On s-quasinormally embedded or weakly s-permutable subgroups of finite groups

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Abstract: Suppose that G is a finite group and H is a subgroup of G. H is said to be s-quasinormally embedded in G if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-quasinormal subgroup of G; H is said to be weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. We fix in every non-cyclic Sylow subgroup P of G some subgroup D satisfying $1 < |D| < |P|$ and study the structure of G under the assumption that every subgroup H of P with $|H| = |D|$ is either s-quasinormally embedded or weakly s-permutable in G. Some recent results are generalized and unified.

Key–Words: s-quasinormally embedded subgroup; Weakly s-permutable subgroup; Solvable groups; Saturated formation; Finite groups.

1 Introduction

All groups considered in this paper are finite. We use conventional notions and notation. G always means a group, $|G|$ denotes the order of G and $\pi(G)$ denotes the set of all primes dividing $|G|$. Let $\mathscr{F}$ be a class of groups. We call $\mathscr{F}$ a formation, provided that (1) if $G \in \mathscr{F}$ and $H \trianglelefteq G$, then $G/H \in \mathscr{F}$, and (2) if G/M and G/N are in $\mathscr{F}$, then $G/(M \cap N)$ is in $\mathscr{F}$ for any normal subgroups M, N of G. A formation $\mathscr{F}$ is said to be saturated if $G/\Phi(G) \in \mathscr{F}$ implies that $G \in \mathscr{F}$. In this paper, $\mathscr{U}$ will denote the class of all supersolvable groups. Clearly, $\mathscr{U}$ is a saturated formation.

A subgroup H of G is called s-quasinormal (or s-permutable, π-quasinormal) in G provided H permutes with all Sylow subgroups of G, i.e, $HP = PH$ for any Sylow subgroup P of G. This concept was introduced by Kegel in [5] and has been studied extensively by Deskins [2] and Schmidt [12]. More recently, Ballester-Bolinches and Pedraza-Aquilera [1] generalized s-quasinormal subgroups to s-quasinormally embedded subgroups. A subgroup H is said to be s-quasinormally embedded in G if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-quasinormal subgroup of G. Clearly, every s-quasinormal subgroup of G is an s-quasinormally embedded subgroup of G, but the converse does not hold. Many authors consider minimal or maximal subgroups of a Sylow subgroup of a group when investigating the structure of G, such as in [1-2], [5-10] and [12-16], etc. For example, Li, Wang and Wei in [10] provide the following result: Let G be a group and P a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is s-quasinormally embedded in G, then G is p-nilpotent. Recently, Wei and Guo in [14] prove the following result: Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if there is a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-quasinormally embedded in G.

As another generalization of the normality, Skiba in [11] introduced the following concept: A subgroup H of G is said to be weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. Clearly, every s-permutable subgroup of G is an weakly s-permutable subgroup of G, but the converse does not hold. He provides the following result: Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that

$1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly s-permutable in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathscr{F}$.

The aim of this article is to unify and improve above Theorems using s-quasinormally embedded or weakly s-permutable subgroups. Our main theorem is the following result: Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-quasinormally embedded or weakly s-permutable in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathscr{F}$.

2 Basic definitions and preliminary results

In this section, we collect some known results that are useful later.

Lemma 1 *([1]) Suppose that U is s-quasinormally embedded in a group G, and let $H \leq G$ and $K \trianglelefteq G$. Then the following assertions hold.*

(i) If $U \leq H$, then U is s-quasinormally embedded in H;

(ii) UK is s-quasinormally embedded in G and UK/K is s-quasinormally embedded in G/K;

(iii) If $K \leq H$ and H/K is s-quasinormally embedded in G/K, then H is s-quasinormally embedded in G.

Lemma 2 *([11]) Let H be a weakly s-permutable subgroup of a group G.*

(i) If $H \leq K \leq G$, then H is weakly s-permutable in K;

(ii) If N is normal in G and $N \leq H \leq G$, then H/N is weakly s-permutable in G/N;

(iii) If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-permutable in G/N;

(iv) Suppose H is a p-group for some prime p and H is not s-permutable in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = HM$.

Lemma 3 *([13]) Let G be a group, K an s-quasinormal subgroup of G and P a Sylow p-subgroup of K, where p is a prime. If either $P \leq O_p(G)$ or $K_G = 1$, then P is s-quasinormal in G.*

Lemma 4 *([12]) If P is an s-quasinormal p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 5 *([13]) Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.*

(i) If N is normal in G of order p, then $N \leq Z(G)$;

(ii) If G has cyclic Sylow p-subgroup, then G is p-nilpotent;

(iii) If $M \leq G$ and $[G : M] = p$, then $M \trianglelefteq G$.

