On the Determinants and Inverses of Skew Circulant and Skew Left Circulant Matrices with Fibonacci and Lucas Numbers

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Abstract: In this paper, we consider the skew circulant and skew left circulant matrices with the Fibonacci and Lucas numbers. Firstly, we discuss the invertibility of the skew circulant matrix and present the determinant and the inverse matrix by constructing the transformation matrices. Furthermore, the invertibility of the skew left circulant matrices are also discussed. We obtain the determinants and the inverse matrices of the skew left circulant matrices by utilizing the relation between skew left circulant matrices and skew circulant matrix, respectively.

Key–Words: Skew circulant matrix, Skew left circulant matrix, Determinant, Inverse, Fibonacci number, Lucas number

1 Introduction

Skew circulant and circulant matrices have important applications in various disciplines including image processing, communications, signal processing, encoding, solving Toeplitz matrix problems, preconditioner, and solving least squares problems. They have been put on firm basis with the work of P. Davis [1] and Z. L. Jiang [2].

The skew-circulant matrices as pre-conditioners for linear multistep formulae(LMF)-based ordinary differential equations(ODEs) codes, Hermitian and skew-Hermitian Toeplitz systems are considered in [3, 4, 5, 6]. Lyness employed a skew-circulant matrix to construct s-dimensional lattice rules in [7]. Spectral decompositions of skew circulant and skew left circulant matrices are discussed in [8].

The Fibonacci and Lucas sequences are defined by the following recurrence relations, respectively:

$$F_{n+1} = F_n + F_{n-1} \quad \text{where} \quad F_0 = 0, F_1 = 1,$$
$$L_{n+1} = L_n + L_{n-1} \quad \text{where} \quad L_0 = 2, L_1 = 1,$$

for $n \geq 0$: The first few values of the sequences are given by the following table:

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34
L_n	2	1	3	4	7	11	18	29	47	76

The $\{F_n\}$ is given by the formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and the $\{L_n\}$ is given by the formula

$$L_n = \alpha^n + \beta^n,$$

where α and β are the roots of the characteristic equation $x^2 - x - 1 = 0$.

Besides, some scholars have given various algorithms for the determinants and inverses of nonsingular circulant matrices [1]. Unfortunately, the computational complexity of these algorithms are very amazing with the order of matrix increasing. However, Some authors gave the explicit determinants and inverse of circulant and skew-circulant involving Fibonacci and Lucas numbers. For example, D. V. Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [9]. D. A. Lind presented the determinants of circulant and skew-circulant involving Fibonacci numbers [10]. D. Z. Lin gave the determinant of the Fibonacci-Lucas quasi-cyclic matrices in [11]. S. Q. Shen considered circulant matrices with Fibonacci and Lucas numbers and presented their explicit determinants and inverses by constructing the transformation matrices [12].

The purpose of this paper is to obtain the better results for the determinants and inverses of skew circulant and skew left circulant matrices by some perfect properties of Fibonacci and Lucas numbers. In this paper, we adopt the following two conventions $0^0 = 1$, and for any sequence $\{a_n\}$, $\sum_{k=i}^{n} a_k = 0$ in the case $i > n$.

Definition 1. [8] *A skew circulant matrix with the first* *row $(a_1, a_2, \ldots, a_n)$ is meant a square matrix of the form*

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ -a_n & a_1 & \ldots & a_{n-1} \\ -a_{n-1} & -a_n & \ldots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_2 & -a_3 & \ldots & a_1 \end{pmatrix}_{n\times n},$$

denoted by $\mathrm{SCirc}(a_1, a_2, ..., a_n)$.

Definition 2. [8] *A skew left circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$ is meant a square matrix of the form*

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ a_2 & a_3 & \ldots & -a_1 \\ a_3 & a_4 & \ldots & -a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & -a_1 & \ldots & -a_{n-1} \end{pmatrix}_{n\times n},$$

denoted by $\mathrm{SLCirc}(a_1, a_2, ..., a_n)$.

Lemma 3. [8, 1] *Let $A = \mathrm{SCirc}(a_1, a_2, ..., a_n)$ be skew circulant matrix, then we have*

(i) *A is invertible if and only if $f(\omega^k \eta) \neq 0 (k = 0, 1, 2, ..., n-1)$, where $f(x) = \sum_{j=1}^{n} a_j x^{j-1}$, $\omega = \exp(\frac{2\pi i}{n})$, and $\eta = \exp(\frac{\pi i}{n})$;*

(ii) *If A is invertible, then the inverse of A is a skew circulant matrix.*

Lemma 4. *With the orthogonal skew left circulant matrix*

$$\Theta := \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & -1 \\ 0 & 0 & \ldots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & \ldots & 0 & 0 \end{pmatrix}_{n\times n},$$

it holds that

$$\mathrm{SCirc}(a_1, a_2, \ldots, a_n) = \Theta\mathrm{SLCirc}(a_1, a_2, \ldots, a_n).$$

Lemma 5. *If*

$$[\mathrm{SCirc}(a_1, a_2, ..., a_n)]^{-1} = \mathrm{SCirc}(b_1, b_2, ..., b_n),$$

then

$$[\mathrm{SLCirc}(a_1, a_2, ..., a_n)]^{-1} = \mathrm{SLCirc}(b_1, -b_n, ..., -b_2).$$

Proof: Let $B = \mathrm{SCirc}(a_1, a_2, ..., a_n)$, $A' = \mathrm{SLCirc}(a_1, a_2, ..., a_n)$, by Lemma 4, we have $B = \Theta A'$, then $B^{-1} = A'^{-1}\Theta^{-1}$. Thus, we obtain

$$A'^{-1} = B^{-1}\Theta = \mathrm{SLCirc}(b_1, -b_n, ..., -b_2),$$

where $B^{-1} = \mathrm{SCirc}(b_1, b_2, ..., b_n)$. □

2 Determinant and inverse of skew circulant matrix with the Fibonacci numbers

In this section, let $A_n = \mathrm{SCirc}(F_1, F_2, ..., F_n)$ be skew circulant matrix. Firstly, we give a determinant explicit formula for the matrix A_n. Afterwards, we prove that A_n is an invertible matrix for $n \geq 2$, and then we find the inverse of the matrix A_n.

