$ and $W_n(x)$, which we call the Chebyshev polynomials of the third and fourth kinds [15, 16].

Definition 1. *The Chebyshev polynomials $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$ of the first, second, third and fourth kinds are polynomials in x of degree n defined respectively by*

$$\begin{aligned} T_n(x) &= \cos n\theta, \\ U_n(x) &= \sin(n+1)\theta/\sin\theta, \\ V_n(x) &= \cos\left(n+\frac{1}{2}\right)\theta\Big/\cos\frac{1}{2}\theta, \\ W_n(x) &= \sin\left(n+\frac{1}{2}\right)\theta\Big/\sin\frac{1}{2}\theta, \end{aligned}$$

when $x=\cos\theta$, $-1\le x\le 1$.

Lemma 2. *The four kinds of Chebyshev polynomial satisfy the same recurrence relation:*

$$X_n(x) = 2xX_{n-1}(x) - X_{n-2}(x),$$

with $X_0(x) = 1$ *in each case and* $X_1(x) = x$, $2x$, $2x-1$, $2x+1$, *respectively. Furthermore, three relationships can be derived from the above relations:*

$$\begin{aligned} 2T_n(x) &= U_n(x) - U_{n-2}(x), \\ V_n(x) &= U_n(x) - U_{n-1}(x), \\ W_n(x) &= U_n(x) + U_{n-1}(x). \end{aligned}$$

In the light of the Laplace expansion, expanding the following determinants along their last rows and using the three-term recurrence for $U_n(x)$ in Lemma 2, we find $U_n(x)$ can be expressed by the determinant, namely,

$$\begin{aligned} U_0(x) &= 1, \\ U_1(x) &= 2x, \\ U_2(x) &= \begin{vmatrix} 2x & z \\ y & 2x \end{vmatrix} = 2xU_1(x) - U_0(x), \\ &\vdots \\ U_n(x) &= \begin{vmatrix} 2x & z & & & \\ y & 2x & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2x & z \\ & & & y & 2x \end{vmatrix} \\ &= 2xU_{n-1}(x) - U_{n-2}(x), \end{aligned}$$

where $yz = 1$.

Lemma 3. *The equality*

$$\sum_{h=1}^{n-1} \cos\frac{kh\pi}{n} = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ -1, & \text{if } k \text{ is even,} \end{cases}$$

holds for every $n \in \mathbb{N}$, $k = 1, \cdots, 2n-1$.

3 Spectral Decomposition

Employing the Laplace expansion, the expression of $U_n(x)$ in terms of determinant, and the relations between the Chebyshev polynomials in Lemma 2, we have the following assertions.

Lemma 4. *If* $\boldsymbol{A}$ *is a tridiagonal matrix of the form* (1)*, then the trace of* $\boldsymbol{A}$ *is*

$$\mathrm{tr}\boldsymbol{A} = nb + 2\alpha,$$

the determinant of $\boldsymbol{A}$ *is*

$$\det \boldsymbol{A} = |\alpha|^{n-1}(2\alpha + b)U_{n-1}\left(\frac{b}{2|\alpha|}\right),$$

and the characteristic polynomial of $\boldsymbol{A}$ *is*

$$p_{\boldsymbol{A}}(\lambda) = |\alpha|^{n-1}(\lambda - b - 2\alpha)U_{n-1}\left(\frac{\lambda - b}{2|\alpha|}\right). \quad (2)$$

Proof: The trace of $\boldsymbol{A}$ is equal to the sum of all the diagonal entries, so we have $\mathrm{tr}\boldsymbol{A} = nb + 2\alpha$ from the form of $\boldsymbol{A}$.

Let $\boldsymbol{B}_n$ (the subscript n denotes the order) be a tridiagonal matrix with constant entries along the diagonal, namely,

$$\boldsymbol{B}_n = \begin{bmatrix} b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ & & & a & b \end{bmatrix}.$$

On the basis of the determinant expression of $U_n(x)$, we deduce an equality concerning the determinant of $\boldsymbol{B}_n$ and Chebyshev polynomial of the second kind by extracting $\sqrt{ac}$ from each row of $\boldsymbol{B}_n$:

$$\begin{aligned} \det \boldsymbol{B}_n &= (\sqrt{ac})^n \begin{vmatrix} \frac{b}{\sqrt{ac}} & \sqrt{\frac{c}{a}} & & & \\ \sqrt{\frac{a}{c}} & \frac{b}{\sqrt{ac}} & \sqrt{\frac{c}{a}} & & \\ & \ddots & \ddots & \ddots & \\ & & \sqrt{\frac{a}{c}} & \frac{b}{\sqrt{ac}} & \sqrt{\frac{c}{a}} \\ & & & \sqrt{\frac{a}{c}} & \frac{b}{\sqrt{ac}} \end{vmatrix} \\ &= (\sqrt{ac})^n U_n\left(\frac{b}{2\sqrt{ac}}\right). \end{aligned} \quad (3)$$

By expanding the determinant of $\boldsymbol{A}$ along the first column and the last column, we have

$$\begin{aligned} \det \boldsymbol{A} &= \begin{vmatrix} \alpha + b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ & & & a & \alpha + b \end{vmatrix}_{n\times n} \\ &= (\alpha + b)^2 \det \boldsymbol{B}_{n-2} - 2(\alpha + b)ac \det \boldsymbol{B}_{n-3} \\ &\quad + a^2c^2 \det \boldsymbol{B}_{n-4}. \end{aligned}$$

According to the equality (3) between $\det \boldsymbol{B}_n$ and $U_n(x)$ and the relations of Chebyshev polynomials in