Lemma 6 *([10]) Let G be a group and P a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is s-quasinormally embedded in G, then G is p-nilpotent.*

Lemma 7 *([3, III, 5.2 and IV, 5.4]) Suppose that p is a prime and G is a minimal non-p-nilpotent group, i.e., G is not a p-nilpotent group but whose proper subgroups are all p-nilpotent.*

(i) G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.

(ii) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(iii) The exponent of P is p or 4.

Lemma 8 *([6]) Let H be a nilpotent subgroup of a group G. Then the following statements are equivalent:*

(i) H is s-quasinormal in G;

(ii) $H \leq F(G)$ and H is s-quasinormally embedded in G.

Lemma 9 *([14]) Let N be an elementary abelian normal p-subgroup of a group G. If there exists a subgroup D in N such that $1 < |D| < |N|$ and every subgroup H of N with $|H| = |D|$ is s-quasinormally embedded in G, then there exists a maximal subgroup M of N such that M is normal in G.*

Lemma 10 *([3, VI, 4.10]) Assume that A and B are two subgroups of a group G and $G \neq AB$. If $AB^g = B^gA$ holds for any $g \in G$, then either A or B is contained in a nontrivial normal subgroup of G.*

The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Its definition and important properties can be found in [4, X, 13]. We would like to give the following basic facts we will use in our proof.

Lemma 11 *([4, X,13]) Let G be a group and M a subgroup of G.*

(i) If M is normal in G, then $F^*(M) \leq F^*(G)$;

(ii) $F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$;

(iii) $F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.

Lemma 12 *([11]) Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly s-permutable in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathscr{F}$.*

3 Main results

Theorem 13 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If every maximal subgroup of P is either s-quasinormally embedded or weakly s-permutable in G. Then G is p-nilpotent.*

Proof. Assume that the theorem is not true and let G be a counterexample of minimal order. We derive a contradiction in several steps.

By Lemmas 1 and 2, the following two steps are obvious.

Step 1. $O_{p'}(G) = 1$.

Step 2. G has a unique minimal normal subgroup N and G/N is p-nilpotent. Moreover, $\Phi(G) = 1$.

Step 3. $O_p(G) = 1$.

If $O_p(G) \neq 1$, then step 2 yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, we conclude that $O_p(G) \cap M$ is normal in G. The uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Furthermore, $P \cap M < P$, and, thus there exists a maximal subgroup P_1 of P such that $P \cap M < P_1$. Hence, $P = NP_1$. By hypothesis, P_1 is s-quasinormally embedded or weakly s-permutable in G. Suppose first P_1 is s-quasinormally embedded in G. Then there is an s-quasinormal subgroup K of G such that $P_1 \in \mathrm{Syl}_p(K)$. If $K_G \neq 1$, then $N \leq K$. Since N is a normal p-subgroup of K and $P_1 \in \mathrm{Syl}_p(K)$, we have that $N \leq P_1$, a contradiction. Hence $K_G = 1$, and so by Lemma 3 P_1 is s-quasinormal in G. By Lemma 4, $O^p(G) \leq N_G(P_1)$, $P_1 \trianglelefteq G$. Then $N \cap P_1 = 1$ and $|N| = p$. By Lemma 5, $N \leq Z(G)$ and hence G is p-nilpotent, a contradiction. Therefore, we may assume that P_1 is weakly s-permutable in G. Then there is a subnormal subgroup T of G such that $G = P_1T$ and

$$P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = N \leq O^p(G)$$

because N is the unique minimal normal subgroup of G. Since $|G : T|$ is a power of p, $O^p(G) \leq T$. Hence,

$$P_1 \cap T \leq (P_1)_{sG} \leq O^p(G) \cap P_1 \leq T \cap P_1,$$

and so

$$P_1 \cap T = (P_1)_{sG} = O^p(G) \cap P_1.$$

Consequently, $G = PO^p(G)$ implies that $(P_1)_{sG}$ is normal in G by Lemma 4. By the minimality of N, we have $(P_1)_{sG} = N$ or $(P_1)_{sG} = 1$. If $(P_1)_{sG} = N$, then $N \leq P_1$ and $P = NP_1 = P_1$, a contradiction. Thus $P_1 \cap T = (P_1)_{sG} = 1$, and so $|T|_p = p$. Then T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is subnormal in G and $T_{p'}$ is a p'-Hall subgroup of G. It follows that $T_{p'}$ is the normal p-complement of G, a contradiction.

