

# A New Fractional Sub-equation Method For Fractional Partial Differential Equations

Chuanbao Wen

Shandong University of Technology  
School of Science

Zhangzhou Road 12, Zibo, 255049  
China

wcb2001171@126.com

Bin Zheng

Shandong University of Technology  
School of Science

Zhangzhou Road 12, Zibo, 255049  
China

Corresponding author:zhengbin2601@126.com

*Abstract:* In this paper, a new fractional sub-equation method is proposed to establish exact solutions for fractional partial differential equations in the sense of modified Riemann-Liouville derivative, which is the fractional version of the known (G'/G) method. For illustrating the validity of this method, we apply it to the space-time fractional generalized Hirota-Satsuma coupled KDV equations and the space-time fractional (2+1)-dimensional breaking soliton equations. As a result, some new exact solutions for them are successfully established.

*Key-Words:* Fractional sub-equation method; Fractional partial differential equation; Exact solution; Fractional complex transformation; Fractional generalized Hirota-Satsuma coupled KDV equations; Fractional (2+1)-dimensional breaking soliton equations

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## 1 Introduction

Fractional differential equations have recently proved to be valuable tools to the modeling of many physical phenomena, and have been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, signal processing, systems identification, control theory, finance and fractional dynamics. Many articles have investigated some aspects of fractional differential equations, such as the existence and uniqueness of solutions to Cauchy type problems, the methods for explicit and numerical solutions, and the stability of solutions [1-8]. Among the investigations for fractional differential equations, research for seeking exact solutions and numerical solutions of fractional differential equations has been an important task of many researchers. Many powerful and efficient methods have been proposed to obtain numerical solutions and exact solutions of fractional differential equations so far. For example, these methods include the Adomian decomposition method [9,10], the variational iterative method [11-13], the homotopy perturbation method [14,15], the differential transformation method [16], the finite difference method [17], the finite element method [18], the fractional sub-

equation method [19-21] and so on. Based on these methods, a variety of fractional differential equations have been investigated.

In this paper, we propose a new fractional sub-equation method to establish exact solutions for fractional partial differential equations in the sense of modified Riemann-Liouville derivative by Jumarie [22], which is the fractional version of the known (G'/G) method [23-29]. The main idea of this method lies in that by a traveling wave transformation  $\xi = \xi(t, x_1, x_2, \dots, x_n)$ , certain fractional partial differential equations expressed in independent variables  $t, x_1, x_2, \dots, x_n$  can be turned into fractional ordinary differential equations in  $\xi$ , the solutions of which are supposed to have the form

$$U(\xi) = \sum_{i=0}^m a_i \left[ \frac{D_\xi^\alpha G(\xi)}{G(\xi)} \right]^i,$$

where  $G(\xi)$  satisfies the following fractional ordinary differential equation:

$$D_\xi^{2\alpha} G(\xi) + \lambda D_\xi^\alpha G(\xi) + \mu G(\xi) = 0, \quad (1)$$

$D_\xi^\alpha G(\xi)$  denotes the modified Riemann-Liouville derivative of order  $\alpha$  for  $G(\xi)$  with respect to  $\xi$ , and the integer  $m$  can be determined by the homogeneous balancing principle. By the general solutions of Eq. (1) we can deduce the expression for

$\frac{D_\xi^\alpha G(\xi)}{G(\xi)}$ , and then the exact solutions for the original fractional partial differential equations can be established.

The Jumarie's modified Riemann-Liouville derivative of order  $\alpha$  is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha < n+1, n \geq 1. \end{cases}$$

We list some important properties for the modified Riemann-Liouville derivative as follows:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \quad (2)$$

$$D_t^\alpha (f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \quad (3)$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)](g'(t))^\alpha. \quad (4)$$

The rest of this paper is organized as follows. In Section 2, we present the expression for  $\frac{D_\xi^\alpha G(\xi)}{G(\xi)}$  related to Eq. (1). In Section 3, we give the description of the fractional sub-equation method for solving fractional partial differential equations. Then in Section 4 we apply this method to establish exact solutions for the space-time fractional generalized Hirota-Satsuma coupled KDV equations and the space-time fractional (2+1)-dimensional breaking soliton equations. Some conclusions are presented at the end of the paper.

