Introduction to the Elliptical Trigonometry Series using two functions *Absolute Elliptic Jes* (*AEjes*) and *Absolute Elliptic Mar* (*AEmar*) of the first form

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Abstract: - The *Elliptical Trigonometry Series* is an original study introduced in mathematical domain, in signal processing and in signal theory; it is a means of representing a periodic signal as a finite or infinite sum of *Absolute Elliptic Jes* (*AEjes*) and *Absolute Elliptic Mar (AEmar)* functions compared to cosine and sine functions in Fourier series. The *Elliptical Trigonometry Series* is more advanced than the Fourier series. The Fourier series is a particular case of the *Elliptical Trigonometry Series* when the value of *AEjes* is equivalent to Cosine and the value of *AEmar* is equivalent to Sine. The new series has many advantages ahead the Fourier series such as we can reduce the number of parameters for a periodic signal formed by the sum of *AEjes* and *AEmar* functions compared to the cosine and sine function in Fourier Series, reduce the circuit size that produce this periodic signal, and reduce the cost of circuits and many other advantages are remarked. In fact, the Elliptical Trigonometry is an original study introduced in Mathematics by the author and it is published by WSEAS journal, and it has enormous applications in mathematics, electronics, signal processing, signal theory and many others domains. This paper emphasizes the importance of this trigonometry in forming what is called the *Elliptical Trigonometry Series.* In fact, this new Series is introduced for electronics applications in order to reduce as possible the circuit size that form a specific signal and therefore reduce the cost, this is not the case of the Fourier series for the same produced signal. Moreover, we can form from only one circuit an infinite number of combined periodic signals which is not the case of the Fourier series in which one circuit can't produce more than one signal.

Key-words: - Elliptical Trigonometry Series, Fourier series, Signal theory, Signal processing, Mathematics, power electronics, Electrical circuit design.

1 Introduction

In mathematics, a Fourier series decomposes periodic functions or periodic signals into the sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines (or complex exponentials) [3-18]. The study of Fourier series is a branch of Fourier analysis.

Early ideas of decomposing a periodic function into the sum of simple oscillating functions date back to the 3rd century BC, when ancient astronomers proposed an empiric model of planetary motions, based on deferent and epicycles.

The Fourier series has many such applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, quantum mechanics, econometrics, thin-walled shell theory, etc. The Fourier series converge to a periodic signal when its number of harmonics tends to infinite [22-23].

In this paper, the author introduced an original study using the Elliptical Trigonometry [1-2]. In fact, the Elliptical trigonometry is also introduced by the author and it is published by WSEAS Journal [1-2]. The main goal of introducing the *Elliptical Trigonometry Series* is to generalize the idea of Fourier series which manipulate two simple functions cosine and sine. The Fourier series is a particular case of the *Elliptical Trigonometry Series* when the value of the function *Absolute Elliptic Jes* (*AEjes*) is equal to cosine and the value of *Absolute Elliptic mar (AEmar)* is equal to sine. In fact, the proposed series has many advantages ahead the Fourier series and it has enormous applications in electronics. So the main advantages are: the number of parameters is reduced, the electronic circuit is reduced and it becomes more efficient, the number of harmonics is also reduced, one circuit has the capability to describe an infinite number of signals by varying the value of some parameters etc…

In this paper, the new concept of the elliptical trigonometry is introduced and few examples are shown and discussed briefly. Figures are drawn and simulated using Matlab.

In the second section, the angular functions are defined, these functions have enormous applications in all domains, and it can be considered as the basis of this trigonometry [1-2]. The definition of the Elliptical trigonometry is presented and discussed briefly in the third section. In the fourth section, a survey on the Elliptical Trigonometric functions is discussed and two different functions are presented and simulated. The *Elliptical Trigonometry Series* is presented in the section 5. In the section 6, an example of the *Elliptical Trigonometry Series* is presented. And finally, a conclusion is presented in the section 6.

2 The angular functions

Angular functions are new mathematical functions that produce a rectangular signal, in which period is function of angles, it is also introduced by the author [20]. Similar to trigonometric functions, the angular functions have the same properties as the precedent, but the difference is that a rectangular signal is obtained instead of a sinusoidal signal [14-16] and moreover, one can change the width of each positive and negative alternate in the same period. This is not the case of any other trigonometric function. In other hand, one can change the frequency, the amplitude and the width of any period of the signal by using the general form of the angular function.
In this section three types of angular functions are presented, they are used in this trigonometry; of course there are more than three types, but in this paper the study is limited to three functions.

