# New Generalized Ostrowski-Grüss Type Inequalities In Two Independent Variables On Time Scales

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*Abstract:* Some new generalized Ostrowski-Grüss type integral inequalities in two independent variables on time scales are established in this paper. The established results generalize some known results in the literature, and unify corresponding continuous and discrete analysis. New bounds for the related Ostrowski-Grüss type inequalities are derived, and some of these bounds are sharp.

*Key–Words:* Ostrowski type inequality, Grüss type inequality, Time scales, Bounds, Sharp inequalities

## 1 Introduction

Recently, the research for the Ostrowski type and Grüss type inequalities has been paid much attention by many authors. The Ostrowski type inequality, which was originally presented by Ostrowski in [1], can be used to estimate the absolute deviation of a function from its integral mean, while the Grüss inequality [2] can be used to estimate the absolute deviation of the integral of the product of two functions from the product of their respective integrals. Among the research for the Ostrowski type and Grüss type inequalities, generalizations of the two inequalities have been a hot topic, and, in the last few decades, various generalizations of the Ostrowski inequality and the Grüss inequality have been established (for example, see [3-13] and the references therein), while some new inequalities are established, one of which is the inequalities of Ostrowski-Grüss type (for example, see [14-26]). The first Ostrowski-Grüss type inequality was presented by Dragomir and Wang in [14], which got improved in [15-18, 21-23]. In [25], Lü extended UJEVIĆ's results [17] to 2D case. In [26], Liu extended the inequality above to a more general form. The most important application for these inequalities mentioned above lies in that they can be used to provide explicit error bounds for some known and some new numerical quadrature formulae, and furthermore can provide sharp bounds related to these inequalities. So establishing new Ostrowski type and Grüss type inequalities is a purposeful work in estimating new error bounds for numerical quadrature formulae.

On the other hand, Hilger [27] initiated the theory of time scales as a theory capable of treating continuous and discrete analysis in a consistent way, based on which some authors have studied the Ostrowski type and Grüss type inequalities on time scales (see [28-38]). But we notice that Ostrowski-Grüss type inequalities in two independent variables on time scales have been paid little attention in the literature.

Motivated by the above works, in this paper, we establish some new Ostrowski-Grüss type inequalities in two independent variables on time scales with more generalized forms than those existing inequalities in the literature. New bounds related to the Ostrowski-Grüss type inequalities are derived, and some of them are sharp. The established results unify continuous and discrete analysis, and extend some known results in the literature.

Throughout this paper, $\mathbf{R}$ denotes the set of real numbers, while $\mathbf{Z}$ denotes the set of integers, and $\mathbf{N}_0$ denotes the set of nonnegative integers. $\mathbf{T}_1,\ \mathbf{T}_2$ denote two arbitrary time scales, and for an interval $[a,b]$, $[a,b]_{\mathbf{T}_i} := [a,b]\bigcap \mathbf{T}_i,\ i=1,2$. For the sake of convenience, we denote the forward jump operators on $\mathbf{T}_1,\ \mathbf{T}_2$ by $\sigma$ uniformly. Finally, a point $t \in \mathbf{T}_i$ is said to be right-dense if $\sigma(t)=t$ and $t \neq \sup \mathbf{T}_i$.

For more details about the calculus of time scales, we refer the reader to [39-40].

## 2 Main results

**Lemma 1** *(Generalized Montgomery Identity). Let* $a,\ b,\ s,\ x \in \mathbf{T}_1,\ c,\ d,\ t,\ y \in \mathbf{T}_2$ *with* $a<b,\ c<d$. $f : [a,b]_{\mathbf{T}_1} \times [c,d]_{\mathbf{T}_2} \to \mathbf{R}$ *is* $\Delta_1\Delta_2$ *differentiable.* $\xi \in [0,1]$ *such that* $a+\xi\frac{b-a}{2},\ b-\xi\frac{b-a}{2} \in$

$\mathbf{T}_1$, $c+\xi\frac{d-c}{2}$, $d-\xi\frac{d-c}{2}\in\mathbf{T}_2$ *and* $x\in[a+\xi\frac{b-a}{2}, b-\xi\frac{b-a}{2}]_{\mathbf{T}_1}$, $y\in[c+\xi\frac{d-c}{2}, d-\xi\frac{d-c}{2}]_{\mathbf{T}_2}$. *Then*

$$\begin{aligned}&(1-\xi)^2 f(x,y)\\&+(1-\xi)\frac{\xi}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)]\\&+\frac{\xi^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)]-\\&\frac{1-\xi}{b-a}\int_a^b f(\sigma(s),y)\Delta_1 s-\frac{1-\xi}{d-c}\int_c^d f(x,\sigma(t))\Delta_2 t\\&-\frac{\xi}{2}\frac{1}{b-a}\int_a^b[f(\sigma(s),c)+f(\sigma(s),d)]\Delta_1 s\\&-\frac{\xi}{2}\frac{1}{d-c}\int_c^d[f(a,\sigma(t))+f(b,\sigma(t))]\Delta_2 t\\&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\\&=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d K(x,y,s,t)\times\\&\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s,\end{aligned}$$

*where* $K(x,y,s,t)=K_1(x,s)K_2(y,t)$, *and*

$$K_1(x,s)=\begin{cases}s-(a+\xi\frac{b-a}{2}), & s\in[a,x)_{\mathbf{T}_1}\\ s-(b-\xi\frac{b-a}{2}), & s\in[x,b]_{\mathbf{T}_1}\end{cases}$$

$$K_2(y,t)=\begin{cases}t-(c+\xi\frac{d-c}{2}), & t\in[c,y)_{\mathbf{T}_2}\\ t-(d-\xi\frac{d-c}{2}), & t\in[y,d]_{\mathbf{T}_2}\end{cases}\qquad(1)$$