## 2 General expression for $\frac{D_\xi^\alpha G(\xi)}{G(\xi)}$

In order to obtain the general solutions for Eq. (1), we suppose  $G(\xi) = H(\eta)$ , and a nonlinear fractional complex transformation  $\eta = \frac{\xi^\alpha}{\Gamma(1+\alpha)}$ . Then by Eq. (2) and the first equality in Eq. (4), Eq. (1) can be turned into the following second ordinary differential equation

$$H''(\eta) + \lambda H'(\eta) + \mu H(\eta) = 0. \quad (5)$$

By the general solutions of Eq. (5) we have

$$\frac{H'(\eta)}{H(\eta)} =$$

$$\begin{cases} -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \times \left( \frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2} \eta + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2} \eta}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2} \eta + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2} \eta} \right), & \lambda^2 - 4\mu > 0, \\ -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \times \left( \frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2} \eta + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2} \eta}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2} \eta + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2} \eta} \right), & \lambda^2 - 4\mu < 0, \\ -\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\eta}, & \lambda^2 - 4\mu = 0, \end{cases} \quad (6)$$

where  $C_1, C_2$  are arbitrary constants.

Since  $D_\xi^\alpha G(\xi) = D_\xi^\alpha H(\eta) = H'(\eta)D_\xi^\alpha \eta = H'(\eta)$ , we obtain

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)} = \begin{cases} -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \times \left( \frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha} \right), & \lambda^2 - 4\mu > 0, \\ -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \times \left( \frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)} \xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)} \xi^\alpha} \right), & \lambda^2 - 4\mu < 0, \\ -\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}, & \lambda^2 - 4\mu = 0. \end{cases} \quad (7)$$

## 3 Description of the fractional sub-equation method

In this section we give the main steps of the fractional sub-equation method for finding exact solutions of fractional partial differential equations.

Suppose that a fractional partial differential equation, say in the independent variables  $t, x_1, x_2, \dots, x_n$ , is given by

$$\begin{aligned} P(u_1, \dots, u_k, D_t^\alpha u_1, \dots, D_t^\alpha u_k, D_{x_1}^\alpha u_1, \dots, D_{x_1}^\alpha u_k, \dots, \\ D_{x_n}^\alpha u_1, \dots, D_{x_n}^\alpha u_k, D_t^{2\alpha} u_1, \dots, D_t^{2\alpha} u_k, D_{x_1}^{2\alpha} u_1, \dots) = 0, \end{aligned} \quad (8)$$

where  $u_i = u_i(t, x_1, x_2, \dots, x_n)$ ,  $i = 1, \dots, k$  are unknown functions,  $P$  is a polynomial in  $u_i$  and their various partial derivatives including fractional derivatives.

*Step 1.* Suppose that

$$u_i(t, x_1, x_2, \dots, x_n) = U_i(\xi),$$

$$\xi = c^\frac{1}{\alpha} t + k_1^\frac{1}{\alpha} x_1 + k_2^\frac{1}{\alpha} x_2 + \dots + k_n^\frac{1}{\alpha} x_n + \xi_0. \quad (9)$$

Then by the second equality in Eq. (4), Eq. (8) can be turned into the following fractional ordinary differential equation with respect to the variable  $\xi$ :

$$\begin{aligned} &\tilde{P}(U_1, \dots, U_k, cD_\xi^\alpha U_1, \dots, cD_\xi^\alpha U_k, k_1 D_\xi^\alpha U_1, \dots, \\ &k_1 D_\xi^\alpha U_k, \dots, k_n D_\xi^\alpha U_1, \dots, k_n D_\xi^\alpha U_k, c^2 D_\xi^{2\alpha} U_1, \dots, \\ &c^2 D_\xi^{2\alpha} U_k, k_1^2 D_\xi^{2\alpha} U_1, \dots) = 0. \end{aligned} \quad (10)$$

*Step 2.* Suppose that the solution of (10) can be expressed by a polynomial in  $(\frac{D_\xi^\alpha G}{G})$  as follows:

$$U_j(\xi) = \sum_{i=0}^{m_j} a_{j,i} \left(\frac{D_\xi^\alpha G}{G}\right)^i, \quad j = 1, 2, \dots, k \quad (11)$$

where  $G = G(\xi)$  satisfies Eq. (1), and  $a_{j,i}$ ,  $i = 0, 1, \dots, m$ ,  $j = 1, 2, \dots, k$  are constants to be determined later with  $a_{j,m} \neq 0$ . The positive integer  $m$  can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (10).

*Step 1.* Substituting (11) into (10) and using (1), collecting all terms with the same order of  $(\frac{D_\xi^\alpha G}{G})$  together, the left-hand side of (10) is converted into another polynomial in  $(\frac{D_\xi^\alpha G}{G})$ . Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for  $\lambda$ ,  $\mu$ ,  $a_{j,i}$ ,  $i = 0, 1, \dots, m$ ,  $j = 1, 2, \dots, k$ .

*Step 4.* Solving the equations system in Step 3, and using (7), we can construct a variety of exact solutions for Eq. (8).