2.1 Angular function $ang_x(x)$

The expression of the angular function related to the (ox) axis is defined, for $K \in \mathbb{Z}$, as:

$$ang_x(\beta(x+\gamma)) = \begin{cases} +1 \ for \ (4K-1)\frac{\pi}{2\beta}-\gamma \le x \le (4K+1)\frac{\pi}{2\beta}-\gamma \\ -1 \ for \ (4K+1)\frac{\pi}{2\beta}-\gamma < x < (4K+3)\frac{\pi}{2\beta}-\gamma \end{cases} \quad (1)$$

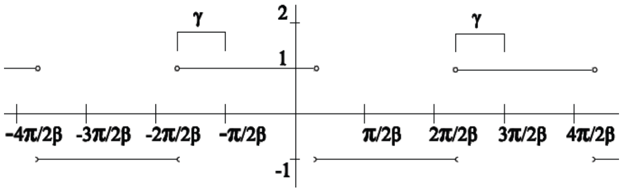


Fig. 1: The $ang_x(\beta(x+\gamma))$ waveform.

For $\beta = 1$ and $\gamma = 0$, the expression of the angular function becomes:

$$ang_x(x) = \begin{cases} +1 \ for \ \cos(x) \ge 0 \\ -1 \ for \ \cos(x) < 0 \end{cases}$$

2.2 Angular function $ang_y(x)$

The expression of the angular function related to the (oy) axis is defined, for $K \in \mathbb{Z}$, as:

$$ang_y(\beta(x+\gamma)) = \begin{cases} +1 \ for \ 2K\pi/\beta-\gamma \le x \le (2K+1)\pi/\beta-\gamma \\ -1 \ for \ (2K+1)\pi/\beta-\gamma < x < (2K+2)\pi/\beta-\gamma \end{cases} \quad (2)$$

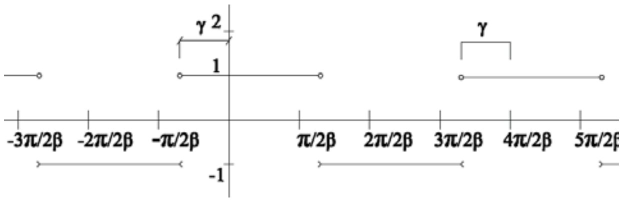


Fig. 2: The $ang_y(\beta(x+\gamma))$ waveform.

For $\beta = 1$ and $\gamma = 0$, the expression of the angular function becomes:

$$ang_y(x) = \begin{cases} +1 \ for \ \sin(x) \ge 0 \\ -1 \ for \ \sin(x) < 0 \end{cases}$$

2.3 Angular function $ang_\alpha(x)$

α (called firing angle) represents the angle width of the positive part of the function in a period. In this case, we can vary the width of the positive and the negative part by varying only α. The firing angle must be positive.

$$ang_\alpha(\beta(x+\gamma)) = \begin{cases} +1 \ for \\ (2K\pi-\alpha)/\beta-\gamma \le x \le (2K\pi+\alpha)/\beta-\gamma \\ \\ -1 \ for \\ (2K\pi+\alpha)/\beta-\gamma < x < (2(K+1)\pi-\alpha)/\beta-\gamma \end{cases} \quad (3)$$

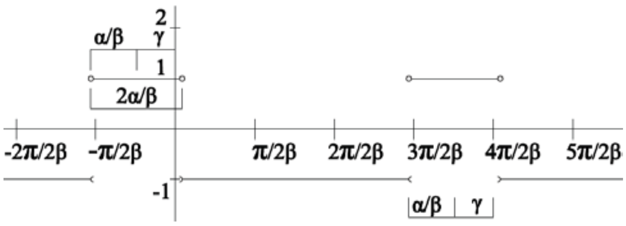


Fig. 3: The $ang_{\alpha}(\beta(x+\gamma))$ waveform.

3 Definition of the Elliptical Trigonometry

3.1 The Elliptical Trigonometry unit

The Elliptical Trigonometry unit is an ellipse with a center O (x = 0, y = 0) and has the equation form:

$$(x/a)^2 + (y/b)^2 = 1 \quad (4)$$

With:
'a' is the radius of the ellipse on the (ox) axis,
'b' is the radius of the ellipse on the (oy) axis.

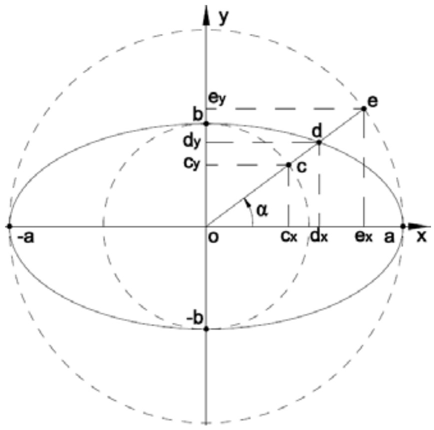


Fig. 4: The elliptical trigonometry unit.

It is essential to note that 'a' and 'b' must be positive. In this paper, 'a' is fixed to 1. One is interested to vary only a single parameter which is 'b'.