Lemma 2, we have

$$\begin{aligned}\det \boldsymbol{A} &= (\alpha+b)^2(\sqrt{ac})^{n-2}U_{n-2}\left(\frac{b}{2\sqrt{ac}}\right)\\ &\quad - 2(\alpha+b)(\sqrt{ac})^{n-1}U_{n-3}\left(\frac{b}{2\sqrt{ac}}\right)\\ &\quad + (\sqrt{ac})^{n}U_{n-4}\left(\frac{b}{2\sqrt{ac}}\right)\\ &= (\alpha+b)^2|\alpha|^{n-2}U_{n-2}\left(\frac{b}{2|\alpha|}\right) - 2(\alpha+b)\\ &\quad \times |\alpha|^{n-1}U_{n-3}\left(\frac{b}{2|\alpha|}\right) + |\alpha|^{n}U_{n-4}\left(\frac{b}{2|\alpha|}\right)\\ &= |\alpha|^n\left[\frac{(\alpha+b)^2}{|\alpha|^2}U_{n-2}\left(\frac{b}{2|\alpha|}\right) - \frac{2(\alpha+b)}{|\alpha|}\right.\\ &\quad \left.\times U_{n-3}\left(\frac{b}{2|\alpha|}\right) + U_{n-4}\left(\frac{b}{2|\alpha|}\right)\right]\\ &= |\alpha|^{n-1}(2\alpha+b)U_{n-1}\left(\frac{b}{2|\alpha|}\right).\end{aligned}$$

Similar to the determinant, the characteristic polynomial

$$p_{\boldsymbol{A}}(\lambda) = |\alpha|^{n-1}(\lambda-b-2\alpha)U_{n-1}\left(\frac{\lambda-b}{2|\alpha|}\right)$$

can be calculated by $p_{\boldsymbol{A}}(\lambda) = \det(\lambda\mathbf{I}-\boldsymbol{A})$, where $\mathbf{I}$ is the identity matrix. □

Consequently, the eigenvalues of $\boldsymbol{A}$ can be obtained through computing the zeros of the characteristic polynomial (2). In view of the roots of $U_{n-1}(x)$ are $x_i = \cos\frac{i\pi}{n}$, $i = 1, 2, \ldots, n-1$, so the eigenvalues of $\boldsymbol{A}$ are

$$\lambda_i = \begin{cases} b + 2|\alpha|\cos\frac{i\pi}{n}, & i = 1, \ldots, n-1,\\ b + 2\alpha, & i = n.\end{cases}$$

From this, we can obtain the following conclusions:

(a) The expression of determinant can be also written as $\det\boldsymbol{A} = (2\alpha+b)\prod_{i=1}^{n-1}\left(b + 2|\alpha|\cos\frac{i\pi}{n}\right)$. Concerning the formula of determinant in Lemma 4, we obtain

$$\begin{aligned}&|\alpha|^{n-1}(2\alpha+b)U_{n-1}\left(\frac{b}{2|\alpha|}\right)\\ &= (2\alpha+b)\prod_{i=1}^{n-1}\left(b+2|\alpha|\cos\frac{i\pi}{n}\right),\end{aligned}$$

namely,

$$|\alpha|^{n-1}U_{n-1}\left(\frac{b}{2|\alpha|}\right) = \prod_{i=1}^{n-1}\left(b+2|\alpha|\cos\frac{i\pi}{n}\right).$$

(b) If n is even, then $\lambda_{n-i} = 2b - \lambda_i$, $i = 1, 2, \ldots, \frac{n}{2}-1$, $\lambda_{\frac{n}{2}} = b$ and $\lambda_n = b + 2\alpha$; If n is odd, then $\lambda_{n-i} = 2b - \lambda_i$, $i = 1, 2, \ldots, \lfloor\frac{n}{2}\rfloor$ and $\lambda_n = b + 2\alpha$. From this, we can again obtain that $\mathrm{tr}\boldsymbol{A} = nb + 2\alpha$. In addition, the spectral radius of $\boldsymbol{A}$ is $b + 2|\alpha|$.

(c) If $b \neq -2|\alpha|\cos\frac{i\pi}{n}(i = 1, 2, \ldots, n-1)$ and $b \neq -2\alpha$, then $\boldsymbol{A}$ is invertible.

It is generally known that the corresponding eigenvectors of $\boldsymbol{A}$ can be attained via solving the equation

$$(\lambda\mathbf{I}-\boldsymbol{A})\boldsymbol{v} = 0, \quad \boldsymbol{v} \neq 0, \tag{4}$$

in which the coefficient matrix $\lambda\mathbf{I}-\boldsymbol{A}$ is nonsymmetric. It is more convenient to solve the equation system if we change the coefficient matrix into a symmetric matrix.

Let $\boldsymbol{D} = \mathrm{diag}(d_0, d_1, \ldots, d_{n-1})$ and $d_k = (a/c)^{k/2}$.

Suppose $\boldsymbol{u}$ solves equation

$$(\lambda\mathbf{I}-\boldsymbol{A})\boldsymbol{D}\boldsymbol{u} = 0, \tag{5}$$

which can be deduced to the linear system of equations with the symmetric tridiagonal matrix, then $\boldsymbol{v} = \boldsymbol{D}\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$.

Let $x_i = \cos\frac{i\pi}{n}$, $i = 1, 2, \ldots, n-1$. When $\alpha = -\sqrt{ac}$, the equation (5) can be written as

$$\begin{aligned}\left(\frac{\lambda-b}{|\alpha|}+1\right)u_1 - u_2 &= 0,\\ -u_1 + \frac{\lambda-b}{|\alpha|}u_2 - u_3 &= 0,\\ -u_2 + \frac{\lambda-b}{|\alpha|}u_3 - u_4 &= 0,\\ &\vdots\\ -u_{n-2} + \frac{\lambda-b}{|\alpha|}u_{n-1} - u_n &= 0,\\ -u_{n-1} + \left(\frac{\lambda-b}{|\alpha|}+1\right)u_n &= 0.\end{aligned}$$

Solving the above equations, we have some solutions

$$\boldsymbol{u}^{(i)} = \begin{cases} [W_0(x_i), W_1(x_i), \ldots, W_{n-1}(x_i)]^{\mathrm{T}}, & \\ \qquad\qquad i = 1, \ldots, n-1, & \\ \left[1, -1, \ldots, (-1)^{n-1}\right]^{\mathrm{T}}, \quad i = n. & \end{cases}$$