**Remark 1** *If we set  $\alpha = 1$  in Eq. (1), then it becomes  $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$ , which is the foundation of the known  $(G'/G)$  method for solving partial differential equations (PDEs). So in this way, the described fractional sub-equation method above is the extension of the  $(G'/G)$  method to fractional case.*

## 4 Applications of the method

### 4.1 Space-time fractional generalized Hirota-Satsuma coupled KDV equations

We consider the following space-time fractional generalized Hirota-Satsuma coupled KDV equa-

tions [20]:

$$\begin{cases} D_t^\alpha u - \frac{1}{2} D_x^{3\alpha} u + 3u D_x^\alpha u - 3D_x^\alpha(vw) = 0 \\ D_t^\alpha v + D_x^{3\alpha} v - 3u D_x^\alpha v = 0 \\ D_t^\alpha w + D_x^{3\alpha} w - 3u D_x^\alpha w = 0 \end{cases} \quad (12)$$

where  $0 < \alpha \leq 1$ . In [20], the authors solved Eqs. (12) by a proposed fractional sub-equation method based on the fractional Riccati equation, and established some exact solutions for it. Now we will apply the described method in Section 3 to Eqs. (12). Suppose  $u(x, t) = U(\xi)$ ,  $v(x, t) = V(\xi)$ ,  $w(x, t) = W(\xi)$ , where  $\xi = k^{\frac{1}{\alpha}} x + c^{\frac{1}{\alpha}} t + \xi_0$ ,  $k$ ,  $c$ ,  $\xi_0$  are all constants with  $k$ ,  $c > 0$ . Then by use of the second equality in Eq. (4), Eqs. (12) can be turned into

$$\begin{cases} cD_\xi^\alpha U - \frac{1}{2} k^3 D_\xi^{3\alpha} U + 3kUD_\xi^\alpha U - 3kD_\xi^\alpha(VW) = 0, \\ cD_\xi^\alpha V + k^3 D_\xi^{3\alpha} V - 3kUD_\xi^\alpha V = 0, \\ cD_\xi^\alpha W + k^3 D_\xi^{3\alpha} W - 3kUD_\xi^\alpha W = 0. \end{cases} \quad (13)$$

Suppose that the solution of Eqs. (13) can be expressed by

$$\begin{cases} U(\xi) = \sum_{i=0}^{m_1} a_i \left(\frac{D_\xi^\alpha G}{G}\right)^i, \\ V(\xi) = \sum_{i=0}^{m_2} b_i \left(\frac{D_\xi^\alpha G}{G}\right)^i, \\ W(\xi) = \sum_{i=0}^{m_3} c_i \left(\frac{D_\xi^\alpha G}{G}\right)^i. \end{cases} \quad (14)$$

Balancing the order of  $D_\xi^{3\alpha} U$  and  $D_\xi^\alpha(VW)$ ,  $D_\xi^{3\alpha} V$  and  $UD_\xi^\alpha V$ ,  $D_\xi^{3\alpha} W$  and  $UD_\xi^\alpha W$  in Eqs. (13) we deduce  $m_1 = m_2 = m_3 = 2$ . So

$$\begin{cases} U(\xi) = a_0 + a_1 \left(\frac{D_\xi^\alpha G}{G}\right)^1 + a_2 \left(\frac{D_\xi^\alpha G}{G}\right)^2, \\ V(\xi) = b_0 + b_1 \left(\frac{D_\xi^\alpha G}{G}\right)^1 + b_2 \left(\frac{D_\xi^\alpha G}{G}\right)^2, \\ W(\xi) = c_0 + c_1 \left(\frac{D_\xi^\alpha G}{G}\right)^1 + c_2 \left(\frac{D_\xi^\alpha G}{G}\right)^2. \end{cases} \quad (15)$$

Substituting (15) into (13), using Eq. (1) and collecting all the terms with the same power of  $(\frac{D_\xi^\alpha G}{G})$  together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations with the aid of the mathematical software Maple, yields the following families of values of  $\lambda$ ,  $\mu$ ,  $a_i$ ,  $b_i$ ,  $c_i$ ,  $i = 0, 1, 2$ :

*Case 1.*

$$a_0 = \frac{2k^3\mu + c + k^3\lambda^2}{3k}, a_1 = 2k^2\lambda, a_2 = 2k^2,$$

$$\begin{aligned}
 b_0 &= b_0, \quad b_1 = b_1, \quad b_2 = 0, \\
 \lambda &= \lambda, \quad \mu = \mu, \\
 c_0 &= -\frac{k(-k^3b_1\lambda^3+4k^3b_1\lambda\mu-4cb_1\lambda+b_0k^3\lambda^2)}{3b_1^2} \\
 &\quad + \frac{4b_0k^3\mu-4b_0c}{3b_1^2}, \\
 c_1 &= \frac{k(k^3\lambda^2-4k^3\mu+4c)}{3b_1}, \quad c_2 = 0,
 \end{aligned}$$

where  $b_0, b_1, \lambda, \mu$  are arbitrary constants with  $b_1 \neq 0$ .