Theorem 6. *Let $A_n = \mathrm{SCirc}(F_1, F_2, ..., F_n)$ be skew circulant matrix, then we have*

$$\det A_n = (1 + F_{n+1})^{n-1} + (-F_n)^{n-2} \sum_{k=1}^{n-1} (-F_k) \left(\frac{1 + F_{n+1}}{-F_n}\right)^{k-1}, \quad (1)$$

where F_n is the nth Fibonacci number.

Proof: Obviously, $\det A_1 = 1$ satisfies the equation (1). In the case $n > 1$, let

$$\Gamma = \begin{pmatrix} 1 & & & & & & & \\ 1 & & & & & & & 1 \\ 1 & & & & & & 1 & -1 \\ 0 & & 0 & & & 1 & -1 & -1 \\ \vdots & & & & \iddots & \iddots & \iddots & \\ 0 & & & 1 & \iddots & \iddots & & \\ 0 & & 1 & -1 & \iddots & & 0 & \\ 0 & 1 & -1 & -1 & & & & \end{pmatrix},$$

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{-F_n}{F_1+F_{n+1}}\right)^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \left(\frac{-F_n}{F_1+F_{n+1}}\right)^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{-F_n}{F_1+F_{n+1}} & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

be two $n \times n$ matrices, then we have

$$\Gamma A_n \Pi_1 = \begin{pmatrix} F_1 & f_n' & b_{13} & \cdots & b_{1,n-1} & b_{1n} \\ 0 & f_n & b_{23} & \cdots & b_{2,n-1} & b_{2n} \\ 0 & 0 & b_{33} & & & \\ 0 & 0 & F_n & \ddots & & \\ \vdots & \vdots & & \ddots & b_{n-1,n-1} & \\ 0 & 0 & & & F_n & b_{nn} \end{pmatrix},$$

where

$$b_{1j} = F_{n-j+2},$$
$$b_{2j} = -F_{n-j+1},$$
$$b_{jj} = F_1 + F_{n+1}, \quad j = 3, 4, \ldots, n,$$
$$f_n = F_1 + F_n + \sum_{k=1}^{n-2}(-F_k)\left(\frac{-F_n}{F_1 + F_{n+1}}\right)^{n-(k+1)},$$

and

$$f_n' = \sum_{k=1}^{n-1} F_{k+1}\left(\frac{-F_n}{F_1 + F_{n+1}}\right)^{n-(k+1)}.$$

So we obtain

$$\begin{aligned}
&\det\Gamma \det A_n \det \Pi_1 \\
&= F_1\left[F_1 + F_n + \sum_{k=1}^{n-2}(-F_k)\left(\frac{-F_n}{F_1 + F_{n+1}}\right)^{n-(k+1)}\right] \\
&\quad \times (F_1 + F_{n+1})^{n-2} \\
&= F_1\left[F_1 + F_{n+1} + \sum_{k=1}^{n-1}(-F_k)\left(\frac{-F_n}{F_1 + F_{n+1}}\right)^{n-(k+1)}\right] \\
&\quad \times (F_1 + F_{n+1})^{n-2} \\
&= (1 + F_{n+1})^{n-1} + (-F_n)^{n-2} \\
&\quad \times \sum_{k=1}^{n-1}(-F_k)\left(\frac{1 + F_{n+1}}{-F_n}\right)^{k-1},
\end{aligned}$$

while

$$\det\Gamma = \det\Pi_1 = (-1)^{\frac{(n-1)(n-2)}{2}},$$

hence, we have

$$\begin{aligned}
\det A_n = {} & (1 + F_{n+1})^{n-1} + \\
& (-F_n)^{n-2}\sum_{k=1}^{n-1}(-F_k)\left(\frac{1 + F_{n+1}}{-F_n}\right)^{k-1}.
\end{aligned}$$

The proof is completed. □

Theorem 7. *Let $A_n = \mathrm{SCirc}(F_1, F_2, ..., F_n)$ be skew circulant matrix, if $n \geq 2$, then A_n is an invertible matrix.*

Proof: When $n = 3$ in Theorem 6, then we have $\det A_3 = 14 \neq 0$, hence A_3 is invertible. In the case $n > 3$, since $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha + \beta = 1, \alpha\beta = -1$, $\omega = \exp(\frac{2\pi i}{n})$ and $\eta = \exp(\frac{\pi i}{n})$, hence we have

$$\begin{aligned}
f(\omega^k\eta) &= \sum_{j=1}^{n} F_j(\omega^k\eta)^{j-1} \\
&= \frac{1}{\alpha - \beta}\sum_{j=1}^{n}(\alpha^j - \beta^j)(\omega^k\eta)^{j-1} \\
&= \frac{1}{\alpha - \beta}\left[\frac{\alpha(1 + \alpha^n)}{1 - \alpha\omega^k\eta} - \frac{\beta(1 + \beta^n)}{1 - \beta\omega^k\eta}\right] \\
&= \frac{1}{\alpha - \beta}\left[\frac{(\alpha - \beta) + (\alpha^{n+1} - \beta^{n+1})}{1 - (\alpha + \beta)\omega^k\eta + \alpha\beta\omega^{2k}\eta^2}\right. \\
&\quad \left. - \frac{\alpha\beta(\alpha^n - \beta^n)\omega^k\eta}{1 - (\alpha + \beta)\omega^k\eta + \alpha\beta\omega^{2k}\eta^2}\right] \\
&= \frac{1 + F_{n+1} + F_n\omega^k\eta}{1 - \omega^k\eta - \omega^{2k}\eta^2} \quad (k = 1, 2, ..., n-1).
\end{aligned}$$