3.2 Intersections and projections of different elements of the Elliptical Trigonometry on the relative axes

From the intersections of the ellipse with the positive parts of the axes (ox) and (oy), define respectively two circles of radii $[oa]$ and $[ob]$. These radii can be variable or constant according to the form of the ellipse.

The points of the intersection of the half-line $[od)$ (figure 4) with the internal and external circles and with the rectangle and their projections on the axes (ox) and (oy) can be described by many functions that have an extremely importance in creating plenty of signals and forms that are very difficult to be created in the traditional trigonometry.

Definition of the letters in the Figure 4:

a: Is the intersection of the ellipse with the positive part of the axis (ox) that gives the relative circle of radius "a". It can be variable.

b: Is the intersection of the ellipse with the positive part of the axis (oy) that gives the relative circle of radius "b". It can be variable.

c: Is the intersection of the half-line $[od)$ with the circle of radius b.

d: Is the intersection of the half-line $[od)$ with the ellipse.

e: Is the intersection of the half-line $[od)$ with the circle of radius a.

c_x: Is the projection of the point c on the ox axis.
d_x: Is the projection of the point d on the ox axis.
e_x: Is the projection of the point e on the ox axis.
c_y: Is the projection of the point c on the oy axis.
d_y: Is the projection of the point d on the oy axis.
e_y: Is the projection of the point e on the oy axis.
α: Is the angle between the (ox) axis and the half-line $[od)$.
o: Is the center (0, 0).

3.3 Definition of the Elliptical Trigonometric functions $\boldsymbol{Efun(\alpha)}$

The traditional trigonometry contains only 6 principal functions: Cosine, Sine, Tangent, Cosec, Sec, Cotan. [6], [16], [17]. But in the Elliptical Trigonometry, there are 32 principal functions and each function has its own characteristics. These functions give a new vision of the world and will be used in all scientific domains and make a new challenge in the reconstruction of the science especially when working on the economical side of the power of electrical circuits, the electrical transmission, the signal theory and many other domains [15],[18].

The functions $Cjes(\alpha), Cmar(\alpha), Cter(\alpha)$ and $Cjes_y(\alpha)$, which are respectively equivalent to cosine, sine, tangent and cotangent. These functions are particular cases of the "*Circular Trigonometry*". The names of the cosine, sine, tangent and cotangent are replaced respectively by *Circular Jes*, *Circular Mar*, *Circular Ter* and *Circular Jes-y*.

$Cjes(\alpha) \Leftrightarrow cos(\alpha)$; $Cmar(\alpha) \Leftrightarrow sin(\alpha)$
$Cter(\alpha) \Leftrightarrow tan(\alpha)$; $Cjes_y(\alpha) \Leftrightarrow cotan(\alpha)$.

The Elliptical Trigonometric functions are denoted using the following abbreviation "$Efun(\alpha)$":
-the first letter "E" is related to the Elliptical trigonometry.
-the word "$fun(\alpha)$" represents the specific function name that is defined hereafter: (refer to Figure 4).

• **Elliptical *Jes* functions:**

El. Jes: $Ejes(\alpha) = \frac{od_x}{oa} = \frac{od_x}{oe}$ (5)

El. Jes-x: $Ejes_x(\alpha) = \frac{od_x}{oe_x} = \frac{Ejes(\alpha)}{Cjes(\alpha)}$ (6)

El. Jes-y: $Ejes_y(\alpha) = \frac{od_x}{oe_y} = \frac{Ejes(\alpha)}{Cmar(\alpha)}$ (7)

• **Elliptical *Mar* functions:**

El. Mar: $Emar(\alpha) = \frac{od_y}{ob} = \frac{od_y}{oc}$ (8)

El. Mar-x: $Emar_x(\alpha) = \frac{od_y}{oc_x} = \frac{Emar(\alpha)}{Cjes(\alpha)}$ (9)

El. Mar-y: $Emar_y(\alpha) = \frac{od_y}{oc_y} = \frac{Emar(\alpha)}{Cmar(\alpha)}$ (10)

• **Elliptical *Ter* functions:**

El. Ter: $Eter(\alpha) = \frac{Emar(\alpha)}{Ejes(\alpha)}$ (11)

El. Ter-x:

$Eter_x(\alpha) = \frac{Emar_x(\alpha)}{Ejes_y(\alpha)} = Eter(\alpha) \cdot Cter(\alpha)$ (12)

El. Ter-y: $Eter_y(\alpha) = \frac{Emar_y(\alpha)}{Ejes_x(\alpha)} = \frac{Eter(\alpha)}{Cter(\alpha)}$ (13)

• **Elliptical *Rit* functions:**

El. Rit: $Erit(\alpha) = \frac{od_x}{ob} = \frac{od_x}{oc} = \frac{Emar(\alpha)}{Cter(\alpha)}$ (14)