Case 2.

$$\begin{aligned}
 a_0 &= \frac{k^3\lambda^2+8k^3\mu+c}{3k}, \quad a_1 = 4k^2\lambda, \\
 a_2 &= 4k^2, \quad \mu = \mu, \\
 b_0 &= \frac{2k(k^3c_2\lambda^2+8k^3c_2\mu-6c_0k^3+4cc_2)}{3c_2^2}, \quad \lambda = \lambda, \\
 b_1 &= \frac{4k^4\lambda}{c_2}, \quad b_2 = \frac{4k^4}{c_2}, \\
 c_0 &= c_0, \quad c_1 = c_2\lambda, \quad c_2 = c_2,
 \end{aligned}$$

where  $c_0, c_2, \lambda, \mu$  are arbitrary constants with  $c_2 \neq 0$ .

Substituting the results above into (15), and combining with (7) we can obtain the following two families of exact solutions to Eqs. (12).

Family 1:

When  $\lambda^2 - 4\mu > 0$ ,

$$\begin{aligned}
 u_1(x, t) &= \frac{2k^3\mu+c+k^3\lambda^2}{3k} \\
 &+ 2k^2\lambda\left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right. \\
 &\quad \left.\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right]^1 \\
 &+ 2k^2\left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right. \\
 &\quad \left.\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right]^2,
 \end{aligned} \tag{16}$$

$$\begin{aligned}
 v_1(x, t) &= b_0 + b_1\left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right. \\
 &\quad \left.\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right]^1,
 \end{aligned} \tag{17}$$

$$\begin{aligned}
 w_1(x, t) &= -\frac{k(-k^3b_1\lambda^3+4k^3b_1\lambda\mu-4cb_1\lambda+b_0k^3\lambda^2)}{3b_1^2} \\
 &+ \frac{4b_0k^3\mu-4b_0c}{3b_1^2} + \frac{k(k^3\lambda^2-4k^3\mu+4c)}{3b_1}\left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right. \\
 &\quad \left.\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right]^1,
 \end{aligned} \tag{18}$$

where  $\xi = k^{\frac{1}{\alpha}}x + c^{\frac{1}{\alpha}}t + \xi_0$ .

When  $\lambda^2 - 4\mu < 0$ ,

$$\begin{aligned}
 u_2(x, t) &= \frac{2k^3\mu+c+k^3\lambda^2}{3k} \\
 &+ 2k^2\lambda\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\right. \\
 &\quad \left.\left(\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right] \\
 &+ 2k^2\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\right. \\
 &\quad \left.\left(\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right]^2,
 \end{aligned} \tag{19}$$

$$\begin{aligned}
 v_2(x, t) &= b_0 \\
 &+ b_1\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\right. \\
 &\quad \left.\left(\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right],
 \end{aligned} \tag{20}$$

$$\begin{aligned}
 w_2(x, t) &= -\frac{k(-k^3b_1\lambda^3+4k^3b_1\lambda\mu-4cb_1\lambda+b_0k^3\lambda^2)}{3b_1^2} \\
 &+ \frac{4b_0k^3\mu-4b_0c}{3b_1^2} + \frac{k(k^3\lambda^2-4k^3\mu+4c)}{3b_1}\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\right. \\
 &\quad \left.\left(\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right],
 \end{aligned} \tag{21}$$

where  $\xi = k^{\frac{1}{\alpha}}x + c^{\frac{1}{\alpha}}t + \xi_0$ .

When  $\lambda^2 - 4\mu = 0$ ,

$$\begin{aligned}
 u_3(x, t) &= \frac{2k^3\mu+c+k^3\lambda^2}{3k} \\
 &+ 2k^2\lambda\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right] \\
 &+ 2k^2\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right]^2,
 \end{aligned} \tag{22}$$

$$v_3(x, t) = b_0 + b_1\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right], \tag{23}$$

$$\begin{aligned}
 w_3(x, t) &= -\frac{k(-k^3b_1\lambda^3+4k^3b_1\lambda\mu-4cb_1\lambda+b_0k^3\lambda^2)}{3b_1^2} \\
 &+ \frac{4b_0k^3\mu-4b_0c}{3b_1^2} \\
 &+ \frac{k(k^3\lambda^2-4k^3\mu+4c)}{3b_1}\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right],
 \end{aligned} \tag{24}$$

where  $\xi = k^{\frac{1}{\alpha}}x + c^{\frac{1}{\alpha}}t + \xi_0$ .