**Proof:** We have the following observations:

$$\begin{aligned}&\int_a^b\int_c^d K(x,y,s,t)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s=\\&\int_a^x\int_c^y[s-(a+\xi\frac{b-a}{2})]\times\\&[t-(c+\xi\frac{d-c}{2})]\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\\&+\int_a^x\int_y^d[s-(a+\xi\frac{b-a}{2})]\times\\&[t-(d-\xi\frac{d-c}{2})]\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\\&+\int_x^b\int_c^y[s-(b-\xi\frac{b-a}{2})]\times\\&[t-(c+\xi\frac{d-c}{2})]\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\\&+\int_x^b\int_y^d[s-(b-\xi\frac{b-a}{2})]\times\\&[t-(d-\xi\frac{d-c}{2})]\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\end{aligned}$$

$$\begin{aligned}&=\int_a^x[s-(a+\xi\frac{b-a}{2})]\{[y-(c+\xi\frac{d-c}{2})]\frac{\partial f(s,y)}{\Delta_1 s}\\&-[c-(c+\xi\frac{d-c}{2})]\frac{\partial f(s,c)}{\Delta_1 s}\}\Delta_1 s\\&-\int_a^x\int_c^y[s-(a+\xi\frac{b-a}{2})]\frac{\partial f(s,\sigma(t))}{\Delta_1 s}\Delta_2 t\Delta_1 s\\&+\int_a^x[s-(a+\xi\frac{b-a}{2})]\{[d-(d-\xi\frac{d-c}{2})]\frac{\partial f(s,d)}{\Delta_1 s}\\&-[y-(d-\xi\frac{d-c}{2})]\frac{\partial f(s,y)}{\Delta_1 s}\}\Delta_1 s\\&-\int_a^x\int_y^d[s-(a+\xi\frac{b-a}{2})]\frac{\partial f(s,\sigma(t))}{\Delta_1 s}\Delta_2 t\Delta_1 s\\&+\int_x^b[s-(b-\xi\frac{b-a}{2})]\{[y-(c+\xi\frac{d-c}{2})]\frac{\partial f(s,y)}{\Delta_1 s}\\&-[c-(c+\xi\frac{d-c}{2})]\frac{\partial f(s,c)}{\Delta_1 s}\}\Delta_1 s\\&-\int_x^b\int_c^y[s-(b-\xi\frac{b-a}{2})]\frac{\partial f(s,\sigma(t))}{\Delta_1 s}\Delta_2 t\Delta_1 s\\&+\int_x^b[s-(b-\xi\frac{b-a}{2})]\{[d-(d-\xi\frac{d-c}{2})]\frac{\partial f(s,d)}{\Delta_1 s}\\&-[y-(d-\xi\frac{d-c}{2})]\frac{\partial f(s,y)}{\Delta_1 s}\}\Delta_1 s\\&-\int_x^b\int_y^d[s-(b-\xi\frac{b-a}{2})]\frac{\partial f(s,\sigma(t))}{\Delta_1 s}\Delta_2 t\Delta_1 s\\&=[y-(c+\xi\frac{d-c}{2})]\{[x-(a+\xi\frac{b-a}{2})]f(x,y)\\&-[a-(a+\xi\frac{b-a}{2})]f(a,y)-\int_a^x f(\sigma(s),y)\Delta_1 s\}\\&-[c-(c+\xi\frac{d-c}{2})]\{[x-(a+\xi\frac{b-a}{2})]f(x,c)\\&-[a-(a+\xi\frac{b-a}{2})]f(a,c)-\int_a^x f(\sigma(s),c)\Delta_1 s\}\\&-\int_c^y\{[x-(a+\xi\frac{b-a}{2})]f(x,\sigma(t))\\&-[a-(a+\xi\frac{b-a}{2})]f(a,\sigma(t))\}\Delta_2 t\\&+\int_a^x\int_c^y f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\\&+[d-(d-\xi\frac{d-c}{2})]\{[x-(a+\xi\frac{b-a}{2})]f(x,d)\\&-[a-(a+\xi\frac{b-a}{2})]f(a,d)-\int_a^x f(\sigma(s),d)\Delta_1 s\}\\&-[y-(d-\xi\frac{d-c}{2})]\{[x-(a+\xi\frac{b-a}{2})]f(x,y)\\&-[a-(a+\xi\frac{b-a}{2})]f(a,y)-\int_a^x f(\sigma(s),y)\Delta_1 s\}\\&-\int_y^d\{[x-(a+\xi\frac{b-a}{2})]f(x,\sigma(t))\end{aligned}$$