Step 4. The final contradiction.

If P has a maximal subgroup P_1 which is weakly s-permutable in G, then there is a subnormal subgroup T of G such that $G = P_1T$ and

$$P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = 1.$$

Then $P_1 \cap T = 1$. Hence $|T|_p = p$. Therefore, T is p-nilpotent. Thus G is p-nilpotent, a contradiction. Now we may assume that all maximal subgroups of P are s-quasinormally embedded in G. Then G is p-nilpotent by Lemma 6, a contradiction. □

The following corollaries is immediate from Theorem 13.

Corollary 14 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If every maximal subgroup of P is s-quasinormally embedded in G. Then G is p-nilpotent.*

Corollary 15 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If every maximal subgroup of P is weakly s-permutable in G. Then G is p-nilpotent.*

Corollary 16 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If every maximal subgroup of P is s-permutable in G. Then G is p-nilpotent.*

Corollary 17 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If every maximal subgroup of P is permutable in G. Then G is p-nilpotent.*

Corollary 18 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If every maximal subgroup of P is normal in G. Then G is p-nilpotent.*

Theorem 19 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-quasinormally embedded or weakly s-permutable in G. Then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

Step 1. $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, Lemma 1 (ii) and Lemma 2 (iii) guarantee that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

Step 2. $|D| > p$.

Suppose that $|D| = p$. Since G is not p-nilpotent, G has a minimal non-p-nilpotent subgroup G_1. By Lemma 7 (i), $G_1 = [P_1]Q$, where $P_1 \in \mathrm{Syl}_p(G_1)$ and $Q \in \mathrm{Syl}_q(G_1)$, $p \neq q$. Let $x \in P_1$ and $L = \langle x \rangle$. Then L is of order p or 4 by Lemma 7 (iii). By the hypotheses, L is either s-quasinormally embedded or weakly s-permutable in G, thus in G_1 by Lemma 1 (i) and 2 (i). First, suppose that L is weakly s-permutable in G_1. Then there is a subnormal subgroup T of G_1 such that $G_1 = LT$ and $L \cap T \leq L_{sG_1}$. Hence $P_1 = P_1 \cap G_1 = P_1 \cap LT = L(P_1 \cap T)$. Since $P_1/\Phi(P_1)$ is abelian, we have $(P_1 \cap T)\Phi(P_1)/\Phi(P_1)$ is normal in $G_1/\Phi(P_1)$. Since $P_1/\Phi(P_1)$ is the minimal normal subgroup of $G_1/\Phi(P_1)$, we have that $P_1 \cap T \leq \Phi(P_1)$ or $P_1 = (P_1 \cap T)\Phi(P_1) = P_1 \cap T$. If $P_1 \cap T \leq \Phi(P_1)$, then $L = P_1$ is normal in G_1. It follows that G_1 is p-nilpotent, a contraction. If $P_1 = P_1 \cap T$, then $T = G_1$ and so $L = L_{sG_1}$ is s-permutable in G_1. For any element x in P_1, now we have $\langle x \rangle Q$ is a proper subgroup of G_1, then $\langle x \rangle Q = \langle x \rangle \times Q$. This implies that $G_1 = P_1 \times Q$, a contradiction. Therefore, $L = \langle x \rangle$ is s-quasinormally embedded in G_1 for every element $x \in P_1$, then by Lemma 8 $\langle x \rangle$ is s-quasinormal in G_1. Thus $LQ \leq G_1$. Therefore, $LQ = L \times Q$. Then $G_1 = P_1 \times Q$, a contradiction.

Step 3. $|P : D| > p$.

By Theorem 13.

Step 4. P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-quasinormally embedded in G. Assume that $H \leq P$ such that $|H| = |D|$ and H is weakly s-permutable in G. Then there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap K \leq H_{sG}$. By Lemma 2 (iv), we may assume G has a normal subgroup M such that $|G : M| = p$ and $G = HM$. Since $|P : D| > p$ by Step 3, M satisfies the hypotheses of the theorem. The choice of G yields that M is p-nilpotent. It is easy to see that G is p-nilpotent, contrary to the choice of G.

Step 5. If $N \leq P$ and N is minimal normal in G, then $|N| \leq |D|$.

Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, N is elementary abelian. By Lemma 9, N has a maximal subgroup which is normal in G, contrary to the minimality of N.

Step 6. Suppose that $N \leq P$ and N is minimal normal in G. Then G/N is p-nilpotent.

If $|N| < |D|$, G/N satisfies the hypotheses of the theorem by Lemma 1 (ii). Thus G/N is p-nilpotent by the minimal choice of G. So we may suppose that $|N| = |D|$ by Step 5. We will show that every cyclic subgroup of P/N of order p or order 4 (when P/N is a non-abelian 2-group) is s-quasinormally embedded in G/N. Let $K \leq P$ and $|K/N| = p$. By Step 2, N is non-cyclic, so are all subgroups containing N. Hence there is a maximal subgroup $L \neq N$ of K such that $K = NL$. Of course, $|N| = |D| = |L|$. Since L is s-quasinormally embedded in G by the hypotheses, $K/N = LN/N$ is s-quasinormally embedded in G/N by Lemma 1 (ii). If $p = 2$ and P/N is non-abelian, take a cyclic subgroup X/N of P/N of order 4. Let K/N be maximal in X/N. Then K is maximal in X and $|K/N|$=2. Since X is non-cyclic and X/N is cyclic, there is a maximal subgroup L of X such that N is not contained in L. Thus $X = LN$ and $|L| = |K| = 2|D|$. By the hypotheses, L is s-quasinormally embedded in G. By Lemma 1 (ii), $X/N = LN/N$ is s-quasinormally embedded in G/N. Hence G/N satisfies the hypotheses. By the minimal choice of G, G/N is p-nilpotent.

Step 7. $O_p(G) = 1$.

Suppose that $O_p(G) \neq 1$. Take a minimal normal subgroup N of G contained in $O_p(G)$. By Step 6, G/N is p-nilpotent. It is easy to see that N is the unique minimal normal subgroup of G contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$.

Hence $O_p(G)$ is an elementary abelian p-group. On the other hand, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. It is easy to deduce that $O_p(G) \cap M = 1$, $N = O_p(G)$ and $M \cong G/N$ is p-nilpotent. Then G can be written as $G = N(M \cap P)M_{p'}$, where $M_{p'}$ is the normal p-complement of M. Pick a maximal subgroup S of $M_p = P \cap M$. Then $NSM_{p'}$ is a subgroup of G with index p. Since p is the minimal prime in $\pi(G)$, we know that $NSM_{p'}$ is normal in G. Now by Step 3 and the induction, we have $NSM_{p'}$ is p-nilpotent. Therefore, G is p-nilpotent, a contradiction.

Step 8. The minimal normal subgroup L of G is not p-nilpotent.

If L is p-nilpotent, then it follows from the fact that $L_{p'}$ char $L \lhd G$ that $L_{p'} \leq O_{p'}(G) = 1$. Thus L is a p-group. Whence $L \leq O_p(G) = 1$ by Step 7, a contradiction.

Step 9. G is a non-abelian simple group.

Suppose that G is not a simple group. Take a minimal normal subgroup L of G. Then $L < G$. If $|L|_p > |D|$, then L is p-nilpotent by the minimal choice of G, contrary to Step 8. If $|L|_p \leq |D|$. Take $P_* \geq L \cap P$ such that $|P_*| = p|D|$. Hence P_* is a Sylow p-subgroup of P_*L. Since every maximal subgroup of P_* is of order $|D|$, every maximal subgroup of P_* is s-quasinormally embedded in G by hypotheses, thus in P_*L by Lemma 1 (i). Now applying Theorem 13, we get P_*L is p-nilpotent. Therefore, L is p-nilpotent, contrary to Step 8.

Step 10. The final contradiction.

Suppose that H is a subgroup of P with $|H| = |D|$ and Q is a Sylow q-subgroup with $q \neq p$. Then $HQ^g = Q^gH$ for any $g \in G$ by the hypotheses that H is s-quasinormally embedded in G and Lemma 8. Since G is simple by Step 9, $G = HQ$ from Lemma 10, the final contradiction. □

The following corollaries is immediate from Theorem 19.