Family 2:

When  $\lambda^2 - 4\mu > 0$ ,

$$\begin{aligned}
 u_4(x, t) = & \frac{k^3\lambda^2+8k^3\mu+c}{3k} \\
 & +4k^2\lambda\left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right. \\
 & \left.\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right] \\
 & +4k^2\left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right. \\
 & \left.\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right]^2,
 \end{aligned} \tag{25}$$

$$\begin{aligned}
 v_4(x, t) = & \frac{2k(k^3c_2\lambda^2+8k^3c_2\mu-6c_0k^3+4cc_2)}{3c_2^2} \\
 & +\frac{4k^4\lambda}{c_2}\left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right. \\
 & \left.\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right] \\
 & +\frac{4k^4}{c_2}\left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right. \\
 & \left.\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right]^2,
 \end{aligned} \tag{26}$$

$$\begin{aligned}
 w_4(x, t) = & c_0 + c_2\lambda\left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right. \\
 & \left.\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right] \\
 & +c_2\left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right. \\
 & \left.\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right]^2,
 \end{aligned} \tag{27}$$

where  $\xi = k^{\frac{1}{\alpha}}x + c^{\frac{1}{\alpha}}t + \xi_0$ .

When  $\lambda^2 - 4\mu < 0$ ,

$$\begin{aligned}
 u_5(x, t) = & \frac{k^3\lambda^2+8k^3\mu+c}{3k} \\
 & +4k^2\lambda\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\right. \\
 & \left.\left(\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right] \\
 & +4k^2\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\right. \\
 & \left.\left(\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right]^2,
 \end{aligned} \tag{28}$$

$$\begin{aligned}
 v_5(x, t) = & \frac{2k(k^3c_2\lambda^2+8k^3c_2\mu-6c_0k^3+4cc_2)}{3c_2^2} \\
 & +\frac{4k^4\lambda}{c_2}\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\right. \\
 & \left.\left(\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right] \\
 & +\frac{4k^4}{c_2}\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\right. \\
 & \left.\left(\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right]^2,
 \end{aligned} \tag{29}$$

$$\begin{aligned}
 w_5(x, t) = & c_0 \\
 & +c_2\lambda\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\right. \\
 & \left.\left(\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right] \\
 & +c_2\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\right. \\
 & \left.\left(\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right]^2,
 \end{aligned} \tag{30}$$

where  $\xi = k^{\frac{1}{\alpha}}x + c^{\frac{1}{\alpha}}t + \xi_0$ .

When  $\lambda^2 - 4\mu = 0$ ,

$$\begin{aligned}
 u_6(x, t) = & \frac{k^3\lambda^2+8k^3\mu+c}{3k} \\
 & +4k^2\lambda\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right] \\
 & +4k^2\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right]^2,
 \end{aligned} \tag{31}$$

$$\begin{aligned}
 v_6(x, t) = & \frac{2k(k^3c_2\lambda^2+8k^3c_2\mu-6c_0k^3+4cc_2)}{3c_2^2} \\
 & +\frac{4k^4\lambda}{c_2}\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right] \\
 & +\frac{4k^4}{c_2}\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right]^2,
 \end{aligned} \tag{32}$$

$$\begin{aligned}
 w_6(x, t) = & c_0 \\
 & +c_2\lambda\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right] \\
 & +c_2\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right]^2,
 \end{aligned} \tag{33}$$

where  $\xi = k^{\frac{1}{\alpha}}x + c^{\frac{1}{\alpha}}t + \xi_0$ .

**Remark 2** As a different method from [20] is used here, the established solutions above for the space-time fractional generalized Hirota-Satsuma coupled KDV equations are essentially different from the results in [20], and furthermore, are new exact solutions so far in the literature.

### 4.2 Space-time fractional (2+1)-dimensional breaking soliton equations

We consider the space-time fractional (2+1)-dimensional breaking soliton equations

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} + a \frac{\partial^{3\alpha} u}{\partial x^{2\alpha} \partial y^\alpha} + 4au \frac{\partial^\alpha v}{\partial x^\alpha} + 4a \frac{\partial^\alpha u}{\partial x^\alpha} v = 0, \\ \frac{\partial^\alpha u}{\partial y^\alpha} = \frac{\partial^\alpha v}{\partial x^\alpha}, \end{cases}, \tag{34}$$

where  $0 < \alpha \leq 1$ . (34) is a variation of the (2+1)-dimensional breaking soliton equations equation [30-33]

$$\begin{cases} u_t + au_{xxy} + 4auv_x + 4au_x v = 0, \\ u_y = v_x. \end{cases} \tag{35}$$

For Eqs. (35), some periodic wave solutions, non-traveling wave solutions, and Jacobi elliptic function solutions were found in [19-22]. But we notice so far no research has been paid for Eqs. (34). In the following, we will apply the described fractional sub-equation method in Section 3 to Eqs. (34).