Hence, solutions of the equation (4), the eigenvectors of $\boldsymbol{A}$ with $\alpha = -\sqrt{ac}$, are

$$\boldsymbol{v}^{(i)} = \begin{cases} [d_0W_0(x_i), d_1W_1(x_i), \ldots, d_{n-1}W_{n-1}(x_i)]^{\mathrm{T}}, & \\ \qquad\qquad i = 1, \ldots, n-1, & \\ \left[d_0, -d_1, \ldots, (-1)^{n-1}d_{n-1}\right]^{\mathrm{T}}, \quad i = n. & \end{cases}$$

When $\alpha = \sqrt{ac}$, the equation (5) can be written as

$$\begin{aligned}
\left(\frac{\lambda-b}{|\alpha|}-1\right)u_1 - u_2 &= 0\\
-u_1 + \frac{\lambda-b}{|\alpha|}u_2 - u_3 &= 0\\
-u_2 + \frac{\lambda-b}{|\alpha|}u_3 - u_4 &= 0\\
&\vdots\\
-u_{n-2} + \frac{\lambda-b}{|\alpha|}u_{n-1} - u_n &= 0\\
-u_{n-1} + \left(\frac{\lambda-b}{|\alpha|}-1\right)u_n &= 0.
\end{aligned}$$

The system has solutions

$$\boldsymbol{u}^{(i)} = \begin{cases} [V_0(x_i), V_1(x_i), \ldots, V_{n-1}(x_i)]^{\mathrm{T}}, \\ \qquad\qquad i = 1, \ldots, n-1, \\ [1, 1, \ldots, 1]^{\mathrm{T}}, \quad i = n. \end{cases}$$

Therefore, the solutions of the equation (4) are

$$\boldsymbol{v}^{(i)} = \begin{cases} [d_0V_0(x_i), d_1V_1(x_i), \ldots, d_{n-1}V_{n-1}(x_i)]^{\mathrm{T}}, \\ \qquad\qquad i = 1, \ldots, n-1, \\ [d_0, d_1, \ldots, d_{n-1}]^{\mathrm{T}}, \quad i = n, \end{cases}$$

which are the eigenvectors of $\boldsymbol{A}$ with $\alpha = \sqrt{ac}$.

Using the above results on the eigenvalues and the corresponding eigenvectors of $\boldsymbol{A}$, we give the spectral decomposition of $\boldsymbol{A}$ and demonstrate it. Note that $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and λ_i $(i = 1, \ldots, n)$ are eigenvalues of $\boldsymbol{A}$ in the remainder of the paper. We introduce the fact about the spectral decomposition in [5] as the following lemma.

Lemma 5. *If $\boldsymbol{A}$ has n linearly independent eigenvectors $\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}, \ldots, \boldsymbol{v}^{(n)}$, form a nonsingular matrix $\boldsymbol{S}$ with them as columns, then $\boldsymbol{A} = \boldsymbol{S\Lambda S}^{-1}$, where*

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

and $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $\boldsymbol{A}$.

Theorem 6. *If $\boldsymbol{A}$ has the form (1) with $\alpha = -\sqrt{ac}$. Then $\boldsymbol{A} = \boldsymbol{S\Lambda TS^T}(\boldsymbol{D}^{-1})^2$, where*

$$\boldsymbol{S} = \begin{bmatrix} d_0W_0(x_1) & d_0W_0(x_2) & \cdots & d_0 \\ d_1W_1(x_1) & d_1W_1(x_2) & \cdots & -d_1 \\ \vdots & \vdots & & \vdots \\ d_{n-1}W_{n-1}(x_1) & d_{n-1}W_{n-1}(x_2) & \cdots & d_{n-1}(-1)^{n-1} \end{bmatrix},$$

$\boldsymbol{T} = \mathrm{diag}(t_1, \ldots, t_n)$ and

$$t_h = \begin{cases} (1-x_h)/n, & h = 1, 2, \ldots, n-1, \\ 1/n, & h = n. \end{cases}$$

Proof: Obviously, the only thing we need to do is to show that $\boldsymbol{STS}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2 = \mathbf{I}$. If $i = j$, then

$$\begin{aligned}
[\boldsymbol{STS}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ii} &= \sum_{h=1}^{n-1} t_h W_{i-1}^2(x_h) + t_n \\
&= \frac{1}{n}\left(n - \sum_{h=1}^{n-1}\cos\frac{(2i-1)h\pi}{n}\right).
\end{aligned}$$

From Lemma 3, we have $[\boldsymbol{STS}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ii} = 1$.

If $i \neq j$ and $i + j$ is even, then

$$\begin{aligned}
&[\boldsymbol{STS}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ij} \\
&= d_{i-j}\left(\sum_{h=1}^{n-1} t_h W_{i-1}(x_h) W_{j-1}(x_h) + t_n\right) \\
&= \frac{d_{i-j}}{n}\left(\sum_{h=1}^{n-1}\cos\frac{(i-j)h\pi}{n} \right.\\
&\qquad \left. - \sum_{h=1}^{n-1}\cos\frac{(i+j-1)h\pi}{n} + 1\right).
\end{aligned}$$

According to Lemma 3, we have

$$[\boldsymbol{STS}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ij} = 0.$$

If $i + j$ is odd, then

$$\begin{aligned}
[\boldsymbol{STS}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ij} &= \frac{d_{i-j}}{n}\left(\sum_{h=1}^{n-1}\cos\frac{(i-j)h\pi}{n}\right. \\
&\qquad \left. - \sum_{h=1}^{n-1}\cos\frac{(i+j-1)h\pi}{n} - 1\right).
\end{aligned}$$

By Lemma 3, we have $[\boldsymbol{STS}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ij} = 0$. And $\boldsymbol{STS}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2 = \mathbf{I}$ follows from the above discussion. $\square$