Corollary 20 *Suppose that G is a group. If every non-cyclic Sylow subgroup of G has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-quasinormally embedded or weakly s-permutable in G, then G has a Sylow tower of supersolvable type.*

Corollary 21 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-quasinormally embedded in G. Then G is p-nilpotent.*

Corollary 22 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly s-permutable in G. Then G is p-nilpotent.*

Corollary 23 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-permutable in G. Then G is p-nilpotent.*

Corollary 24 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is permutable in G. Then G is p-nilpotent.*

Corollary 25 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is normal in G. Then G is p-nilpotent.*

Corollary 26 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. Suppose that every cyclic subgroup of P of prime order or order 4 is either s-quasinormally embedded or weakly s-permutable in G. Then G is p-nilpotent.*

Corollary 27 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. Suppose that every cyclic subgroup of P of prime order or order 4 is s-quasinormally embedded in G. Then G is p-nilpotent.*

Corollary 28 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. Suppose that every cyclic subgroup of P of prime order or order 4 is weakly s-permutable in G. Then G is p-nilpotent.*

Corollary 29 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. Suppose that every cyclic subgroup of P of prime order or order 4 is s-permutable in G. Then G is p-nilpotent.*

Corollary 30 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. Suppose that every cyclic subgroup of P of prime order or order 4 is permutable in G. Then G is p-nilpotent.*

Corollary 31 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. Suppose that every cyclic subgroup of P of prime order or order 4 is normal in G. Then G is p-nilpotent.*

Theorem 32 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-quasinormally embedded or weakly s-permutable in G. Then $G \in \mathscr{F}$.*

Proof. Suppose that P is a non-cyclic Sylow p-subgroup of E, $\forall p \in \pi(E)$. Since P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-quasinormally embedded or weakly s-permutable in G by hypotheses, thus in E by Lemma 1 (i). Applying Corollary 20, we conclude that E has a Sylow tower of supersolvable type. Let q be the maximal prime divisor of $|E|$ and $Q \in \mathrm{Syl}_q(E)$. Then $Q \trianglelefteq G$. Since $(G/Q, E/Q)$ satisfies the hypotheses of the theorem, by induction, $G/Q \in \mathscr{F}$. For any subgroup H of Q with $|H| = |D|$, since $Q \leq O_q(G)$, H is either s-quasinormal or weakly s-permutable in G by Lemma 8. Since s-quasinormal implies weakly s-permutable and $F^*(Q) = Q$ by Lemma 11, we get $G \in \mathscr{F}$ by applying Lemma 12. □

The following corollaries is immediate from Theorem 32.

Corollary 33 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-quasinormally embedded in G. Then $G \in \mathscr{F}$.*

Corollary 34 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly s-permutable in G. Then $G \in \mathscr{F}$.*

Corollary 35 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-permutable in G. Then $G \in \mathscr{F}$.*

Corollary 36 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is permutable in G. Then $G \in \mathscr{F}$.*

Corollary 37 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is normal in G. Then $G \in \mathscr{F}$.*

Corollary 38 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of E is either s-quasinormally embedded or weakly s-permutable in G. Then $G \in \mathscr{F}$.*

Corollary 39 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every cyclic subgroup of any non-cyclic Sylow subgroup of E of prime order or order 4 is either s-quasinormally embedded or weakly s-permutable in G. Then $G \in \mathscr{F}$.*

Corollary 40 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of E is s-quasinormally embedded in G. Then $G \in \mathscr{F}$.*

Corollary 41 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal* *subgroup E such that $G/E \in \mathscr{F}$. Suppose that every cyclic subgroup of any non-cyclic Sylow subgroup of E of prime order or order 4 is s-quasinormally embedded in G. Then $G \in \mathscr{F}$.*

Corollary 42 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of E is weakly s-permutable in G. Then $G \in \mathscr{F}$.*

Corollary 43 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every cyclic subgroup of any non-cyclic Sylow subgroup of E of prime order or order 4 is weakly s-permutable in G. Then $G \in \mathscr{F}$.*

Corollary 44 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of E is s-quasinormal in G. Then $G \in \mathscr{F}$.*

Corollary 45 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every cyclic subgroup of any non-cyclic Sylow subgroup of E of prime order or order 4 is s-quasinormal in G. Then $G \in \mathscr{F}$.*

Corollary 46 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of E is quasinormal in G. Then $G \in \mathscr{F}$.*

Corollary 47 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every cyclic subgroup of any non-cyclic Sylow subgroup of E of prime order or order 4 is quasinormal in G. Then $G \in \mathscr{F}$.*

Corollary 48 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of E is normal in G. Then $G \in \mathscr{F}$.*

Corollary 49 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every cyclic subgroup of any non-cyclic Sylow subgroup of E of prime order or order 4 is normal in G. Then $G \in \mathscr{F}$.*

Theorem 50 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-quasinormally embedded or weakly s-permutable in G. Then $G \in \mathscr{F}$.*

Proof. We distinguish two cases:

Case 1. $\mathscr{F} = \mathscr{U}$.