To begin with, we suppose

$$u(x, y, t) = U(\xi), \quad v(x, y, t) = V(\xi),$$

where  $\xi = k_1^{\frac{1}{\alpha}} x + k_2^{\frac{1}{\alpha}} y + c^{\frac{1}{\alpha}} t + \xi_0$ ,  $k_1, k_2, c, \xi_0$  are all constants with  $k_1, k_2, c > 0$ . Then by use of the second equality in Eq. (4), Eqs. (34) can be turned into

$$\begin{cases} cD_\xi^\alpha U + ak_1^2 k_2 D_\xi^{3\alpha} U + 4ak_1 U D_\xi^\alpha V \\ \quad + 4ak_1 V D_\xi^\alpha U = 0, \\ k_2 D_\xi^\alpha U = k_1 D_\xi^\alpha V. \end{cases} \tag{36}$$

Suppose that the solution of Eqs. (36) can be expressed by

$$\begin{cases} U(\xi) = \sum_{i=0}^{m_1} a_i \left(\frac{D_\xi^\alpha G}{G}\right)^i, \\ V(\xi) = \sum_{i=0}^{m_2} b_i \left(\frac{D_\xi^\alpha G}{G}\right)^i. \end{cases} \tag{37}$$

Balancing the order of  $D_\xi^{3\alpha} U$  and  $U D_\xi^\alpha V$ ,  $D_\xi^\alpha U$  and  $D_\xi^\alpha V$  in (37) we have  $m_1 = n_1 = 2$ . So

$$\begin{cases} U(\xi) = a_0 + a_1 \left(\frac{D_\xi^\alpha G}{G}\right)^1 + a_2 \left(\frac{D_\xi^\alpha G}{G}\right)^2, \\ V(\xi) = b_0 + b_1 \left(\frac{D_\xi^\alpha G}{G}\right)^1 + b_2 \left(\frac{D_\xi^\alpha G}{G}\right)^2. \end{cases} \tag{38}$$

Substituting (38) into (36), using Eq. (1) and collecting all the terms with the same power of

$\left(\frac{D_\xi^\alpha G}{G}\right)$  together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations yields:

$$\begin{aligned} a_0 &= a_0, \quad a_1 = -\frac{3}{2}k_1^2 \lambda, \quad a_2 = -\frac{3}{2}k_1^2, \\ b_0 &= -\frac{8ak_1^2 k_2 \mu + ak_1^2 k_2 \lambda^2 + c + 4aa_0 k_2}{4ak_1}, \\ b_1 &= -\frac{3}{2}k_1 k_2 \lambda, \quad b_2 = -\frac{3}{2}k_1 k_2, \\ \lambda &= \lambda, \quad \mu = \mu, \end{aligned}$$

where  $\lambda, \mu, a_0$  are arbitrary constants.

Substituting the result above into Eqs. (38), and combining with (7) we can obtain the following exact solutions to Eqs. (34).

When  $\lambda^2 - 4\mu > 0$ ,

$$\begin{aligned} u_1(x, y, t) &= a_0 \\ &- \frac{3}{2}k_1^2 \lambda \left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}\right] \\ &\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha}\right) \\ &- \frac{3}{2}k_1^2 \left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}\right] \\ &\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha}\right)^2, \end{aligned} \tag{39}$$

$$\begin{aligned} v_1(x, y, t) &= -\frac{8ak_1^2 k_2 \mu + ak_1^2 k_2 \lambda^2 + c + 4aa_0 k_2}{4ak_1} \\ &- \frac{3}{2}k_1 k_2 \lambda \left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}\right] \\ &\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha}\right) \\ &- \frac{3}{2}k_1 k_2 \left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}\right] \\ &\left(\frac{C_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha}{C_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} \xi^\alpha}\right)^2, \end{aligned} \tag{40}$$

where  $\xi = k_1^{\frac{1}{\alpha}} x + k_2^{\frac{1}{\alpha}} y + c^{\frac{1}{\alpha}} t + \xi_0$ .