If there exists $\omega^l\eta(l = 1, 2, ..., n-1)$ such that $f(\omega^l\eta) = 0$, then we obtain $1 + F_{n+1} + F_n\omega^l\eta = 0$ for $1 - \omega^l\eta - \omega^{21}\eta^2 \neq 0$, thus, $\omega^l\eta = \frac{1+F_{n+1}}{-F_n}$ is a real number. While

$$\begin{aligned}
\omega^l\eta &= \exp\left(\frac{(2l + 1)\pi i}{n}\right) \\
&= \cos\frac{(2l + 1)\pi}{n} + i\sin\frac{(2l + 1)\pi}{n},
\end{aligned}$$

hence, $\sin\frac{(2l+1)\pi}{n} = 0$, so we have $\omega^l\eta = -1$ for $0 < \frac{(2l+1)\pi}{n} < 2\pi$. But $x = -1$ isn't the root of the equation $1 + F_{n+1} + F_n x = 0$ $(n > 3)$. Hence, we obtain $f(\omega^k\eta) \neq 0$ for any $\omega^k\eta$ $(k = 1, 2, ..., n-1)$, while $f(\eta) = \sum_{j=1}^{n} F_j\eta^{j-1} = \frac{1+F_{n+1}+F_n\eta}{1-\eta-\eta^2} \neq 0$. Hence, by Lemma 3, the conclusion is obtained. □

Lemma 8. *Let the matrix $\mathcal{G} = [g_{i,j}]_{i,j=1}^{n-2}$ be of the form*

$$g_{ij} = \begin{cases} F_1 + F_{n+1}, & i = j, \\ F_n, & i = j + 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then the inverse $\mathcal{G}^{-1} = [g'_{i,j}]_{i,j=1}^{n-2}$ of the matrix $\mathcal{G}$ is equal to

$$g'_{i,j} = \begin{cases} \frac{(-F_n)^{i-j}}{(F_1+F_{n+1})^{i-j+1}}, & i \geq j, \\ 0, & i < j. \end{cases}$$

Proof: Let $c_{ij} = \sum_{k=1}^{n-2} g_{ik}g'_{kj}$. Then $c_{i,j} = 0$ for $i < j$. In the case $i = j$, we obtain

$$c_{ii} = g_{ii}g'_{ii} = (F_1 + F_{n+1}) \cdot \frac{1}{F_1 + F_{n+1}} = 1.$$

For $i \geq j+1$, we obtain

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n-2} g_{ik} g'_{kj} = g_{i,i-1} g'_{i-1,j} + g_{i,i} g'_{i,j} \\ &= F_n \cdot \frac{(-F_n)^{i-j-1}}{(F_1+F_{n+1})^{i-j}} \\ &\quad + (F_1+F_{n+1}) \cdot \frac{(-F_n)^{i-j}}{(F_1+F_{n+1})^{i-j+1}} = 0. \end{aligned}$$

Hence, we verify $\mathcal{G}\mathcal{G}^{-1} = I_{n-2}$, where I_{n-2} is $(n-2)\times(n-2)$ identity matrix. Similarly, we can verify $\mathcal{G}^{-1}\mathcal{G} = I_{n-2}$. Thus, the proof is completed. □

Theorem 9. *Let $A_n = \mathrm{SCirc}(F_1, F_2, ..., F_n)$ be skew circulant matrix, if $n \geq 2$, then we have*

$$A_n^{-1} = \frac{1}{f_n} \mathrm{SCirc}(x'_1, x'_2, \ldots, x'_n),$$

where

$$\begin{aligned} x'_1 &= 1 - \sum_{i=1}^{n-2} \frac{F_{n-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i}, \\ x'_2 &= -1 - \sum_{i=1}^{n-2} \frac{F_{n-1-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i}, \\ x'_k &= -\frac{(-F_n)^{k-3}}{(F_1+F_{n+1})^{k-2}}, \qquad k = 3,4,\ldots,n, \\ f_n &= F_1 + F_n + \sum_{k=1}^{n-2} (-F_k) \left(\frac{-F_n}{F_1+F_{n+1}} \right)^{n-(k+1)}. \end{aligned}$$

Proof: Let

$$\Pi_2 = \begin{pmatrix} 1 & -f'_n & \pi_{13} & \pi_{14} & \cdots & \pi_{1n} \\ 0 & 1 & \frac{F_{n-2}}{f_n} & \frac{F_{n-3}}{f_n} & \cdots & \frac{F_1}{f_n} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

where

$$\pi_{1j} = \frac{-f'_n}{f_n} F_{n-j+1} - F_{n-j+2}, \quad j = 3,4,\ldots,n$$

$$f_n = F_1 + F_n + \sum_{k=1}^{n-2} (-F_k) \left(\frac{-F_n}{F_1+F_{n+1}} \right)^{n-(k+1)},$$

and

$$f'_n = \sum_{k=1}^{n-1} F_{k+1} \left(\frac{-F_n}{F_1+F_{n+1}} \right)^{n-(k+1)}.$$

Then we have

$$\Gamma A_n \Pi_1 \Pi_2 = \mathcal{D}_1 \oplus \mathcal{G},$$

where $\mathcal{D}_1 = \mathrm{diag}(F_1, f_n)$ is a diagonal matrix, and $\mathcal{D}_1 \oplus \mathcal{G}$ is the direct sum of $\mathcal{D}_1$ and $\mathcal{G}$. If we denote $\Pi = \Pi_1 \Pi_2$, then we obtain

$$A_n^{-1} = \Pi(\mathcal{D}_1^{-1} \oplus \mathcal{G}^{-1})\Gamma.$$

Since the last row elements of the matrix Π are $0, 1, \frac{F_{n-2}}{f_n}, \frac{F_{n-3}}{f_n}, ..., \frac{F_2}{f_n}, \frac{F_1}{f_n}$. Hence by Lemma 8 if let $A_n^{-1} = \mathrm{SCirc}(x_1, x_2, ..., x_n)$, then its last row elements are given by the following equations:

$$\begin{aligned} -x_2 &= \frac{1}{f_n} + \frac{1}{f_n} \sum_{i=1}^{n-2} \frac{F_{n-1-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i}, \\ -x_3 &= \frac{1}{f_n} \frac{F_1}{(F_1+F_{n+1})}, \\ -x_4 &= \frac{1}{f_n} \sum_{i=1}^{2} \frac{F_{3-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} - \frac{F_1}{f_n(F_1+F_{n+1})}, \\ -x_5 &= \frac{1}{f_n} \sum_{i=1}^{3} \frac{F_{4-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} - \frac{1}{f_n} \sum_{i=1}^{2} \frac{F_{3-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} \\ &\quad - \frac{F_1}{f_n(F_1+F_{n+1})}, \\ &\vdots \\ -x_n &= \frac{1}{f_n} \sum_{i=1}^{n-2} \frac{F_{n-1-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} \\ &\quad - \frac{1}{f_n} \sum_{i=1}^{n-3} \frac{F_{n-2-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} \\ &\quad - \frac{1}{f_n} \sum_{i=1}^{n-4} \frac{F_{n-3-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i}, \\ x_1 &= \frac{1}{f_n} - \frac{1}{f_n} \sum_{i=1}^{n-2} \frac{F_{n-1-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} \\ &\quad - \frac{1}{f_n} \sum_{i=1}^{n-3} \frac{F_{n-2-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i}. \end{aligned}$$