$$-[a-(a+\xi\frac{b-a}{2})]f(a,\sigma(t))\}\Delta_2 t$$
$$+\int_a^x\int_y^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s$$
$$+[y-(c+\xi\frac{d-c}{2})]\{[b-(b-\xi\frac{b-a}{2})]f(b,y)$$
$$-[x-(b-\xi\frac{b-a}{2})]f(x,y)-\int_x^b f(\sigma(s),y)\Delta_1 s\}$$
$$-[c-(c+\xi\frac{d-c}{2})]\{[b-(b-\xi\frac{b-a}{2})]f(b,c)$$
$$-[x-(b-\xi\frac{b-a}{2})]f(x,c)-\int_x^b f(\sigma(s),c)\Delta_1 s\}$$
$$-\int_c^y\{[b-(b-\xi\frac{b-a}{2})]f(b,\sigma(t))$$
$$-[x-(b-\xi\frac{b-a}{2})]f(x,\sigma(t))\}\Delta_2 t$$
$$+\int_x^b\int_c^y f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s$$
$$+[d-(d-\xi\frac{d-c}{2})]\{[b-(b-\xi\frac{b-a}{2})]f(b,d)$$
$$-[x-(b-\xi\frac{b-a}{2})]f(x,d)-\int_x^b f(\sigma(s),d)\Delta_1 s\}$$
$$-[y-(d-\xi\frac{d-c}{2})]\{[b-(b-\xi\frac{b-a}{2})]f(b,y)$$
$$-[x-(b-\xi\frac{b-a}{2})]f(x,y)-\int_x^b f(\sigma(s),y)\Delta_1 s\}$$
$$-\int_y^d\{[b-(b-\xi\frac{b-a}{2})]f(b,\sigma(t))$$
$$-[x-(b-\xi\frac{b-a}{2})]f(x,\sigma(t))\}\Delta_2 t$$
$$+\int_x^b\int_y^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s$$
$$=(1-\xi)^2(b-a)(d-c)f(x,y)$$
$$+(b-a)(d-c)(1-\xi)\times$$
$$\frac{\xi}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)]$$
$$+(b-a)(d-c)\frac{\xi^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)]$$
$$-(d-c)(1-\xi)\int_a^b f(\sigma(s),y)\Delta_1 s$$
$$-(b-a)(1-\xi)\int_c^d f(x,\sigma(t))\Delta_2 t$$
$$\frac{\xi}{2}(d-c)\int_a^b[f(\sigma(s),c)+f(\sigma(s),d)]\Delta_1 s$$
$$-\frac{\xi}{2}(b-a)\int_c^d[f(a,\sigma(t))+f(b,\sigma(t))]\Delta_2 t$$
$$+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s,$$

that is the desired result. □

**Theorem 2** *Under the conditions of Lemma 1, if* $f^{\Delta_1\Delta_2}\in L_2((a,b)_{\mathbf{T}_1}\times(c,d)_{\mathbf{T}_2})$*, then we have*

$$|(1-\xi)^2 f(x,y)+$$
$$(1-\xi)\frac{\xi}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)]$$
$$+\frac{\xi^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)]$$
$$-\frac{1}{b-a}(1-\xi)\int_a^b f(\sigma(s),y)\Delta_1 s$$
$$-\frac{1}{d-c}(1-\xi)\int_c^d f(x,\sigma(t))\Delta_2 t$$
$$-\frac{\xi}{2}\frac{1}{b-a}\int_a^b[f(\sigma(s),c)+f(\sigma(s),d)]\Delta_1 s$$
$$-\frac{\xi}{2}\frac{1}{d-c}\int_c^d[f(a,\sigma(t))+f(b,\sigma(t))]\Delta_2 t$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s$$
$$-\frac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)^2(d-c)^2}\times$$
$$[h_2(x,a+\xi\frac{b-a}{2})-h_2(a,a+\xi\frac{b-a}{2})$$
$$+h_2(b,b-\xi\frac{b-a}{2})-h_2(x,b-\xi\frac{b-a}{2})]\times$$
$$[h_2(y,c+\xi\frac{d-c}{2})-h_2(c,c+\xi\frac{d-c}{2})$$
$$+h_2(d,d-\xi\frac{d-c}{2})-h_2(y,d-\xi\frac{d-c}{2})]|$$
$$\le\{[\frac{b^3-a^3}{3}-2(a+\xi\frac{b-a}{2})(h_2(x,a+\xi\frac{b-a}{2})$$
$$-h_2(a,a+\xi\frac{b-a}{2}))-(a+\xi\frac{b-a}{2})^2(x-a)$$
$$-2(b-\xi\frac{b-a}{2})(h_2(b,b-\xi\frac{b-a}{2})$$
$$-h_2(x,b-\xi\frac{b-a}{2}))-(b-\xi\frac{b-a}{2})^2(b-x)]\times$$
$$[\frac{d^3-c^3}{3}-2(c+\xi\frac{d-c}{2})(h_2(y,c+\xi\frac{d-c}{2})$$
$$-h_2(c,c+\xi\frac{d-c}{2}))-(c+\xi\frac{d-c}{2})^2(y-c)$$
$$-2(d-\xi\frac{d-c}{2})(h_2(d,d-\xi\frac{d-c}{2})$$
$$-h_2(y,d-\xi\frac{d-c}{2}))-(d-\xi\frac{d-c}{2})^2(d-y)]$$
$$-\frac{1}{(b-a)(d-c)}[(h_2(x,a+\xi\frac{b-a}{2})$$
$$-h_2(a,a+\xi\frac{b-a}{2})$$

$$+h_2(b,b-\xi\frac{b-a}{2})-h_2(x,b-\xi\frac{b-a}{2}))\times$$
$$(h_2(y,c+\xi\frac{d-c}{2})-h_2(c,c+\xi\frac{d-c}{2})$$
$$+h_2(d,d-\xi\frac{d-c}{2})$$
$$-h_2(y,d-\xi\frac{d-c}{2}))]^2\}^{\frac{1}{2}}\sqrt{T(f^{\Delta_1\Delta_2})}, \quad (2)$$

*where*

$$T(f) = \int_a^b\int_c^d f^2(s,t)\Delta_2t\Delta_1s - \frac{1}{(b-a)(d-c)}(\int_a^b\int_c^d f(s,t)\Delta_2t\Delta_1s)^2.$$

*The inequality (2) is sharp in the sense that the coefficient constant 1 of the right-hand side of it can not be replaced by a smaller one.*