Let G be a minimal counter-example.

Step 1. Every proper normal subgroup N of G containing $F^*(E)$ (if it exists) is supersolvable.

If N is a proper normal subgroup of G containing $F^*(E)$, then $N/N \cap E \cong NE/E$ is supersolvable. By Lemma 11 (iii), $F^*(E) = F^*(F^*(E)) \leq F^*(E \cap N) \leq F^*(E)$, so $F^*(E \cap N) = F^*(E)$. For any Sylow subgroup P of $F^*(E \cap N) = F^*(E)$, P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-quasinormally embedded or weakly s-permutable in G by hypotheses, thus in N by Lemma 1 (i) and Lemma 2 (i). So N and $N \cap H$ satisfy the hypotheses of the theorem, the minimal choice of G implies that N is supersolvable.

Step 2. $E = G$.

If $E < G$, then $E \in \mathscr{U}$ by Step 1. Hence $F^*(E) = F(E)$ by Lemma 11. It follows that every Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 8, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-quasinormal or weakly s-permutable in G. Applying Lemma 12 for the special case $\mathscr{F} = \mathscr{U}$, $G \in \mathscr{U}$, a contradiction.

Step 3. $F^*(G) = F(G) < G$.

If $F^*(G) = G$, then $G \in \mathscr{F}$ by Theorem 32, contrary to the choice of G. So $F^*(G) < G$. By Step 1, $F^*(G) \in \mathscr{U}$ and $F^*(G) = F(G)$ by Lemma 11.

Step 4. The final contradiction.

Since $F^*(G) = F(G)$, each non-cyclic Sylow subgroup of $F^*(G)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-quasinormal or weakly s-permutable in G by Lemma 8. Applying Lemma 12, $G \in \mathscr{U}$, a contradiction.

Case 2. $\mathscr{F} \neq \mathscr{U}$.

By hypotheses, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-quasinormally embedded or weakly s-permutable in G, thus in E Lemma 1 (i) and Lemma 2 (i). Applying Case 1, $E \in \mathscr{U}$. Then $F^*(E) = F(E)$ by Lemma 11. It follows that each Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 8, each non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-quasinormal or weakly s-permutable in G. Applying Lemma 12, $G \in \mathscr{F}$. These complete the proof of the theorem. □

The following corollaries are immediate from Theorem 50.

Corollary 51 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-quasinormally embedded in G. Then $G \in \mathscr{F}$.*

Corollary 52 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly s-permutable in G. Then $G \in \mathscr{F}$.*

Corollary 53 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-permutable in G. Then $G \in \mathscr{F}$.*

Corollary 54 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is permutable in G. Then $G \in \mathscr{F}$.*

Corollary 55 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is normal in G. Then $G \in \mathscr{F}$.*

Corollary 56 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is either s-quasinormally embedded or weakly s-permutable in G.*

Corollary 57 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is either s-quasinormally embedded or weakly s-permutable in G.*

Corollary 58 *([9, Theorem 1.1]) Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is s-quasinormally embedded in G.*

Corollary 59 *([9, Theorem 1.2]) Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is s-quasinormally embedded in G.*

Corollary 60 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is weakly s-permutable in G.*

Corollary 61 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is weakly s-permutable in G.*

Corollary 62 *([7, Theorem 3.4]) Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is s-quasinormal in G.*

Corollary 63 *([8, Theorem 3.3]) Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is s-quasinormal in G.*

Corollary 64 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is quasinormal in G.*

Corollary 65 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is quasinormal in G.*

Corollary 66 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is normal in G.*

Corollary 67 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is normal in G.*

4 Conclusion

The results explained in the previous sections show that the method that we replace conditions for all maximal subgroups or all minimal subgroups of Sylow subgroups of G by conditions referring to only some subgroups of Sylow subgroups of G in order to investigate the structure of a finite group is very useful. Results of this type are interesting. In addition, there are many other generalizations of the normality, for example, SS-quasinormal subgroups in [6]; c^*-normality in [13]; X-semipermutable subgroups in [17]; c-supplemented subgroups in [18]. As an application, we may consider using the above special subgroups to characterize the structure of finite groups.

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