Let $C_n^{(j)} = \sum_{i=1}^{j} \frac{F_{j+1-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i}$ $(j = 1, 2, ..., n-2)$, then we have

$$\begin{aligned} C_n^{(2)} - C_n^{(1)} &= -\frac{F_1}{F_1+F_{n+1}} + \sum_{i=1}^{2} \frac{F_{3-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} \\ &= \frac{-F_n}{(F_1+F_{n+1})^2}, \end{aligned}$$

$$\begin{aligned}
&C_n^{(n-2)} + C_n^{(n-3)} \\
&= \sum_{i=1}^{n-2} \frac{F_{n-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} + \sum_{i=1}^{n-3} \frac{F_{n-i-2}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} \\
&= \frac{F_1(-F_n)^{n-3}}{(F_1+F_{n+1})^{n-2}} + \sum_{i=1}^{n-3} \frac{F_{n-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} \\
&= \sum_{i=1}^{n-2} \frac{F_{n-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i},
\end{aligned}$$

and

$$\begin{aligned}
&C_n^{(j+2)} - C_n^{(j+1)} - C_n^{(j)} \\
&= \sum_{i=1}^{j+2} \frac{F_{j+3-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} \\
&\quad - \sum_{i=1}^{j+1} \frac{F_{j+2-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} - \sum_{i=1}^{j} \frac{F_{j+1-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i} \\
&= \frac{F_2(-F_n)^j}{(F_1+F_{n+1})^{j+1}} + \frac{F_1(-F_n)^{j+1}}{(F_1+F_{n+1})^{j+2}} \\
&\quad - \frac{F_1(-F_n)^j}{(F_1+F_{n+1})^{j+1}} \\
&\quad + \sum_{i=1}^{j} \frac{(F_{j+5-i} - 2F_{j+4-i} + F_{j+2-i})(-F_n)^{i-1}}{(F_1+F_{n+1})^i} \\
&= \frac{(-F_n)^{j+1}}{(F_1+F_{n+1})^{j+2}} \ (j = 1, 2, ..., n-4).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
x_1 &= \frac{1 - C_n^{(n-2)} - C_n^{(n-3)}}{f_n} \\
&= \frac{1}{f_n}\left(1 - \sum_{i=1}^{n-2} \frac{F_{n-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i}\right), \\
x_2 &= -\frac{C_n^{(n-2)} + 1}{f_n} = \frac{1}{f_n}\left(-1 - \sum_{i=1}^{n-2} \frac{F_{n-1-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i}\right), \\
x_3 &= -\frac{C_n^{(1)}}{f_n} = \frac{1}{f_n}\left(-\frac{1}{F_1+F_{n+1}}\right), \\
x_4 &= -\frac{C_n^{(2)} - C_n^{(1)}}{f_n} = \frac{1}{f_n}\left(\frac{F_n}{(F_1+F_{n+1})^2}\right), \\
x_5 &= -\frac{C_n^{(3)} - C_n^{(2)} - C_n^{(1)}}{f_n} = \frac{1}{f_n}\left(-\frac{(-F_n)^2}{(F_1+F_{n+1})^3}\right), \\
&\ \vdots \\
x_n &= -\frac{C_n^{(n-2)} - C_n^{(n-3)} - C_n^{(n-4)}}{f_n} \\
&= \frac{1}{f_n}\left(-\frac{(-F_n)^{n-3}}{(F_1+F_{n+1})^{n-2}}\right),
\end{aligned}$$

and

$$A_n^{-1} = \frac{1}{f_n}\mathrm{SCirc}(x_1', x_2', \ldots, x_n'),$$

where

$$\begin{aligned}
x_1' &= 1 - \sum_{i=1}^{n-2} \frac{F_{n-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i}, \\
x_2' &= -1 - \sum_{i=1}^{n-2} \frac{F_{n-1-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i}, \\
x_k' &= -\frac{(-F_n)^{k-3}}{(F_1+F_{n+1})^{k-2}}, \qquad k = 3, 4, \ldots, n.
\end{aligned}$$

The proof is completed. □

3 Determinant and inverse of skew circulant matrix with the Lucas numbers

In this section, let $B_n = \mathrm{SCirc}(L_1, L_2, \ldots, L_n)$ be skew circulant matrix. Firstly, we give a determinant explicit formula for the matrix B_n. Afterwards, we prove that B_n is an invertible matrix for any positive integer n , and then we find the inverse of the matrix B_n.

Theorem 10. *Let $B_n = \mathrm{SCirc}(L_1, L_2, \ldots, L_n)$ be skew circulant matrix, then we have*

$$\begin{aligned}
\det B_n &= (1 + L_{n+1})^{n-1} + (-L_n - 2)^{n-2} \\
&\times \sum_{k=1}^{n-1} (3L_{k+1} - L_{k+2}) \left(-\frac{1 + L_{n+1}}{2 + L_n}\right)^{k-1}, \quad (2)
\end{aligned}$$

where L_n is the $n$the lucas number.