**Proof:** From the definition of $K(x,y,s,t)$ we obtain

$$\int_a^b\int_c^d K(x,y,s,t)\Delta_2t\Delta_1s =$$
$$\int_a^b K_1(x,s)\Delta_1s\int_c^d K_2(y,t)\Delta_2t =$$
$$\{\int_a^x[s-(a+\xi\frac{b-a}{2})]\Delta_1s+\int_x^b[s-(b-\xi\frac{b-a}{2})]\Delta_1s\}$$
$$\times\{\int_c^y[t-(c+\xi\frac{d-c}{2})]\Delta_2t+\int_y^d[t-(d-\xi\frac{d-c}{2})]\Delta_2t\}$$
$$=[h_2(x,a+\xi\frac{b-a}{2})-h_2(a,a+\xi\frac{b-a}{2})$$
$$+h_2(b,b-\xi\frac{b-a}{2})-h_2(x,b-\xi\frac{b-a}{2})]\times$$
$$[h_2(y,c+\xi\frac{d-c}{2})-h_2(c,c+\xi\frac{d-c}{2})$$
$$+h_2(d,d-\xi\frac{d-c}{2})-h_2(y,d-\xi\frac{d-c}{2})], \quad (3)$$

and

$$\int_a^b\int_c^d K^2(x,y,s,t)\Delta_2t\Delta_1s$$
$$=\int_a^b K_1^2(x,s)\Delta_1s\int_c^d K_2^2(y,t)\Delta_2t$$
$$=\{\int_a^x[s-(a+\xi\frac{b-a}{2})]^2\Delta_1s$$
$$+\int_x^b[s-(b-\xi\frac{b-a}{2})]^2\Delta_1s\}$$
$$\times\{\int_c^y[t-(c+\xi\frac{d-c}{2})]^2\Delta_2t$$
$$+\int_y^d[t-(d-\xi\frac{d-c}{2})]^2\Delta_2t\}$$
$$=\{\int_a^x[s^2-2(a+\xi\frac{b-a}{2})(s-(a+\xi\frac{b-a}{2}))$$
$$-(a+\xi\frac{b-a}{2})^2]\Delta_1s$$
$$+\int_x^b[s^2-2(b-\xi\frac{b-a}{2})(s-(b-\xi\frac{b-a}{2}))$$
$$-(b-\xi\frac{b-a}{2})^2]\Delta_1s\}$$
$$\times\{\int_c^y[t^2-2(c+\xi\frac{d-c}{2})(t-(c+\xi\frac{d-c}{2}))$$
$$-(c+\xi\frac{d-c}{2})^2]\Delta_2t$$
$$+\int_y^d[t^2-2(d-\xi\frac{d-c}{2})(t-(d-\xi\frac{d-c}{2}))$$
$$-(b-\xi\frac{d-c}{2})^2]\Delta_2t\}$$
$$\le\{\int_a^x[\frac{s^2+s\sigma(s)+(\sigma(s))^2}{3}$$
$$-2(a+\xi\frac{b-a}{2})(s-(a+\xi\frac{b-a}{2}))$$
$$-(a+\xi\frac{b-a}{2})^2]\Delta_1s+\int_x^b[\frac{s^2+s\sigma(s)+(\sigma(s))^2}{3}$$
$$-2(b-\xi\frac{b-a}{2})(s-(b-\xi\frac{b-a}{2}))$$
$$-(b-\xi\frac{b-a}{2})^2]\Delta_1s\}\{\int_c^y[\frac{t^2+t\sigma(t)+(\sigma(t))^2}{3}$$
$$-2(c+\xi\frac{d-c}{2})(t-(c+\xi\frac{d-c}{2}))$$
$$-(c+\xi\frac{d-c}{2})^2]\Delta_2t+\int_y^d[\frac{t^2+t\sigma(t)+(\sigma(t))^2}{3}$$
$$-2(d-\xi\frac{d-c}{2})(t-(d-\xi\frac{d-c}{2}))$$
$$-(d-\xi\frac{d-c}{2})^2]\Delta_2t\}$$
$$=\{\frac{x^3-a^3}{3}-2(a+\xi\frac{b-a}{2})[h_2(x,a+\xi\frac{b-a}{2})$$
$$-h_2(a,a+\xi\frac{b-a}{2})]-(a+\xi\frac{b-a}{2})^2(x-a)$$
$$+\frac{b^3-x^3}{3}-2(b-\xi\frac{b-a}{2})[h_2(b,b-\xi\frac{b-a}{2})$$
$$-h_2(x,b-\xi\frac{b-a}{2})]-(b-\xi\frac{b-a}{2})^2(b-x)\}\times$$
$$\{\frac{y^3-c^3}{3}-2(c+\xi\frac{d-c}{2})[h_2(y,c+\xi\frac{d-c}{2})$$
$$-h_2(c,c+\xi\frac{d-c}{2})]-(c+\xi\frac{d-c}{2})^2(y-c)$$
$$+\frac{d^3-y^3}{3}-2(d-\xi\frac{d-c}{2})[h_2(d,d-\xi\frac{d-c}{2})$$

$$-h_2(y, d - \xi\frac{d-c}{2})] - (d - \xi\frac{b-a}{2})^2(d-y)\}. \tag{4}$$

Furthermore, we have

$$\begin{aligned}
&\int_a^b\int_c^d \{[K(x,y,s,t) - \frac{1}{(b-a)(d-c)}\\
&\int_a^b\int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s]\\
&\times[\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t} - \frac{1}{(b-a)(d-c)}\\
&\int_a^b\int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s]\}\Delta_2 t\Delta_1 s\\
&= \int_a^b\int_c^d K(x,y,s,t)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\\
&-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s\\
&\int_a^b\int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\\
&= \int_a^b\int_c^d K(x,y,s,t)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\\
&-\frac{[f(b,d) - f(a,d) - f(b,c) + f(a,c)]}{(b-a)(d-c)}\times\\
&\int_a^b\int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s.
\end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned}
&\int_a^b\int_c^d \{[K(x,y,s,t) - \frac{1}{(b-a)(d-c)}\\
&\int_a^b\int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s]\times\\
&[\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t} - \frac{1}{(b-a)(d-c)}\\
&\int_a^b\int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s]\}\Delta_2 t\Delta_1 s\\
&\le \|K(x,y,.,.) - \frac{1}{(b-a)(d-c)}\\
&\int_a^b\int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s\|_2\times\\
&\|\frac{\partial^2 f(.,.)}{\Delta_1 s\Delta_2 t} - \frac{1}{(b-a)(d-c)}\times\\
&\int_a^b\int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\|_2\\
&= [\int_a^b\int_c^d K^2(x,y,s,t)\Delta_2 t\Delta_1 s\\
&-\frac{1}{(b-a)(d-c)}(\int_a^b\int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s)^2]^{\frac12}\times\\
&[\int_a^b\int_c^d (\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t})^2\Delta_2 t\Delta_1 s\\
&-\frac{1}{(b-a)(d-c)}(\int_a^b\int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s)^2]^{\frac12}\\
&= [\int_a^b\int_c^d K^2(x,y,s,t)\Delta_2 t\Delta_1 s\\
&-\frac{1}{(b-a)(d-c)}(\int_a^b\int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s)^2]^{\frac12}\\
&\sqrt{T(f^{\Delta_1\Delta_2})}.
\end{aligned} \tag{6}$$