Corollary 7. *Let $\boldsymbol{A}$ be a tridiagonal matrix of the form (1) with $\alpha = -\sqrt{ac}$. If $|a| = |c|$, then $\boldsymbol{A}$ can be unitarily diagonalizable.*

Proof: A scalar multiple of an eigenvector of $\boldsymbol{A}$ is still an eigenvector of $\boldsymbol{A}$, so $\boldsymbol{v}_1^{(i)} = \sqrt{\frac{1-x_i}{n}}[d_0W_0(x_i), d_1W_1(x_i), \ldots, d_{n-1}W_{n-1}(x_i)]^{\mathrm{T}}$, $i = 1, 2, \ldots, n-1$, and $\boldsymbol{v}_1^{(n)} = \sqrt{\frac{1}{n}}\left[d_0, -d_1, \ldots, (-1)^{n-1}d_{n-1}\right]^{\mathrm{T}}$ are still eigenvectors of $\boldsymbol{A}$. Let $\boldsymbol{U}$ be a matrix with $\boldsymbol{v}_1^{(1)}, \ldots, \boldsymbol{v}_1^{(n)}$ as columns. If we want to prove that $\boldsymbol{A} = \boldsymbol{U\Lambda U}^*$ ($\boldsymbol{U}^*$ is the Hermitian adjoint, whose entries are the conjugate transpose of entries of $\boldsymbol{U}$), then what we need to do is to verify that $\boldsymbol{UU}^* = \boldsymbol{I}$. Obviously,

$$[\boldsymbol{UU}^*]_{ij} = d_{i-1}\overline{d_{j-1}}\left(\sum_{h=1}^{n-1}\frac{1-x_h}{n}W_{i-1}(x_h) \times W_{j-1}(x_h) + \frac{1}{n}\right).$$

If $i = j$, then

$$[\boldsymbol{UU}^*]_{ii} = d_{i-1}\overline{d_{i-1}}\left(\sum_{h=1}^{n-1}\frac{1-x_h}{n}W_{i-1}^2(x_h) + \frac{1}{n}\right) = |d_{i-1}|^2\left(\sum_{h=1}^{n-1}\frac{1-x_h}{n}W_{i-1}^2(x_h) + \frac{1}{n}\right).$$

From the proof of Theorem 6, we have

$$\sum_{h=1}^{n-1}\frac{1-x_h}{n}W_{i-1}^2(x_h) + \frac{1}{n} = 1.$$

Since $|a| = |c|$, $|d_{i-1}|^2 = |\frac{a}{c}|^{i-1} = 1$. Thus, $[\boldsymbol{UU}^*]_{ii} = 1$. If $i \neq j$, then

$$\sum_{h=1}^{n-1}\frac{1-x_h}{n}W_{i-1}(x_h)W_{j-1}(x_h) + \frac{1}{n} = 0$$

by the proof of Theorem 6. From the above discussion, we know that the transformation matrix $\boldsymbol{U}$ is unitary and $\boldsymbol{A}$ with $\alpha = -\sqrt{ac}$ can be unitarily diagonalizable when $|a| = |c|$. □

Theorem 8. *$\boldsymbol{A}$ is a tridiagonal matrix of the form (1) with $\alpha = \sqrt{ac}$. Then $\boldsymbol{A} = \boldsymbol{P\Lambda QP}^T(\boldsymbol{D}^{-1})^2$, where $\boldsymbol{P}$ consists of the eigenvectors of $\boldsymbol{A}$, i.e.,*

$$\boldsymbol{P} = \begin{bmatrix} d_0V_0(x_1) & d_0V_0(x_2) & \cdots & d_0 \\ d_1V_1(x_1) & d_1V_1(x_2) & \cdots & d_1 \\ \vdots & \vdots & & \vdots \\ d_{n-1}V_{n-1}(x_1) & d_{n-1}V_{n-1}(x_2) & \cdots & d_{n-1} \end{bmatrix},$$

$\boldsymbol{Q} = \mathrm{diag}(q_1, \ldots, q_n)$ and

$$q_h = \begin{cases} (1+x_h)/n, & h = 1, 2, \ldots, n-1, \\ 1/n, & h = n. \end{cases}$$

Proof: The technique used in the proof is the same as Theorem 6 and we need to demonstrate that $\boldsymbol{PQP}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2 = \mathbf{I}$. First, we have

$$\begin{aligned}[\boldsymbol{PQP}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ij} &= d_{i-1}d_{j-1}^{-1}\left(\sum_{h=1}^{n-1}q_hV_{i-1}(x_h)V_{j-1}(x_h) + q_n\right) \\ &= \frac{d_{i-j}}{n}\left(\sum_{h=1}^{n-1}\cos\frac{(i-j)h\pi}{n} + \sum_{h=1}^{n-1}\cos\frac{(i+j-1)h\pi}{n} + 1\right).\end{aligned}$$

According to Lemma 3, we obtain the following conclusions: If $i = j$, then

$$\sum_{h=1}^{n-1}\cos\frac{(2i-1)h\pi}{n} = 0,$$

and $[\boldsymbol{PQP}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ii} = \frac{1}{n}(n-1+1) = 1$. If $i \neq j$, then

$$\sum_{h=1}^{n-1}\cos\frac{(i-j)h\pi}{n} + \sum_{h=1}^{n-1}\cos\frac{(i+j-1)h\pi}{n} = -1,$$

and $[\boldsymbol{PQP}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2]_{ij} = 0$. Therefore, we have $\boldsymbol{PQP}^{\mathrm{T}}(\boldsymbol{D}^{-1})^2 = \mathbf{I}$. □

Corollary 9. *Let $\boldsymbol{A}$ be a tridiagonal matrix of the form (1) with $\alpha = \sqrt{ac}$. If $|a| = |c|$, then $\boldsymbol{A}$ can be unitarily diagonalizable.*