Proof: Obviously, $\det B_1 = 1$ satisfies the equation (2), When $n > 1$, let

$$\Sigma = \begin{pmatrix}
1 & & & & & & \\
3 & & & & & & 1 \\
1 & & & & & 1 & -1 \\
0 & & 0 & & 1 & -1 & -1 \\
\vdots & & & \iddots & \iddots & \iddots & \\
0 & & 1 & \iddots & \iddots & & \\
0 & 1 & -1 & \iddots & & 0 & \\
0 \ \ 1 & -1 & -1 & & & &
\end{pmatrix},$$

$$\Omega_1 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & (-\frac{L_n+2}{L_1+L_{n+1}})^{n-2} & 0 & \cdots & 0 & 1 \\
0 & (-\frac{L_n+2}{L_1+L_{n+1}})^{n-3} & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -\frac{L_n+2}{L_1+L_{n+1}} & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0
\end{pmatrix},$$

be two $n \times n$ matrices, then we have

$$\Sigma B_n \Omega_1 = \begin{pmatrix} L_1 & l_n' & L_{n-1} & \cdots & L_3 & L_2 \\ 0 & l_n & c_{23} & \cdots & c_{2,n-1} & c_{2n} \\ 0 & 0 & c_{33} & & & \\ 0 & 0 & 2+L_n & \ddots & & \\ \vdots & \vdots & & \ddots & c_{n-1,n-1} & \\ 0 & 0 & & & 2+L_n & c_{nn} \end{pmatrix},$$

where

$$c_{2j} = 3L_{n-j+2} - L_{n-j+3},$$
$$c_{jj} = L_1 + L_{n+1}, \quad j = 3, 4, \ldots, n,$$
$$l_n = L_1 + 3L_n + \sum_{k=1}^{n-2}(3L_{k+1} - L_{k+2}) \times \left(-\frac{L_n+2}{L_1+L_{n+1}}\right)^{n-(k+1)},$$

and

$$l_n' = \sum_{k=1}^{n-1} L_{k+1}\left(-\frac{L_n+2}{L_1+L_{n+1}}\right)^{n-(k+1)}.$$

Hence, we obtain

$$\begin{aligned} &\det\Sigma \det B_n \det\Omega_1 \\ &= L_1\Bigg[L_1 + 3L_n + \sum_{k=1}^{n-2}(3L_{k+1} - L_{k+2}) \times \left(-\frac{L_n+2}{L_1+L_{n+1}}\right)^{n-(k+1)}\Bigg](L_1+L_{n+1})^{n-2} \\ &= L_1\Bigg[L_1 + L_{n+1} + \sum_{k=1}^{n-1}(3L_{k+1} - L_{k+2}) \times \left(-\frac{L_n+2}{1+L_{n+1}}\right)^{n-(k+1)}\Bigg](L_1+L_{n+1})^{n-2} \\ &= (1+L_{n+1})^{n-1} + (-L_n-2)^{n-2} \times \sum_{k=1}^{n-1}(3L_{k+1} - L_{k+2})\left(-\frac{1+L_{n+1}}{2+L_n}\right)^{k-1}, \end{aligned}$$

while

$$\det\Sigma = \det\Omega_1 = (-1)^{\frac{(n-1)(n-2)}{2}}.$$

Thus, we have

$$\det B_n = (1+L_{n+1})^{n-1} + (-L_n-2)^{n-2} \times \sum_{k=1}^{n-1}(3L_{k+1} - L_{k+2})\left(-\frac{1+L_{n+1}}{2+L_n}\right)^{k-1}.$$

The proof is completed. □

Theorem 11. *Let $B_n = \mathrm{SCirc}(L_1, L_2, \ldots, L_n)$ be skew circulant matrix, then B_n is invertible for any positive integer n.*

Proof: Since $L_n = \alpha^n + \beta^n$, where $\alpha+\beta = 2, \alpha\beta = -1$, $\omega = \exp(\frac{2\pi i}{n})$ and $\eta = \exp(\frac{\pi i}{n})$, hence we have

$$\begin{aligned} f(\omega^k\eta) &= \sum_{j=1}^{n} L_j(\omega^k\eta)^{j-1} = \sum_{j=1}^{n}(\alpha^j + \beta^j)(\omega^k\eta)^{j-1} \\ &= \frac{\alpha(1+\alpha^n)}{1-\alpha\omega^k\eta} + \frac{\beta(1+\beta^n)}{1-\beta\omega^k\eta} \\ &= \frac{(\alpha+\beta) + (\alpha^{n+1}+\beta^{n+1})}{1-(\alpha+\beta)\omega^k\eta + \alpha\beta\omega^{2k}\eta^2} \\ &\quad - \frac{\alpha\beta(\alpha^n+\beta^n)\omega^k\eta + 2\alpha\beta\omega^k\eta}{1-(\alpha+\beta)\omega^k\eta + \alpha\beta\omega^{2k}\eta^2} \\ &= \frac{2+L_{n+1} + (2+L_n)\omega^k\eta}{1-2\omega^k\eta - \omega^{2k}\eta^2} \\ &\quad (k = 1, 2, ..., n-1). \end{aligned}$$

If there exists $\omega^l\eta (l = 1, 2, \ldots, n-1)$ such that $f(\omega^l\eta) = 0$, then we obtain $2+L_{n+1}+(2+L_n)\omega^l\eta = 0$ for $1-2\omega^l\eta - \omega^{2l}\eta^2 \neq 0$, thus, $\omega^l\eta = -\frac{2+L_{n+1}}{L_n+2}$ is a real number, While

$$\begin{aligned} \omega^l\eta &= \exp\left(\frac{(2l+1)\pi i}{n}\right) \\ &= \cos\frac{(2l+1)\pi}{n} + i\sin\frac{(2l+1)\pi}{n}. \end{aligned}$$

Hence, $\sin\frac{(2l+1)\pi}{n} = 0$, so we have $\omega^l\eta = -1$ for $0 < \frac{(2l+1)\pi}{n} < 2\pi$. But $x = -1$ isn't the root of the equation $2 + L_{n+1} + (2+L_n)x = 0$ for any positive integer n. Hence, we obtain $f(\omega^k\eta) \neq 0$ for any $\omega^k\eta (k = 1, 2, \ldots, n-1)$, while $f(\eta) = \sum_{j=1}^{n} L_j\eta^{j-1} = \frac{2+L_{n+1}+(2+L_n)\eta}{1-2\eta-\eta^2} \neq 0$. Thus, by Lemma 3, the conclusion is obtained. □

Lemma 12. *Let the matrix $\mathcal{H} = [h_{ij}]_{i,j=1}^{n-2}$ be of the form*

$$h_{ij} = \begin{cases} L_1 + L_{n+1}, & i = j, \\ 2 + L_n, & i = j+1, \\ 0, & \text{otherwise.} \end{cases}$$

then the inverse $\mathcal{H}^{-1} = [h'_{i,j}]_{i,j=1}^{n-2}$ of the matrix $\mathcal{H}$ is equal to

$$h'_{ij} = \begin{cases} \frac{(-L_n-2)^{i-j}}{(L_1+L_{n+1})^{i-j+1}}, & i \geq j, \\ 0, & i < j. \end{cases}$$