Combining (3)-(6) we can get the desired inequality (2).

To prove of the sharpness of (2), let $\mathbf{T}$ be right dense and $f(s,t) = f_1(s)f_2(t)$, where

$$f_1(s) = \begin{cases} h_2(s, a+\xi\frac{b-a}{2}) - h_2(x, a+\xi\frac{b-a}{2}), \\ \quad s\in[a,x)_{\mathbf{T}_1} \\ h_2(s, b-\xi\frac{b-a}{2}) - h_2(x, b-\xi\frac{b-a}{2}), \\ \quad s\in[x,b]_{\mathbf{T}_1} \end{cases}$$

$$f_2(t) = \begin{cases} h_2(t, c+\xi\frac{d-c}{2}) - h_2(y, c+\xi\frac{d-c}{2}), \\ \quad t\in[c,y)_{\mathbf{T}_2} \\ h_2(t, d-\xi\frac{d-c}{2}) - h_2(y, d-\xi\frac{d-c}{2}), \\ \quad t\in[x,b]_{\mathbf{T}_2} \end{cases}.$$

Then

$$\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t} = \begin{cases} [s-(a+\xi\frac{b-a}{2})][t-(c+\xi\frac{d-c}{2})], \\ \quad s\in[a,x)_{\mathbf{T}_1},\ t\in[c,y)_{\mathbf{T}_2} \\ [s-(a+\xi\frac{b-a}{2})][t-(d-\xi\frac{d-c}{2})], \\ \quad s\in[a,x)_{\mathbf{T}_1},\ t\in[y,d)_{\mathbf{T}_2} \\ [s-(b-\xi\frac{b-a}{2})][t-(c+\xi\frac{d-c}{2})], \\ \quad s\in[x,b)_{\mathbf{T}_1},\ t\in[c,y)_{\mathbf{T}_2} \\ [s-(b-\xi\frac{b-a}{2})][t-(d-\xi\frac{d-c}{2})], \\ \quad s\in[x,b)_{\mathbf{T}_1},\ t\in[y,d)_{\mathbf{T}_2} \end{cases} \tag{7}$$

which implies $\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t} = K(x,y,s,t)$. So (4) and (6) hold equality, which implies (2) holds equality, and the proof is complete. □

In Theorem 2, if we take $\mathbf{T}_1,\ \mathbf{T}_2$ for some special time scales, then we immediately obtain the following two corollaries.

**Corollary 3** *If we take $\mathbf{T}_1 = \mathbf{T}_2 = \mathbf{R}$ in Theorem 2, then we obtain the following inequality*

$$\begin{aligned}
&|(1-\xi)^2 f(x,y)\\
&+(1-\xi)\frac{\xi}{2}[f(a,y) + f(b,y) + f(x,c) + f(x,d)]
\end{aligned}$$

$$+\frac{\xi^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)]$$
$$-\frac{1}{b-a}(1-\xi)\int_a^b f(\sigma(s),y)\Delta_1 s$$
$$-\frac{1}{d-c}(1-\xi)\int_c^d f(x,t)dt$$
$$-\frac{\xi}{2}\frac{1}{b-a}\int_a^b [f(s,c)+f(s,d)]ds$$
$$-\frac{\xi}{2}\frac{1}{d-c}\int_c^d [f(a,t)+f(b,t)]dt$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dtds$$
$$-\frac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)^2(d-c)^2}\times$$
$$[\frac{(x-(a+\xi\frac{b-a}{2}))^2}{2}-\frac{(a-(a+\xi\frac{b-a}{2}))^2}{2}$$
$$+\frac{(b-(b-\xi\frac{b-a}{2}))^2}{2}-\frac{(x-(b-\xi\frac{b-a}{2}))^2}{2}]\times$$
$$[\frac{(y-(c+\xi\frac{d-c}{2}))^2}{2}-\frac{(c-(c+\xi\frac{d-c}{2}))^2}{2}$$
$$+\frac{(d-(d-\xi\frac{d-c}{2}))^2}{2}-\frac{(y-(d-\xi\frac{d-c}{2}))^2}{2}]|$$
$$\leq\{[\frac{b^3-a^3}{3}-2(a+\xi\frac{b-a}{2})(\frac{(x-(a+\xi\frac{b-a}{2}))^2}{2}$$
$$-\frac{(a-(a+\xi\frac{b-a}{2}))^2}{2})-(a+\xi\frac{b-a}{2})^2(x-a)$$
$$-2(b-\xi\frac{b-a}{2})(\frac{(b-(b-\xi\frac{b-a}{2}))^2}{2}$$
$$-\frac{(x-(b-\xi\frac{b-a}{2}))^2}{2})-(b-\xi\frac{b-a}{2})^2(b-x)]\times$$
$$[\frac{d^3-c^3}{3}-2(c+\xi\frac{d-c}{2})(\frac{(y-(c+\xi\frac{d-c}{2}))^2}{2}$$
$$-\frac{(c-(c+\xi\frac{d-c}{2}))^2}{2})-(c+\xi\frac{d-c}{2})^2(y-c)$$
$$-2(d-\xi\frac{d-c}{2})(\frac{(d-(d-\xi\frac{d-c}{2}))^2}{2}$$
$$-\frac{(y-(d-\xi\frac{d-c}{2}))^2}{2})-(d-\xi\frac{d-c}{2})^2(d-y)]$$
$$-\frac{1}{(b-a)(d-c)}[(\frac{(x-(a+\xi\frac{b-a}{2}))^2}{2}$$
$$-\frac{(a-(a+\xi\frac{b-a}{2}))^2}{2}+\frac{(b-(b-\xi\frac{b-a}{2}))^2}{2}$$
$$-\frac{(x-(b-\xi\frac{b-a}{2}))^2}{2})\times$$
$$(\frac{(y-(c+\xi\frac{d-c}{2}))^2}{2}-\frac{(c-(c+\xi\frac{d-c}{2}))^2}{2}$$
$$+\frac{(d-(d-\xi\frac{d-c}{2}))^2}{2}-\frac{(y-(d-\xi\frac{d-c}{2}))^2}{2})]^2\}^{\frac{1}{2}}$$
$$\sqrt{T(f''_{st})},$$