When  $\lambda^2 - 4\mu < 0$ ,

$$\begin{aligned} u_2(x, y, t) &= a_0 \\ &- \frac{3}{2}k_1^2 \lambda \left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2}\right] \\ &\left(\frac{-C_1 \sin \frac{\sqrt{4\mu - \lambda^2}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \cos \frac{\sqrt{4\mu - \lambda^2}}{2\Gamma(1+\alpha)} \xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu - \lambda^2}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \sin \frac{\sqrt{4\mu - \lambda^2}}{2\Gamma(1+\alpha)} \xi^\alpha}\right) \\ &- \frac{3}{2}k_1^2 \left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2}\right] \\ &\left(\frac{-C_1 \sin \frac{\sqrt{4\mu - \lambda^2}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \cos \frac{\sqrt{4\mu - \lambda^2}}{2\Gamma(1+\alpha)} \xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu - \lambda^2}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \sin \frac{\sqrt{4\mu - \lambda^2}}{2\Gamma(1+\alpha)} \xi^\alpha}\right)^2, \end{aligned} \tag{41}$$

$$\begin{aligned}
 v_2(x, y, t) = & -\frac{8ak_1^2k_2\mu+ak_1^2k_2\lambda^2+c+4aa_0k_2}{4ak_1} \\
 & -\frac{3}{2}k_1k_2\lambda\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\right. \\
 & \left. \left(\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right] \\
 & -\frac{3}{2}k_1k_2\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\right. \\
 & \left. \left(\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)\right]^2,
 \end{aligned} \tag{42}$$

where  $\xi = k_1^{\frac{1}{\alpha}}x + k_2^{\frac{1}{\alpha}}y + c^{\frac{1}{\alpha}}t + \xi_0$ .

When  $\lambda^2 - 4\mu = 0$ ,

$$\begin{aligned}
 u_3(x, y, t) = & a_0 - \frac{3}{2}k_1^2\lambda\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right] \\
 & -\frac{3}{2}k_1^2\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right]^2,
 \end{aligned} \tag{43}$$

$$\begin{aligned}
 v_3(x, y, t) = & -\frac{8ak_1^2k_2\mu+ak_1^2k_2\lambda^2+c+4aa_0k_2}{4ak_1} \\
 & -\frac{3}{2}k_1k_2\lambda\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right] \\
 & -\frac{3}{2}k_1k_2\left[-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right]^2,
 \end{aligned} \tag{44}$$

where  $\xi = k_1^{\frac{1}{\alpha}}x + k_2^{\frac{1}{\alpha}}y + c^{\frac{1}{\alpha}}t + \xi_0$ .

**Remark 3** *The established solutions for the space-time fractional (2+1)-dimensional breaking soliton equations are new to our best knowledge.*

## 5 Conclusions

We have proposed a new fractional sub-equation method for solving fractional partial differential equations successfully, which is the fractional version of the known (G'/G) method. As applications, some new exact solutions for the space-time fractional generalized Hirota-Satsuma coupled KDV equations and the space-time fractional (2+1)-dimensional breaking soliton equations have been established. As this method is based on the homogeneous balancing principle, so it can also be applied to other fractional partial differential equations where the homogeneous balancing principle is satisfied.

### References:

[1] A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractional-order differential equations, *Comput. Math. Appl.*, 59, 2010, pp.1326-1336.

[2] Y. Zhou, F. Jiao, J. Li, Existence and uniqueness for p-type fractional neutral differential equations, *Nonlinear Anal.*, 71, 2009 pp.2724-2733.

[3] L. Galeone, R. Garrappa, Explicit methods for fractional differential equations and their stability properties, *J. Comput. Appl. Math.*, 228, 2009, pp.548-560.

[4] J. C. Trigeassou, N. Maamri, J. Sabatier, A. Oustaloup, A Lyapunov approach to the stability of fractional differential equations, *Signal Process.*, 91, 2011, pp.437-445.

[5] W. Deng, Smoothness and stability of the solutions for nonlinear fractional differential equations, *Nonlinear Anal.*, 72, 2010, pp.1768-1777.

[6] F. Ghoreishi, S. Yazdani, An extension of the spectral Tau method for numerical solution of multi-order fractional differential equations with convergence analysis, *Comput. Math. Appl.*, 61, 2011, pp.30-43.

[7] J. T. Edwards, N. J. Ford, A. C. Simpson, The numerical solution of linear multi-term fractional differential equations: systems of equations, *J. Comput. Appl. Math.*, 148, 2002, pp.401-418.

[8] M. Muslim, Existence and approximation of solutions to fractional differential equations, *Math. Comput. Model.*, 49, 2009, pp.1164-1172.

[9] A. M. A. El-Sayed, M. Gaber, The Adomian decomposition method for solving partial differential equations of fractal order in finite domains, *Phys. Lett. A*, 359, 2006, pp.175-182.