Proof: Let $r_{ij} = \sum_{k=1}^{n-2} h_{ik}h'_{kj}$. Obviously, $r_{ij} = 0$ for $i < j$. In the case $i = j$, we obtain

$$r_{ii} = h_{ii}h'_{ii} = (L_1 + L_{n+1}) \cdot \frac{1}{L_1 + L_{n+1}} = 1.$$

For $i \geq j + 1$, we obtain

$$\begin{aligned} r_{ij} &= \sum_{k=1}^{n-2} h_{ik}h'_{kj} = h_{i,i-1}h'_{i-1,j} + h_{ii}h'_{ij} \\ &= (2 + L_n) \cdot \frac{(-L_n - 2)^{i-j-1}}{(L_1 + L_{n+1})^{i-j}} \\ &\quad + (L_1 + L_{n+1}) \cdot \frac{(-L_n - 2)^{i-j}}{(L_1 + L_{n+1})^{i-j+1}} = 0. \end{aligned}$$

Hence, $\mathcal{H}\mathcal{H}^{-1} = I_{n-2}$, where I_{n-2} is $(n-2)\times(n-2)$ identity matrix. Similarly, we can verify $\mathcal{H}^{-1}\mathcal{H} = I_{n-2}$. Thus, the proof is completed. □

Theorem 13. *Let $B_n = \mathrm{SCirc}(L_1, L_2, \ldots, L_n)$ be skew circulant matrix, then we have*

$$B_n^{-1} = \frac{1}{l_n}\mathrm{SCirc}(y'_1, y'_2, \ldots, y'_n),$$

where

$$\begin{aligned} y'_1 &= 1 - \sum_{i=1}^{n-2} \frac{(L_{n+2-i} - 3L_{n+1-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i}, \\ y'_2 &= -3 - \sum_{i=1}^{n-2} \frac{(L_{n+1-i} - 3L_{n-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i}, \\ y'_k &= \frac{5(-L_n - 2)^{k-3}}{(L_1 + L_{n+1})^{k-2}}, \quad k = 3, 4, \ldots, n. \end{aligned}$$

Proof: Let

$$\Omega_2 = \begin{pmatrix} 1 & -l'_n & \omega_{13} & \omega_{14} & \cdots & \omega_{1n} \\ 0 & 1 & \omega_{23} & \omega_{24} & \cdots & \omega_{2n} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

where

$$\begin{aligned} \omega_{1j} &= \frac{l'_n(3L_{n-j+2} - L_{n-j+3})}{l_n} - L_{n-j+2}, \\ \omega_{2j} &= \frac{L_{n-j+3} - 3L_{n-j+2}}{l_n}, \quad j = 3, 4, \ldots, n, \\ l_n &= L_1 + 3L_n + \sum_{k=1}^{n-2}(3L_{k+1} - L_{k+2}) \\ &\quad \times \left(-\frac{L_n + 2}{L_1 + L_{n+1}}\right)^{n-(k+1)}, \end{aligned}$$

and

$$l'_n = \sum_{k=1}^{n-1} L_{k+1}\left(-\frac{L_n + 2}{L_1 + L_{n+1}}\right)^{n-(k+1)}.$$

Then we have

$$\Sigma B_n \Omega_1 \Omega_2 = \mathcal{D}_2 \oplus \mathcal{H},$$

where $\mathcal{D}_2 = \mathrm{diag}(L_1, l_n)$ is a diagonal matrix, and $\mathcal{D}_2 \oplus \mathcal{H}$ is the direct sum of $\mathcal{D}_2$ and $\mathcal{H}$. If we denote $\Omega = \Omega_1\Omega_2$, then we obtain

$$B_n^{-1} = \Omega(\mathcal{D}_2^{-1} \oplus \mathcal{H}^{-1})\Sigma.$$

Since the last row elements of the matrix Ω are $0, 1, \frac{L_n - 3L_{n-1}}{l_n}, \frac{L_{n-1} - 3L_{n-2}}{l_n}, \ldots, \frac{L_3 - 3L_2}{l_n}$. Hence, by Lemma 12, if let $B_n^{-1} = \mathrm{SCirc}(y_1, y_2, \ldots, y_n)$ then its last row elements are given by the following equations:

$$\begin{aligned} -y_2 &= \frac{3}{l_n} + \frac{1}{l_n}\sum_{i=1}^{n-2} \frac{(L_{n+1-i} - 3L_{n-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i}, \\ -y_3 &= \frac{1}{l_n}\frac{L_3 - 3L_2}{(L_1 + L_{n+1})}, \\ -y_4 &= -\frac{1}{l_n}\frac{L_3 - 3L_2}{(L_1 + L_{n+1})} \\ &\quad + \frac{1}{l_n}\sum_{i=1}^{2} \frac{(L_{5-i} - 3L_{4-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i}, \\ -y_5 &= -\frac{1}{l_n}\frac{L_3 - 3L_2}{(L_1 + L_{n+1})} \\ &\quad - \frac{1}{l_n}\sum_{i=1}^{2} \frac{(L_{5-i} - 3L_{4-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \\ &\quad + \frac{1}{l_n}\sum_{i=1}^{3} \frac{(L_{6-i} - 3L_{5-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i}, \\ &\vdots \\ -y_n &= -\frac{1}{l_n}\sum_{i=1}^{n-4} \frac{(L_{n-1-i} - 3L_{n-2-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \\ &\quad - \frac{1}{l_n}\sum_{i=1}^{n-3} \frac{(L_{n-i} - 3L_{n-1-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \\ &\quad + \frac{1}{l_n}\sum_{i=1}^{n-2} \frac{(L_{n+1-i} - 3L_{n-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i}, \\ y_1 &= \frac{1}{l_n} - \frac{1}{l_n}\sum_{i=1}^{n-3} \frac{(L_{n-i} - 3L_{n-1-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \\ &\quad - \frac{1}{l_n}\sum_{i=1}^{n-2} \frac{(L_{n+1-i} - 3L_{n-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i}. \end{aligned}$$