Proof: Firstly, we know that the vectors $\boldsymbol{v}_2^{(i)} = \sqrt{\frac{1+x_i}{n}}[d_0V_0(x_i), d_1V_1(x_i), \ldots, d_{n-1}V_{n-1}(x_i)]^{\mathrm{T}}$, $i = 1, \ldots, n-1$, and $\boldsymbol{v}_2^{(n)} = \sqrt{\frac{1}{n}}\left[d_0, d_1, \ldots, d_{n-1}\right]^{\mathrm{T}}$ are still eigenvectors of $\boldsymbol{A}$. Let $\boldsymbol{V}$ be a matrix with $\boldsymbol{v}_2^{(1)}, \ldots, \boldsymbol{v}_2^{(n)}$ as columns. If we want to prove that $\boldsymbol{A} = \boldsymbol{V\Lambda V}^*$ ($\boldsymbol{V}^*$ is the Hermitian adjoint, whose entries are the conjugate transpose of entries of $\boldsymbol{V}$), then what we need to do is to verify that $\boldsymbol{VV}^* = \boldsymbol{I}$. Obviously,

$$[\boldsymbol{VV}^*]_{ij} = d_{i-1}\overline{d_{j-1}}\left(\sum_{h=1}^{n-1}\frac{1+x_h}{n}V_{i-1}(x_h) \times V_{j-1}(x_h) + \frac{1}{n}\right).$$

If $i = j$, then

$$[\boldsymbol{V}\boldsymbol{V}^*]_{ii} = d_{i-1}\overline{d_{i-1}}\left(\sum_{h=1}^{n-1}\frac{1+x_h}{n}V_{i-1}^2(x_h)+\frac{1}{n}\right) = |d_{i-1}|^2\left(\sum_{h=1}^{n-1}\frac{1+x_h}{n}V_{i-1}^2(x_h)+\frac{1}{n}\right).$$

From the proof of Theorem 8, we have

$$\sum_{h=1}^{n-1}\frac{1+x_h}{n}V_{i-1}^2(x_h)+\frac{1}{n} = 1.$$

Since $|a| = |c|$, $|d_{i-1}|^2 = |\frac{a}{c}|^{i-1} = 1$. Thus, $[\boldsymbol{V}\boldsymbol{V}^*]_{ii} = 1$. If $i \neq j$, then

$$\sum_{h=1}^{n-1}\frac{1+x_h}{n}V_{i-1}(x_h)V_{j-1}(x_h)+\frac{1}{n} = 0$$

by the proof of Theorem 8. From the above discussion, we know that the transformation matrix $\boldsymbol{V}$ is unitary and $\boldsymbol{A}$ with $\alpha = \sqrt{ac}$ can be unitarily diagonalizable when $|a| = |c|$. □

Corollary 10. *Let $\boldsymbol{A}$ be a tridiagonal matrix of the form (1) with $\alpha = -\sqrt{ac}$ or $\alpha = \sqrt{ac}$. If $a = c$, then two arbitrary tridiagonal matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ with this kind of form are simultaneously diagonalizable, that is, there is a single similarity matrix $\boldsymbol{S}$ such that $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$ and $\boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{S}$ are both diagonal.*

Proof: If $a = c$, then $\boldsymbol{D}$ is the identity matrix in Theorem 6 and Theorem 8. The conclusion can be obtained directly from Theorem 6 and Theorem 8. □

Corollary 11. *Let $\mathfrak{F}$ be a family of the matrices of the form (1) with $a = c$, $\alpha = -|a|$ or $\alpha = |a|$. Then $\mathfrak{F}$ is a simultaneously diagonalizable family and a commuting family.*

Proof: From Corollary 10, we know that $\mathfrak{F}$ is a simultaneously diagonalizable family, that is, for any $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{F}$, there exists a single similarity matrix $\boldsymbol{S}$ such that $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{\Lambda}_1$ and $\boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{S} = \boldsymbol{\Lambda}_2$, where $\boldsymbol{\Lambda}_1$, $\boldsymbol{\Lambda}_2$ are diagonal matrices. Then

$$\boldsymbol{A}\boldsymbol{B} = \boldsymbol{S}\boldsymbol{\Lambda}_1\boldsymbol{S}^{-1}\boldsymbol{S}\boldsymbol{\Lambda}_2\boldsymbol{S}^{-1} = \boldsymbol{S}\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2\boldsymbol{S}^{-1} = \boldsymbol{S}\boldsymbol{\Lambda}_2\boldsymbol{\Lambda}_1\boldsymbol{S}^{-1} = \boldsymbol{S}\boldsymbol{\Lambda}_2\boldsymbol{S}^{-1}\boldsymbol{S}\boldsymbol{\Lambda}_1\boldsymbol{S}^{-1} = \boldsymbol{B}\boldsymbol{A}.$$

Therefore, $\mathfrak{F}$ is not only a simultaneously diagonalizable family but also a commuting family. □

4 Powers and Inverse

As we all know, if the matrix $\boldsymbol{A}$ has spectral decomposition $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$, then the lth ($l \in \mathbb{N}$) power of $\boldsymbol{A}$ can be obtained by $\boldsymbol{A}^l = \boldsymbol{S}\boldsymbol{\Lambda}^l\boldsymbol{S}^{-1}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix, the diagonal entries of which are the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{S}$ is the transforming matrix formed by eigenvectors of $\boldsymbol{A}$ with them as columns [5]. In the previous section, we have stated the spectral decomposition of $\boldsymbol{A}$. In this section, we calculate the powers, inverse and a square root of $\boldsymbol{A}$.