*where*

$$T(f)=\int_a^b\int_c^d f^2(s,t)dtds$$
$$-\frac{1}{(b-a)(d-c)}(\int_a^b\int_c^d f(s,t)dtds)^2.$$

**Corollary 4** *If we take* $\mathbf{T}_1=\mathbf{T}_2=\mathbf{Z}$ *in Theorem 2, then we obtain the following inequality*

$$|(1-\xi)^2 f(x,y)$$
$$+(1-\xi)\frac{\xi}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)]$$
$$+\frac{\xi^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)]$$
$$-\frac{1}{b-a}(1-\xi)\sum_{s=a}^{b-1}f(s+1,y)$$
$$-\frac{1}{d-c}(1-\xi)\sum_{t=c}^{d-1}f(x,t+1)$$
$$-\frac{\xi}{2}\frac{1}{b-a}\sum_{s=a}^{b-1}[f(s+1,c)+f(s+1,d)]$$
$$-\frac{\xi}{2}\frac{1}{d-c}\sum_{t=c}^{d-1}[f(a,t+1)+f(b,t+1)]$$
$$+\frac{1}{(b-a)(d-c)}\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}f(s+1,t+1)$$
$$-\frac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)^2(d-c)^2}\times$$
$$[\frac{(x-(a+\xi\frac{b-a}{2}))(x-(a+\xi\frac{b-a}{2})-1)}{2}$$
$$-\frac{(a-(a+\xi\frac{b-a}{2}))(a-(a+\xi\frac{b-a}{2})-1)}{2}$$
$$+\frac{(b-(b-\xi\frac{b-a}{2}))(b-(b+\xi\frac{b-a}{2})-1)}{2}$$
$$-\frac{(x-(b-\xi\frac{b-a}{2}))(b-(b-\xi\frac{b-a}{2})-1)}{2}]\times$$
$$[\frac{(y-(c+\xi\frac{d-c}{2}))(y-(c+\xi\frac{d-c}{2})-1)}{2}$$
$$-\frac{(c-(c+\xi\frac{d-c}{2}))(c-(c+\xi\frac{d-c}{2})-1)}{2}$$
$$+\frac{(d-(d-\xi\frac{d-c}{2}))(d-(d+\xi\frac{d-c}{2})-1)}{2}$$
$$-\frac{(y-(d-\xi\frac{d-c}{2}))(d-(d-\xi\frac{d-c}{2})-1)}{2}]|$$

$$
\begin{aligned}
&\leq \{[\frac{b^3-a^3}{3} - 2(a+\xi\frac{b-a}{2}) \\
&(\frac{(x-(a+\xi\frac{b-a}{2}))(x-(a+\xi\frac{b-a}{2})-1)}{2} \\
&-\frac{(a-(a+\xi\frac{b-a}{2}))(a-(a+\xi\frac{b-a}{2})-1)}{2}) \\
&-(a+\xi\frac{b-a}{2})^2(x-a) - 2(b-\xi\frac{b-a}{2}) \\
&(\frac{(b-(b-\xi\frac{b-a}{2}))(b-(b+\xi\frac{b-a}{2})-1)}{2} \\
&-\frac{(x-(b-\xi\frac{b-a}{2}))(b-(b-\xi\frac{b-a}{2})-1)}{2}) \\
&-(b-\xi\frac{b-a}{2})^2(b-x)] \times \\
&[\frac{d^3-c^3}{3} - 2(c+\xi\frac{d-c}{2}) \\
&(\frac{(y-(c+\xi\frac{d-c}{2}))(y-(c+\xi\frac{d-c}{2})-1)}{2} \\
&-\frac{(c-(c+\xi\frac{d-c}{2}))(c-(c+\xi\frac{d-c}{2})-1)}{2}) \\
&-(c+\xi\frac{d-c}{2})^2(y-c) - 2(d-\xi\frac{d-c}{2}) \\
&(\frac{(d-(d-\xi\frac{d-c}{2}))(d-(d+\xi\frac{d-c}{2})-1)}{2} \\
&-\frac{(y-(d-\xi\frac{d-c}{2}))(d-(d-\xi\frac{d-c}{2})-1)}{2}) \\
&-(d-\xi\frac{d-c}{2})^2(d-y)] - \frac{1}{(b-a)(d-c)} \\
&[(\frac{(x-(a+\xi\frac{b-a}{2}))(x-(a+\xi\frac{b-a}{2})-1)}{2} \\
&-\frac{(a-(a+\xi\frac{b-a}{2}))(a-(a+\xi\frac{b-a}{2})-1)}{2} \\
&+\frac{(b-(b-\xi\frac{b-a}{2}))(b-(b+\xi\frac{b-a}{2})-1)}{2} \\
&-\frac{(x-(b-\xi\frac{b-a}{2}))(b-(b-\xi\frac{b-a}{2})-1)}{2}) \times \\
&(\frac{(y-(c+\xi\frac{d-c}{2}))(y-(c+\xi\frac{d-c}{2})-1)}{2} \\
&-\frac{(c-(c+\xi\frac{d-c}{2}))(c-(c+\xi\frac{d-c}{2})-1)}{2} \\
&+\frac{(d-(d-\xi\frac{d-c}{2}))(d-(d+\xi\frac{d-c}{2})-1)}{2} \\
&-\frac{(y-(d-\xi\frac{d-c}{2}))(d-(d-\xi\frac{d-c}{2})-1)}{2})]^2\}^{\frac{1}{2}} \\
&\sqrt{T(\Delta_2\Delta_1 f)},
\end{aligned}
$$