Let

$$D_n^{(j)} = \sum_{i=1}^{j} \frac{(L_{j+3-i} - 3L_{j+2-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i},$$
$$j = 1, 2, \ldots, n-2,$$

then we have

$$\begin{aligned} D_n^{(1)} - D_n^{(2)} &= \frac{L_3 - 3L_2}{L_1 + L_{n+1}} \\ &\quad - \sum_{i=1}^{2} \frac{(L_{5-i} - 3L_{4-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \\ &= -\frac{5(L_n + 2)}{(L_1 + L_{n+1})^2}, \end{aligned}$$

$$\begin{aligned} & D_n^{(n-3)} + D_n^{(n-2)} \\ &= \sum_{i=1}^{n-3} \frac{(L_{n-i} - 3L_{n-i-1})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \\ &\quad + \sum_{i=1}^{n-2} \frac{(L_{n+1-i} - 3L_{n-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \\ &= \frac{(L_4 - 3L_3)(-L_n - 2)^{n-3}}{(L_1 + L_{n+1})^{n-2}} \\ &\quad + \sum_{i=1}^{n-3} \frac{(L_{n+2-i} - 3L_{n+1-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \\ &= \sum_{i=1}^{n-2} \frac{(L_{n+2-i} - 3L_{n+1-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i}, \end{aligned}$$

and

$$\begin{aligned} & D_n^{(j)} + D_n^{(j+1)} - D_n^{(j+2)} \\ &= \sum_{i=1}^{j} \frac{(L_{j+3-i} - 3L_{j+2-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \\ &\quad + \sum_{i=1}^{j+1} \frac{(L_{j+4-i} - 3L_{j+3-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \\ &\quad - \sum_{i=1}^{j+2} \frac{(L_{j+5-i} - 3L_{j+4-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \\ &= \frac{(L_3 - 3L_2)(-L_n - 2)^j}{(L_1 + L_{n+1})^{j+1}} - \frac{(L_4 - 3L_3)(-L_n - 2)^j}{(L_1 + L_{n+1})^{j+1}} \\ &\quad - \frac{(L_3 - 3L_2)(-L_n - 2)^{j+1}}{(L_1 + L_{n+1})^{j+2}} \\ &= \frac{5(-L_n - 2)^{j+1}}{(L_1 + L_{n+1})^{j+2}} \ (j = 1, 2, \ldots, n-4). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} y_1 &= \frac{1 - D_n^{(n-3)} - D_n^{(n-2)}}{l_n} \\ &= \frac{1}{l_n} \left(1 - \sum_{i=1}^{n-2} \frac{(L_{n+2-i} - 3L_{n+1-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \right), \\ y_2 &= -\frac{D_n^{(n-2)} + 3}{l_n} \\ &= \frac{1}{l_n} \left(-3 - \sum_{i=1}^{n-2} \frac{(L_{n+1-i} - 3L_{n-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i} \right), \\ y_3 &= -\frac{D_n^{(1)}}{l_n} = \frac{1}{l_n} \frac{5}{L_1 + L_{n+1}}, \\ y_4 &= \frac{D_n^{(1)} - D_n^{(2)}}{l_n} = \frac{1}{l_n} \frac{5(-L_n - 2)}{(L_1 + L_{n+1})^2}, \\ y_5 &= \frac{D_n^{(1)} + D_n^{(2)} - D_n^{(3)}}{l_n} = \frac{1}{l_n} \frac{5(-L_n - 2)^2}{(L_1 + L_{n+1})^3}, \\ &\vdots \\ y_n &= \frac{D_n^{(n-4)} + D_n^{(n-3)} - D_n^{(n-2)}}{l_n} \\ &= \frac{1}{l_n} \frac{5(-L_n - 2)^{n-3}}{(L_1 + L_{n+1})^{n-2}}, \end{aligned}$$

and

$$B_n^{-1} = \frac{1}{l_n} \mathrm{SCirc}(y_1', y_2', \ldots, y_n'),$$

where

$$\begin{aligned} y_1' &= 1 - \sum_{i=1}^{n-2} \frac{(L_{n+2-i} - 3L_{n+1-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i}, \\ y_2' &= -3 - \sum_{i=1}^{n-2} \frac{(L_{n+1-i} - 3L_{n-i})(-L_n - 2)^{i-1}}{(L_1 + L_{n+1})^i}, \\ y_k' &= \frac{5(-L_n - 2)^{k-3}}{(L_1 + L_{n+1})^{k-2}}, \quad k = 3, 4, \ldots, n. \end{aligned}$$

The proof is completed. □

4 Determinant and inverse of skew left circulant matrix with the Fibonacci numbers

In this section, let $A_n' = \mathrm{SLCirc}(F_1, F_2, ..., F_n)$ be skew left circulant matrix. By using the obtained conclusions in Section 2, we give a determinant explicit formula for the matrix A_n'. Afterwards, we prove that

A'_n is an invertible matrix for any positive integer n. The inverse of the matrix A'_n is also presented.

According to Lemma 4, Lemma 5, Theorem 6, Theorem 7 and Theorem 9, we can obtain the following theorems.

Theorem 14. *Let* $A'_n = \mathrm{SLCirc}(F_1, F_2, ..., F_n)$ *be skew left circulant matrix, then we have*

$$\det A'_n = (-1)^{\frac{n(n-1)}{2}} \Bigg[(1+F_{n+1})^{n-1} + (-F_n)^{n-2} \times \sum_{k=1}^{n-1}(-F_k)\left(\frac{1+F_{n+1}}{-F_n}\right)^{k-1}\Bigg],$$

where F_n *is the* n*th Fibonacci number.*

Theorem 15. *Let* $A'_n = \mathrm{SLCirc}(F_1, F_2, ..., F_n)$ *be skew left circulant matrix, if* $n > 2$*, then* A'_n *is an invertible matrix.*