Theorem 12. *If $\boldsymbol{A}$ has the form (1) with $\alpha = -\sqrt{ac}$ and $x_h = \cos\frac{h\pi}{n}$, $h = 1, 2, \ldots, n-1$. Then the i, j entry of $\boldsymbol{A}^l$ is*

$$[\boldsymbol{A}^l]_{ij} = \frac{d_{i-j}}{n}\left\{\sum_{h=1}^{n-1}\left[(b+2|\alpha|x_h)^l(1-x_h) \times W_{i-1}(x_h)W_{j-1}(x_h)\right] + (-1)^{i+j}(b+2\alpha)^l\right\}.$$

Proof: According to Theorem 6, we have the following results: If $i+j$ is even, then

$$\begin{aligned}[\boldsymbol{A}^l]_{ij} &= \sum_{h=1}^{n-1} d_{i-j}W_{i-1}(x_h)\lambda_h^l t_h W_{j-1}(x_h) + d_{i-j}t_n\lambda_n^l \\ &= \frac{d_{i-j}}{n}\left\{\sum_{h=1}^{n-1}\left[(b+2|\alpha|x_h)^l(1-x_h)W_{i-1}(x_h) \times W_{j-1}(x_h)\right] + (b+2\alpha)^l\right\}.\end{aligned}$$

If $i+j$ is odd, then

$$\begin{aligned}[\boldsymbol{A}^l]_{ij} &= \sum_{h=1}^{n-1} d_{i-j}W_{i-1}(x_h)\lambda_h^l t_h W_{j-1}(x_h) - d_{i-j}t_n\lambda_n^l \\ &= \frac{d_{i-j}}{n}\left\{\sum_{h=1}^{n-1}\left[(b+2|\alpha|x_h)^l(1-x_h)W_{i-1}(x_h) \times W_{j-1}(x_h)\right] - (b+2\alpha)^l\right\}.\end{aligned}$$

The conclusion follows from the above discussion. □

Corollary 13. *Let $\boldsymbol{A}$ be a tridiagonal matrix of the form (1) with $\alpha = -\sqrt{ac}$ and $x_h = \cos\frac{h\pi}{n}$, $h = 1, 2, \ldots, n-1$. If $b \neq 2\alpha x_h$, $h = 1, 2, \ldots, n-1$, and $b \neq -2\alpha$, that is, $\boldsymbol{A}$ is invertible, then l can be taken* *negative integer in Theorem 12. In particular,*

$$[\boldsymbol{A}^{-1}]_{ij} = \frac{d_{i-j}}{n}\left\{\sum_{h=1}^{n-1}\left[\frac{1-x_h}{b+2|\alpha|x_h}W_{i-1}(x_h) \times W_{j-1}(x_h)\right] + \frac{(-1)^{i+j}}{b+2\alpha}\right\},$$

and the matrix $\boldsymbol{C}$, whose i, j entry is

$$[\boldsymbol{C}]_{ij} = \frac{d_{i-j}}{n}\left\{\sum_{h=1}^{n-1}\left[(1-x_h)\sqrt{b+2|\alpha|x_h}W_{i-1}(x_h) \times W_{j-1}(x_h)\right] + (-1)^{i+j}\sqrt{b+2\alpha}\right\},$$

is a square root of $\boldsymbol{A}$ with $\alpha = -\sqrt{ac}$.

Theorem 14. *If $\boldsymbol{A}$ has the form (1) with $\alpha = \sqrt{ac}$ and $x_h = \cos\frac{h\pi}{n}$, $h = 1, 2, \ldots, n-1$. Then the i, j entry of $\boldsymbol{A}^l$ is*

$$[\boldsymbol{A}^l]_{ij} = \frac{d_{i-j}}{n}\left\{\sum_{h=1}^{n-1}\left[(b+2\alpha x_h)^l(1+x_h)V_{i-1}(x_h) \times V_{j-1}(x_h)\right] + (b+2\alpha)^l\right\}.$$

Proof: By reference to the proof of Theorem 12 and employing Theorem 8, the theorem can be proved. □

Corollary 15. *Let $\boldsymbol{A}$ be a tridiagonal matrix of the form (1) with $\alpha = \sqrt{ac}$ and $x_h = \cos\frac{h\pi}{n}$, $h = 1, 2, \ldots, n-1$. If $b \neq -2\alpha x_h$, $h = 1, 2, \ldots, n-1$, and $b \neq -2\alpha$, then l can be taken negative integer in Theorem 14. Furthermore,*

$$[\boldsymbol{A}^{-1}]_{ij} = \frac{d_{i-j}}{n}\left\{\sum_{h=1}^{n-1}\left[\frac{1+x_h}{b+2\alpha x_h}V_{i-1}(x_h) \times V_{j-1}(x_h)\right] + \frac{1}{b+2\alpha}\right\},$$

and the matrix $\boldsymbol{D}$, whose i, j entry is

$$[\boldsymbol{D}]_{ij} = \frac{d_{i-j}}{n}\left\{\sum_{h=1}^{n-1}\left[(1+x_h)\sqrt{b+2\alpha x_h}V_{i-1}(x_h) \times V_{j-1}(x_h)\right] + \sqrt{b+2\alpha}\right\},$$

is a square root of $\boldsymbol{A}$ with $\alpha = \sqrt{ac}$.

5 Numerical examples

In this section, Example 16 indicates that the conclusion presented in [10, 11] is a special case of Theorem 14. Example 18 explains the application of our work by solving a simple homogeneous difference system.

Example 16. *Consider the matrix*

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \end{bmatrix}.$$

On the basis of the conclusions in preceding part, we derive the following assertions:

(a) *The eigenvalues of $\boldsymbol{B}$ are $\lambda_i = 2\cos\frac{i\pi}{n}$, $i = 1, 2, \ldots, n-1$ and $\lambda_n = 2$. The corresponding eigenvectors are*

$$\boldsymbol{v}^{(i)} = \begin{cases} [V_0(x_i), V_1(x_i), \ldots, V_{n-1}(x_i)]^{\mathrm{T}}, & i = 1, \ldots, n-1, \\ [1, 1, \ldots, 1]^{\mathrm{T}}, & i = n, \end{cases}$$

where $x_i = \cos\frac{i\pi}{n}$. Moreover, if n is even, then $\lambda_{n-i} = -\lambda_i$, $i = 1, 2, \ldots, \frac{n}{2}-1$, $\lambda_{\frac{n}{2}} = 0$ and $\lambda_n = 2$; If n is odd, then $\lambda_{n-i} = -\lambda_i$, $i = 1, 2, \ldots, \lfloor\frac{n}{2}\rfloor$ and $\lambda_n = 2$. From this, we deduce that if n is odd, then $\boldsymbol{B}$ is invertible and if n is even, then $\boldsymbol{B}$ is singular.