*where* $\Delta_2\Delta_1 f$ *denotes the difference on* $f$ *with order two, and* $T(f) = \sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f^2(s,t) - \frac{1}{(b-a)(d-c)}(\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f(s,t))^2$.

**Remark 5** *Corollary 3 is equivalent to [26, Theorem 3], and is the generalization of [22, Theorem 5] to 2D case. If we take* $\xi = 0$ *, then Corollary 3 reduces to [25, Theorem 4], and is the 2D generalization of [17, Theorem 4]. If we take* $\xi = \frac{1}{3}$, $x = \frac{a+b}{2}$, $y = \frac{c+d}{2}$, *then Corollary 3 reduces to [25, Theorem 3], and is the 2D generalization of [17, Theorem 1]. So in this way, Theorem 2 is the further extension of some known results in the literature to arbitrary time scales.*

**Theorem 6** *Under the conditions of Lemma 1, if we assume* $f^{\Delta_1\Delta_2} \in L_\infty((a,b)_{\mathbf{T}_1} \times (c,d)_{\mathbf{T}_2})$, *then we have*

$$
\begin{aligned}
&|(1-\xi)^2 f(x,y) \\
&+(1-\xi)\frac{\xi}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)] \\
&+\frac{\xi^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)] \\
&-\frac{1}{b-a}(1-\xi)\int_a^b f(\sigma(s),y)\Delta_1 s \\
&-\frac{1}{d-c}(1-\xi)\int_c^d f(x,\sigma(t))\Delta_2 t \\
&-\frac{\xi}{2}\frac{1}{b-a}\int_a^b [f(\sigma(s),c)+f(\sigma(s),d)]\Delta_1 s \\
&-\frac{\xi}{2}\frac{1}{d-c}\int_c^d [f(a,\sigma(t))+f(b,\sigma(t))]\Delta_2 t \\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s \\
&-\frac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)^2(d-c)^2} \\
&[h_2(x,a+\xi\frac{b-a}{2}) - h_2(a,a+\xi\frac{b-a}{2}) \\
&+h_2(b,b-\xi\frac{b-a}{2}) - h_2(x,b-\xi\frac{b-a}{2})] \times \\
&[h_2(y,c+\xi\frac{d-c}{2}) - h_2(c,c+\xi\frac{d-c}{2}) \\
&+h_2(d,d-\xi\frac{d-c}{2}) - h_2(y,d-\xi\frac{d-c}{2})]| \\
&\leq \sqrt{b-a}\{\{\frac{x^3-a^3}{3} - 2(a+\xi\frac{b-a}{2}) \\
&[h_2(x,a+\xi\frac{b-a}{2}) - h_2(a,a+\xi\frac{b-a}{2})] \\
&-(a+\xi\frac{b-a}{2})^2(x-a) \\
&+\frac{b^3-x^3}{3} - 2(b-\xi\frac{b-a}{2})[h_2(b,b-\xi\frac{b-a}{2})
\end{aligned}
$$

$$-h_2(x,b-\xi\frac{b-a}{2})]-(b-\xi\frac{b-a}{2})^2(b-x)\}\times$$
$$\{\frac{y^3-c^3}{3}-2(c+\xi\frac{d-c}{2})[h_2(y,c+\xi\frac{d-c}{2})$$
$$-h_2(c,c+\xi\frac{d-c}{2})]-(c+\xi\frac{d-c}{2})^2(y-c)$$
$$+\frac{d^3-y^3}{3}-2(d-\xi\frac{d-c}{2})$$
$$[h_2(d,d-\xi\frac{d-c}{2})-h_2(y,d-\xi\frac{d-c}{2})]$$
$$-(d-\xi\frac{b-a}{2})^2(d-y)\}-\frac{1}{(b-a)(d-c)}$$
$$\{[h_2(x,a+\xi\frac{b-a}{2})-h_2(a,a+\xi\frac{b-a}{2})$$
$$+h_2(b,b-\xi\frac{b-a}{2})-h_2(x,b-\xi\frac{b-a}{2})]\times$$
$$[h_2(y,c+\xi\frac{d-c}{2})-h_2(c,c+\xi\frac{d-c}{2})$$
$$+h_2(d,d-\xi\frac{d-c}{2})$$
$$-h_2(y,d-\xi\frac{d-c}{2})]]\}^2\}^{\frac{1}{2}}\|f^{\Delta_1\Delta_2}\|_\infty. \quad (8)$$