Theorem 16. *Let* $A'_n = \mathrm{SLCirc}(F_1, F_2, ..., F_n)$ $(n > 2)$ *be skew left circulant matrix, then we have*

$$A'^{-1}_n = \frac{1}{f_n}\mathrm{SLCirc}(x''_1, x''_2, \ldots, x''_n),$$

where

$$f_n = F_1 + F_n + \sum_{k=1}^{n-2}(-F_k)\left(\frac{-F_n}{F_1+F_{n+1}}\right)^{n-(k+1)},$$

$$x''_1 = 1 - \sum_{i=1}^{n-2}\frac{F_{n-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i},$$

$$x''_k = \frac{(-F_n)^{n-k-1}}{(F_1+F_{n+1})^{n-k}}, \quad k = 2, 3, \ldots, n-1,$$

$$x''_n = 1 + \sum_{i=1}^{n-2}\frac{F_{n-1-i}(-F_n)^{i-1}}{(F_1+F_{n+1})^i}.$$

5 Determinant and inverse of skew left circulant matrix with Lucas numbers

In this section, let $B'_n = \mathrm{SLCirc}(L_1, L_2, \ldots, L_n)$ be skew left circulant matrix. By using the obtained conclusions in Section 3, we give a determinant explicit formula for the matrix B'_n. Afterwards, we prove that B'_n is an invertible matrix for any positive integer n, and then we also find the inverse of the matrix B'_n.

According to Lemma 4, Lemma 5, Theorem 10, Theorem 11 and Theorem 13, we can obtain the following results.

Theorem 17. *Let* $B'_n = \mathrm{SLCirc}(L_1, L_2, \ldots, L_n)$ *be skew left circulant matrix, then we have*

$$\det B'_n = (-1)^{\frac{n(n-1)}{2}} \Bigg[(1+L_{n+1})^{n-1} + (-L_n-2)^{n-2} \times \sum_{k=1}^{n-1}(3L_{k+1}-L_{k+2})\left(-\frac{1+L_{n+1}}{2+L_n}\right)^{k-1}\Bigg],$$

where L_n *is the* n*th Lucas number.*

Theorem 18. *Let* $B'_n = \mathrm{SLCirc}(L_1, L_2, \ldots, L_n)$ *be skew left circulant matrix, then* B'_n *is invertible for any positive integer* n*.*

Theorem 19. *Let* $B'_n = \mathrm{SLCirc}(L_1, L_2, \ldots, L_n)$ *be skew left circulant matrix, then we have*

$$B'^{-1}_n = \frac{1}{l_n}\mathrm{SLCirc}(y''_1, y''_2, \ldots, y''_n),$$

where

$$l_n = L_1 + 3L_n + \sum_{k=1}^{n-2}(3L_{k+1}-L_{k+2}) \times \left(-\frac{L_n+2}{L_1+L_{n+1}}\right)^{n-(k+1)},$$

$$y''_1 = 1 - \sum_{i=1}^{n-2}\frac{(L_{n+2-i}-3L_{n+1-i})(-L_n-2)^{i-1}}{(L_1+L_{n+1})^i},$$

$$y''_k = -\frac{5(-L_n-2)^{n-k-1}}{(L_1+L_{n+1})^{n-k}}, \quad k = 2, 3, \ldots, n-1,$$

$$y''_n = 3 + \sum_{i=1}^{n-2}\frac{(L_{n+1-i}-3L_{n-i})(-L_n-2)^{i-1}}{(L_1+L_{n+1})^i}.$$

6 Conclusions

Besides, some scholars have given various algorithms for the determinants and inverses of nonsingular skew circulant matrices [1,2]. For example, the most commonly implemented algorithms for computing the determinant and inverse of nonsingular skew circulant matrix $A = \mathrm{SCirc}(a_1, a_2, ..., a_n)$ are given by the following formulas:

$$\det A = \prod_{j=1}^{n} f(\omega^{j-\frac{1}{2}})$$

and

$$A^{-1} = \mathrm{SCirc}(b_1, b_2, ..., b_n),$$

where $b_s = \frac{1}{n}\sum_{r=1}^{n} f(\omega^{r-\frac{1}{2}})^{-1}(\omega^{r-\frac{1}{2}})^{-(s-1)}$, $s = 1, 2, ..., n$, $f(x) = \sum_{i=1}^{n} a_i x^{i-1}$ and $\omega = \exp(\frac{2\pi i}{n})$.

Unfortunately, the computational complexity of these algorithms are very amazing with the order of matrix increasing. For a general nonsingular skew circulant matrices, its determinants and inverses are hard determined only by its first row. The purpose of this paper is to obtain the explicit determinants and inverses of skew circulant matrices by some perfect properties of Fibonacci and Lucas numbers.

References:

[1] P. J. Davis, *Circulant Matrices,* John Wiley & Sons, New York, 1979.

[2] Z. L. Jiang and Z. X. Zhou, *Circulant Matrices,* Chengdu Technology University Publishing Company, Chengdu, 1999.

[3] D. Bertaccini and M. K. Ng, Skew-circulant preconditioners for systems of LMF-based ODE codes, *Numer. Anal. Appl.* LNCS, 2001, pp. 93–101.

[4] R. Chan, X. Q. Jin, Circulant and skew-circulant preconditioners for skew-Hermitian type Toeplitz systems, *BIT* 31, 1991, pp. 632–646.

[5] R. Chan, M. K. Ng, Toeplitz preconditioners for Hermitian Toeplitz sys- tems, *Linear Algebra Appl.* 190, 1993, pp. 181–208.

[6] T. Huclke, Circulant and skew-circulant matrices for solving Toeplitz matrix problems, *SIAM J. Matrix Anal. Appl.* 13, 1992, pp. 767–777.

[7] J. N. Lyness and T. S ö revik, Four-dimensional lattice rules generated by skew-circulant matrices, *Math. Comput.* 73, 2004, pp. 279–295.

[8] H. Karner and J. Schneid and C. W. Ueberhuber, Spectral decomposition of real circulant matrices, *Linear Algebra Appl.* 367, 2003, pp. 301–311.

[9] D. V. Jaiswal, On determinants involving generalized Fibonacci numbers, *Fibonacci Quart.* 7, 1969, pp. 319–330.

[10] D. A. Lind, A Fibonacci circulant, *Fibonacci Quart.* 8, 1970, pp. 449–455.

[11] D. Z. Lin, *Fibonacci-Lucas quasi-cyclic matrices*, *Fibonacci Quart.* 40, 2002, pp. 280–286.

[12] S. Q. Shen and J. M. Cen and Y. Hao, On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers, *Appl. Math. Comput.* 217, 2011, pp. 9790–9797.