(b) *The trace of $\boldsymbol{B}$ is $\mathrm{tr}\boldsymbol{B} = 2$. The determinant of $\boldsymbol{B}$ is $\det\boldsymbol{B} = 2U_{n-1}(0) = 2\prod_{i=1}^{n-1} 2\cos\frac{i\pi}{n}$. And if n is even, then $\det\boldsymbol{B} = 0$; If $n \equiv 1 \pmod 4$, then $\det\boldsymbol{B} = 2$; If $n \equiv 3 \pmod 4$, then $\det\boldsymbol{B} = -2$.*

(c) *Let $x_h = \cos\frac{h\pi}{n}$, $h = 1, 2, \ldots, n-1$. Then the i, j entry of $\boldsymbol{B}^l$ is*

$$[\boldsymbol{B}^l]_{ij} = \frac{1}{n}\sum_{h=1}^{n-1}(2x_h)^l(1+x_h)V_{i-1}(x_h)V_{j-1}(x_h) + \frac{2^l}{n}.$$

If n is odd, then the i, j entry of the inverse of $\boldsymbol{B}$ is

$$[\boldsymbol{B}^{-1}]_{ij} = \frac{1}{n}\sum_{h=1}^{n-1}\frac{1+x_h}{2x_h}V_{i-1}(x_h)V_{j-1}(x_h) + \frac{1}{2n}.$$

The matrix $\boldsymbol{B}_1$, whose i, j entry is

$$[\boldsymbol{B}_1]_{ij} = \frac{1}{n}\sum_{h=1}^{n-1}\left[\sqrt{2x_h}(1+x_h)V_{i-1}(x_h) \times V_{j-1}(x_h)\right] + \frac{\sqrt{2}}{n},$$

is a square root of $\boldsymbol{B}$.

Proof: We demonstrate that the result (c) is equivalent to the expression given in [10, 11]. Let $\theta_h = \frac{h\pi}{n}$, $\lambda_h = -2\cos\frac{h\pi}{n}$ and $x_h = \cos\frac{h\pi}{n}$, $h = 1, 2, \ldots, n-1$, then $\lambda_h = -2x_h$, $x_h = \cos\theta_h$. Since $\lambda_{n-i} = -\lambda_i$, $i = 1, 2, \ldots, \lfloor\frac{n}{2}\rfloor$, we have

$$\begin{aligned}
[\boldsymbol{B}^l]_{ij} &= \frac{1}{n}\sum_{h=1}^{n-1} 2\lambda_h^l T_{\frac{2i-1}{2}}\left(\frac{\lambda_h}{2}\right) T_{\frac{2j-1}{2}}\left(\frac{\lambda_h}{2}\right) \\
&\quad + \frac{1}{n}\lambda_n^l T_{\frac{2i-1}{2}}\left(\frac{\lambda_n}{2}\right) T_{\frac{2j-1}{2}}\left(\frac{\lambda_n}{2}\right) \\
&= \frac{1}{n}\sum_{h=1}^{n-1} 2(2x_h)^l T_{\frac{2i-1}{2}}(x_h) T_{\frac{2j-1}{2}}(x_h) + \frac{2^l}{n} \\
&= \frac{1}{n}\sum_{h=1}^{n-1} 2(2x_h)^l \cos\frac{2i-1}{2}\theta_h \cos\frac{2j-1}{2}\theta_h + \frac{2^l}{n} \\
&= \frac{1}{n}\sum_{h=1}^{n-1}\left[(2x_h)^l(1+\cos\theta_h)\frac{\cos\left(i-\frac{1}{2}\right)\theta_h}{\cos\frac{\theta_h}{2}}\right. \\
&\quad \left.\times \frac{\cos\left(j-\frac{1}{2}\right)\theta_h}{\cos\frac{\theta_h}{2}}\right] + \frac{2^l}{n} \\
&= \frac{1}{n}\sum_{h=1}^{n-1}(2x_h)^l(1+x_h)V_{i-1}(x_h)V_{j-1}(x_h) + \frac{2^l}{n}.
\end{aligned}$$

The proof is completed. □

In view of the matrix $\boldsymbol{B}$ in [10, 11], we consider the matrix $\boldsymbol{C}$ of the similar form with $\boldsymbol{B}$ and give the related facts.

Example 17. *Consider the matrix*

$$\boldsymbol{C} = \begin{bmatrix} -1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & -1 \end{bmatrix}.$$

We derive the following results:

(a) *The eigenvalues of $\boldsymbol{C}$ are $\lambda_i = 2\cos\frac{i\pi}{n}$, $i = 1, 2, \ldots, n-1$ and $\lambda_n = -2$. The corresponding eigenvectors are*

$$\boldsymbol{v}^{(i)} = \begin{cases} [W_0(x_i), W_1(x_i), \ldots, W_{n-1}(x_i)]^{\mathrm{T}}, \\ \qquad\qquad i = 1, \ldots, n-1, \\ \left[1, -1, \ldots, (-1)^{n-1}\right]^{\mathrm{T}}, \quad i = n, \end{cases}$$

where $x_i = \cos\frac{i\pi}{n}$. Moreover, if n is even, then $\lambda_{n-i} = -\lambda_i$, $i = 1, 2, \ldots, \frac{n}{2}-1$, $\lambda_{\frac{n}{2}} = 0$ and $\lambda_n = -2$; If n is odd, then $\lambda_{n-i} = -\lambda_i$, $i = 1, 2, \ldots, \lfloor\frac{n}{2}\rfloor$ and $\lambda_n = -2$. From this, we deduce that if n is odd, then $\boldsymbol{C}$ is invertible and if n is even, then $\boldsymbol{C}$ is singular.