**Proof:** First we have the following observation:

$$\int_a^b\int_c^d[K(x,y,s,t)-\frac{1}{(b-a)(d-c)}\times$$
$$\int_a^b\int_c^d K(x,y,s,t)\Delta_2t\Delta_1s]\frac{\partial^2 f(s,t)}{\Delta_1s\Delta_2t}\Delta_2t\Delta_1s$$
$$=\int_a^b\int_c^d K(x,y,s,t)\frac{\partial^2 f(s,t)}{\Delta_1s\Delta_2t}\Delta_2t\Delta_1s$$
$$-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d K(x,y,s,t)\Delta_2t\Delta_1s$$
$$\int_a^b\int_c^d\frac{\partial^2 f(s,t)}{\Delta_1s\Delta_2t}\Delta_2t\Delta_1s$$
$$=\int_a^b\int_c^d K(x,y,s,t)\frac{\partial^2 f(s,t)}{\Delta_1s\Delta_2t}\Delta_2t\Delta_1s$$
$$-\frac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)(d-c)}\times$$
$$\int_a^b\int_c^d K(x,y,s,t)\Delta_2t\Delta_1s. \quad (9)$$

Then

$$|\int_a^b\int_c^d[K(x,y,s,t)-\frac{1}{(b-a)(d-c)}$$
$$\int_a^b\int_c^d K(x,y,s,t)\Delta_2t\Delta_1s]\frac{\partial^2 f(s,t)}{\Delta_1s\Delta_2t}\Delta_2t\Delta_1s|$$
$$\le\int_a^b\int_c^d|K(x,y,s,t)-\frac{1}{(b-a)(d-c)}\times$$
$$\int_a^b\int_c^d K(x,y,s,t)\Delta_2t\Delta_1s|\Delta_2t\Delta_1s\|f^{\Delta_1\Delta_2}\|_\infty$$
$$\le\sqrt{(b-a)(d-c)}[\int_a^b\int_c^d|K(x,y,s,t)$$
$$-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d K(x,y,s,t)\Delta_2t\Delta_1s|^2$$
$$\Delta_2t\Delta_1s]^{\frac{1}{2}}\|f^{\Delta_1\Delta_2}\|_\infty$$
$$=\sqrt{(b-a)(d-c)}[\int_a^b\int_c^d K^2(x,y,s,t)\Delta_2t\Delta_1s-$$
$$\frac{1}{(b-a)(d-c)}(\int_a^b\int_c^d K(x,y,s,t)\Delta_2t\Delta_1s)^2]^{\frac{1}{2}}\times$$
$$\|f^{\Delta_1\Delta_2}\|_\infty. \quad (10)$$

Then combining (3), (4) and (10) we get the desired result.

**Lemma 7** *[41, Lemma 2.8](2D Grüss' inequality on time scales). Let $f,\ g\in C_{rd}([a,b]_{\mathbf{T}_1}\times[c,d]_{\mathbf{T}_2},\mathbf{R})$ such that $\phi\le f(x,y)\le\Phi$ and $\gamma\le g(x,y)\le\Gamma$ for all $x\in[a,b]_{\mathbf{T}_1},\ y\in[c,d]_{\mathbf{T}_2}$, where $\phi,\ \Phi,\ \gamma,\ \Gamma$ are constants. Then we have*

$$|\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(s,t)g(s,t)\Delta_2t\Delta_1s$$
$$-\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(s,t)\Delta_2t\Delta_1s$$
$$\times\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d g(s,t)\Delta_2t\Delta_1s|$$
$$\le\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma). \quad (11)$$

**Theorem 8** *Under the conditions of Lemma 1, if there exist constants $K_1,\ K_2$ such that $K_1\le\frac{\partial^2 f(s,t)}{\Delta_1s\Delta_2t}\le K_2$, then we have*

$$|(1-\xi)^2f(x,y)$$
$$+(1-\xi)\frac{\xi}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)]$$
$$+\frac{\xi^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)]$$
$$-\frac{1}{b-a}(1-\xi)\int_a^b f(\sigma(s),y)\Delta_1s$$
$$-\frac{1}{d-c}(1-\xi)\int_c^d f(x,\sigma(t))\Delta_2t$$
$$-\frac{\xi}{2}\frac{1}{b-a}\int_a^b[f(\sigma(s),c)+f(\sigma(s),d)]\Delta_1s$$

$$
\begin{aligned}
&-\frac{\xi}{2}\frac{1}{d-c}\int_c^d [f(a,\sigma(t))+f(b,\sigma(t))]\Delta_2 t \\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s \\
&-\frac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)^2(d-c)^2} \\
&[h_2(x,a+\xi\frac{b-a}{2})-h_2(a,a+\xi\frac{b-a}{2}) \\
&+h_2(b,b-\xi\frac{b-a}{2})-h_2(x,b-\xi\frac{b-a}{2})]\times \\
&[h_2(y,c+\xi\frac{d-c}{2})-h_2(c,c+\xi\frac{d-c}{2}) \\
&+h_2(d,d-\xi\frac{d-c}{2})-h_2(y,d-\xi\frac{d-c}{2})]| \\
&\leq \frac{1}{4}(K_2-K_1).
\end{aligned} \tag{12}
$$

**Proof:** From the definition of $K(x,y,s,t)$ we have $\sup(K(x,y,s,t))-\inf(K(x,y,s,t))\leq(b-a)(d-c)$. So by Lemma 7 we obtain

$$
\begin{aligned}
&|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d K(x,y,s,t)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s \\
&-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s \\
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s| \\
&\leq \frac{1}{4}(b-a)(d-c)(K_2-K_1).
\end{aligned}
$$

The desired result can be obtained by the combination of (3) and Lemma 1. □

**Remark 9** *If we take $\xi=0$ in Theorem 8, then Theorem 8 becomes the 2D extension of [28, Theorem 4].*

# 3 Conclusions

In this paper, we establish some generalized Ostrowski-Grüss type inequalities in two independent variables on time scales. Some of the estimates for the established inequalities are sharp. The established results unify continuous and discrete analysis, and are further extension of some known results in the literature. How to extend the results in this paper to the case in n independent variables is our task in the future.

**Acknowledgements:** The research was supported by Natural Science Foundation of Shandong Province (grant No. ZR2013AQ009).

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