[10] A. M. A. El-Sayed, S. H. Behiry, W. E. Raslan, Adomian's decomposition method for solving an intermediate fractional advection-dispersion equation, *Comput. Math. Appl.*, 59, 2010, pp.1759-1765.

[11] J. H. He, A new approach to nonlinear partial differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, 2, 1997, pp.230-235.

[12] G. Wu, E. W. M. Lee, Fractional Variational Iteration Method And Its Application, *Phys. Lett. A*, 374, 2010, pp.2506-2509.

[13] S. Guo, L. Mei, The fractional variational iteration method using He's polynomials, *Phys. Lett. A*, 375, 2011, pp.309-313.

[14] J. H. He, Homotopy perturbation technique, *Comput. Methods Appl. Mech. Engrg.*, 178, 1999, pp.257-262.

- [15] J. H. He, A coupling method of homotopy technique and a perturbation technique for non-linear problems, *Inter. J. Non-Linear Mech.*, 35, 2000, pp.37-43.
- [16] Z. Odibat, S. Momani, Fractional Green function for linear time-fractional equations of fractional order, *Appl. Math. Lett.*, 21, 2008, pp.194-199.
- [17] M. Cui, Compact finite difference method for the fractional diffusion equation, *J. Comput. Phys.*, 228, 2009, pp.7792-7804.
- [18] Q. Huang, G. Huang, H. Zhan, A finite element solution for the fractional advection-dispersion equation, *Adv. Water Resour.*, 31, 2008, pp.1578-1589.
- [19] S. Zhang, H. Q. Zhang, Fractional sub-equation method and its applications to nonlinear fractional PDEs, *Phys. Lett. A*, 375, 2011, pp.1069-1073.
- [20] S. M. Guo, L. Q. Mei, Y. Li, Y. F. Sun, The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics, *Phys. Lett. A*, 376, 2012, pp.407-411.
- [21] B. Lu, Bäcklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations, *Phys. Lett. A*, 376, 2012, pp.2045-2048.
- [22] G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results, *Comput. Math. Appl.*, 51, 2006, pp.1367-1376.
- [23] M. L. Wang, X. Z. Li, J. L. Zhang, The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Phys. Lett. A*, 372, 2008, pp.417-423.
- [24] E. M. E. Zayed, M. Abdelaziz, Exact traveling wave solutions of nonlinear variable coefficients evolution equations with forced terms using the generalized (G'/G)-expansion method, *WSEAS Transactions on Mathematics*, 10(3), 2011, pp.115-124.
- [25] M. L. Wang, J. L. Zhang, X. Z. Li, Application of the (G'/G)-expansion to travelling wave solutions of the Broer-Kaup and the approximate long water wave equations. *Appl. Math. Comput.*, 206, 2008, pp.321-326.
- [26] E. M. E. Zayed, A further improved (G'/G)-expansion method and the extended tanh-method for finding exact solutions of nonlinear PDEs, *WSEAS Transactions on Mathematics*, 10(2), 2011, pp.56-64.
- [27] I. Aslan, Discrete exact solutions to some nonlinear differential-difference equations via the (G'/G)-expansion method, *Appl. Math. Comput.*, 215, 2009, pp.3140-3147.
- [28] B. Ayhan, A. Bekir, The (G'/G)-expansion method for the nonlinear lattice equations, *Commun. Nonlinear Sci. Numer. Simulat.*, 17, 2012, pp.3490-3498.
- [29] E.M.E. Zayed, K.A. Gepreel, The modified (G'/G)-expansion method and its applications to construct exact solutions for nonlinear PDEs, *WSEAS Transactions on Mathematics*, 10(8), 2011, pp.270-278.
- [30] S. Zhang, New exact non-traveling wave and coefficient function solutions of the (2+1)-dimensional breaking soliton equations, *Phys. Lett. A*, 368, 2007, pp.470-475.
- [31] Y. Z. Peng, E. V. Krishnan, Two classes of new exact solutions to (2+1)-dimensional breaking soliton equation, *Commun. Theor. Phys. (Beijing, China)*, 44, 2005, pp.807-809.
- [32] Y. J. Ren, S. T. Liu, H. Q. Zhang, On a generalized improved F-expansion method, *Commun. Theor. Phys. (Beijing, China)*, 45, 2006, pp.15-28.
- [33] Y. Chen, B. Li, H. Q. Zhang, Symbolic Computation and Construction of Soliton-Like Solutions to the (2+1)-Dimensional Breaking Soliton Equation, *Commun. Theor. Phys. (Beijing, China)*, 40, 2003, pp.137-142.