(b) *The trace of $\boldsymbol{C}$ is $\mathrm{tr}\boldsymbol{C} = -2$. The determinant of $\boldsymbol{C}$ is $\det\boldsymbol{C} = -2U_{n-1}(0) = -2\prod_{i=1}^{n-1} 2\cos\frac{i\pi}{n}$. And if n is even, then $\det\boldsymbol{C} = 0$; If $n \equiv 1 \pmod 4$, then $\det\boldsymbol{C} = -2$; If $n \equiv 3 \pmod 4$, then $\det\boldsymbol{C} = 2$.*

(c) *Let $x_h = \cos\frac{h\pi}{n}$, $h = 1, 2, \ldots, n-1$. Then the i, j entry of $\boldsymbol{C}^l$ is*

$$\begin{aligned}
[\boldsymbol{C}^l]_{ij} &= \frac{1}{n}\sum_{h=1}^{n-1}\Big[(2x_h)^l(1-x_h)W_{i-1}(x_h) \\
&\quad \times W_{j-1}(x_h)\Big] + \frac{(-1)^{i+j}}{n}(-2)^l.
\end{aligned}$$

If n is odd, then the i, j entry of the inverse of $\boldsymbol{C}$ is

$$\begin{aligned}
[\boldsymbol{C}^{-1}]_{ij} &= \frac{1}{n}\sum_{h=1}^{n-1}\Big[\frac{1-x_h}{2x_h}W_{i-1}(x_h) \\
&\quad \times W_{j-1}(x_h)\Big] + \frac{(-1)^{i+j}}{-2n}.
\end{aligned}$$

The matrix $\boldsymbol{C}_1$, whose i, j entry is

$$\begin{aligned}
[\boldsymbol{C}_1]_{ij} &= \frac{1}{n}\sum_{h=1}^{n-1}\Big[\sqrt{2x_h}(1-x_h)W_{i-1}(x_h) \\
&\quad \times W_{j-1}(x_h)\Big] + \frac{(-1)^{i+j}}{n}\sqrt{-2},
\end{aligned}$$

is a square root of $\boldsymbol{C}$.

Example 18. *Consider the homogeneous difference system $\boldsymbol{u}(l+1) = \boldsymbol{A}\boldsymbol{u}(l)$, $l \in \mathbb{Z}$, where the matrix $\boldsymbol{A}$ is given by*

$$\begin{bmatrix} 5 & 8 & 0 & 0 & 0 \\ 2 & 1 & 8 & 0 & 0 \\ 0 & 2 & 1 & 8 & 0 \\ 0 & 0 & 2 & 1 & 8 \\ 0 & 0 & 0 & 2 & 5 \end{bmatrix}.$$

Solving the homogeneous difference system, we know that the general solution can be written as $\boldsymbol{u}(l) = \boldsymbol{A}^l\boldsymbol{c}$, $l \in \mathbb{Z}$, where $\boldsymbol{c}$ is an arbitrary constant vector. By using Theorem 14 and Maple 13 programme, we *can calculate $\boldsymbol{A}^l$, so the general solution $\boldsymbol{u}(l) = \boldsymbol{A}^l\boldsymbol{c}$ is also computed. For example, we get*

$$\boldsymbol{u}(5) = \begin{bmatrix} 20181.00 & 34920.00 & 44160.00 & 56320.00 & 53248.00 \\ 8730.00 & 13761.00 & 26920.00 & 29312.00 & 56320.00 \\ 2760.00 & 6730.00 & 10049.00 & 26920.00 & 44160.00 \\ 880.00 & 1832.00 & 6730.00 & 13761.00 & 34920.00 \\ 208.00 & 880.00 & 2760.00 & 8730.00 & 20181.00 \end{bmatrix} \boldsymbol{c}$$

by Maple 13 programme.

Note that Theorem 12 and Theorem 14 can be executed by Maple 13 programme. The algorithm of Theorem 12 is as follows:

```
> restart:
> with(linalg):
> n:=n: l:=l: a:=a; b:=b: c:=c:
  Al:=array(1..n,1..n):
  x:=cos(h*pi/n):
> for i from 1 by 1 to n do
    for j from 1 by 1 to n do
      Al[i,j]:=evalf((sum((b+2
            *sqrt(a*c)*x)^l*(1-x)
            *(ChebyshevU(i-1,x)
            +ChebyshevU(i-2,x))
            *(ChebyshevU(j-1,x)
            +ChebyshevU(j-2,x)),
            h=1..n-1)+(-1)^(i+j)
            *(b-2*sqrt(a*c))^l)
            *sqrt(a/c)^(i-j)/n)
    end do
  end do;
> print(Al);
```

The algorithm of Theorem 14 is as follows:

```
> restart:
> with(linalg):
> n:=n: l:=l: a:=a; b:=b: c:=c:
  Al:=array(1..n,1..n):
  x:=cos(h*pi/n):
> for i from 1 by 1 to n do
    for j from 1 by 1 to n do
      Al[i,j]:=evalf((sum((b+2
            *sqrt(a*c)*x)^l*(1+x)
            *(ChebyshevU(i-1,x)
            -ChebyshevU(i-2,x))
            *(ChebyshevU(j-1,x)
            -ChebyshevU(j-2,x)),
            h=1..n-1)+(b+2*sqrt(a*c))
               ^l)*sqrt(a/c)^(i-j)/n)
    end do
  end do;
> print(Al);
```

where a, b, c are the entries of $\boldsymbol{A}$, n is the order of $\boldsymbol{A}$, l is the power index. The lth power of $\boldsymbol{A}$ can be obtained if we input a, b, c, n and l.

6 Conclusion

Being inspired by the research done by J. Rimas and J. Gutirrez-Gutirrez, we not only generalize their work concerning the positive integer powers of tridiagonal matrices, but also explore other basic properties including trace, determinant, eigenvalues, eigenvectors and so on. Unfortunately, In this paper, we consider only two kinds of tridiagonal matrices. If possible, we can discuss more general tridiagonal matrices.

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