El. Rit-y: $Erit_y(\alpha) = \frac{od_x}{oc_y} = \frac{Erit(\alpha)}{Cmar(\alpha)}$ (15)

• **Elliptical *Raf* functions:**

El. Raf: $Eraf(\alpha) = \frac{od_y}{oa} = Cter(\alpha).Ejes(\alpha)$ (16)

El. Raf-x: $Eraf_x(\alpha) = \frac{od_y}{oe_x} = \frac{Eraf(\alpha)}{Cjes(\alpha)}$ (17)

• **Elliptical *Ber* functions:**

El. Ber: $Eber(\alpha) = \frac{Eraf(\alpha)}{Erit(\alpha)}$ (18)

El. Ber-x:

$Eber_x(\alpha) = \frac{Eraf_x(\alpha)}{Erit_y(\alpha)} = Eber(\alpha) \cdot Cter(\alpha)$ (19)

El. Ber-y: $Eber_y(\alpha) = \frac{Eraf_y(\alpha)}{Erit_x(\alpha)} = \frac{Eber(\alpha)}{Cter(\alpha)}$ (20)

3.4 The reciprocal of the Elliptical Trigonometric function

$Efun^{-1}(\alpha)$ is defined as the inverse function of $Efun(\alpha)$. $(Efun^{-1}(\alpha) = 1/Efun(\alpha))$. In this way the reduced number of functions is equal to 32 principal functions.

E.g.: $Ejes^{-1}(\alpha) = \frac{1}{Ejes(\alpha)}$

3.5 Definition of the Absolute Elliptical Trigonometric functions $\bar{E}fun(\alpha)$

The *Absolute Elliptical Trigonometry* is introduced to create the absolute value of a function by varying only one parameter without using the absolute value "| |". The advantage is that we can change and control the sign of an Elliptical Trigonometric function without using the absolute value in an expression. Some functions are treated to get an idea about the importance of this new definition. To obtain the Absolute Elliptical Trigonometry for a specified function (e.g., $Ejes(\alpha)$) we must multiply it by the corresponding Angular Function (e.g.,$\left(ang_x(\alpha)\right)^i$) in a way to obtain the original function if i is even, and to obtain the absolute value of the function if i is odd (e.g.$|Ejes(\alpha)|$).
If the function doesn't have a negative part (not alternative), we multiply it by $\left(ang_x(\beta(\alpha-\gamma)\right)^i$ to obtain an alternating signal which form depends on the value of the frequency "β" and the translation value "γ". By varying the last parameters, one can get a multi form signals.

• $\bar{E}jes_i(\alpha) = \left(ang_x(\alpha)\right)^i \cdot Ejes(\alpha)$ (21)

$$= \begin{cases} \left(ang_x(\alpha)\right)^1 \cdot Ejes(\alpha) = |Ejes(\alpha)| & if\ i = 1 \\ \left(ang_x(\alpha)\right)^2 \cdot Ejes(\alpha) = Ejes(\alpha) & if\ i = 2 \end{cases}$$

• $\bar{E}jes_{i,x}(\alpha) = \left(ang_x(\alpha-\gamma)\right)^i \cdot Ejes_x(\alpha)$ (22)

$$= \begin{cases} ang_x(\alpha-\gamma) \cdot Ejes_x(\alpha) & if\ i = 1 \\ Ejes_x(\alpha) & if\ i = 2 \end{cases}$$

$$\bullet\ \bar{E}jes_{i,y}(\alpha) = \left(ang_y(2\alpha)\right)^i \cdot Ejes_y(\alpha) \quad (23)$$

$$= \begin{cases} ang_y(2\alpha) \cdot Ejes_y(\alpha) = |Ejes_y(\alpha)| & if\ i = 1 \\ Ejes_y(\alpha) & if\ i = 2 \end{cases}$$

$$\bullet\ \bar{E}mar_i(\alpha) = \left(ang_y(\alpha)\right)^i \cdot Emar(\alpha) \quad (24)$$

$$\bullet\ \bar{E}mar_{i,x}(\alpha) = \left(ang_y(2\alpha)\right)^i \cdot Emar_x(\alpha) \ (25)$$

$$\bullet\ \bar{E}mar_{i,y}(\alpha) = \left(ang_x(\alpha - \gamma)\right)^i \cdot Emar_y(\alpha) \ (26)$$

$$\bullet\ \bar{E}rit_i(\alpha) = \left(ang_x(\alpha)\right)^i \cdot Erit(\alpha) \quad (27)$$

And so on…

4 A survey on the Elliptical Trigonometric functions

As previous sections, a brief study on the Elliptical Trigonometry is given. Two functions of 32 are treated with examples to show multi form signals made using the characteristic of this trigonometry.
Elliptic cosine and Elliptic sine that appear in the previous articles [1] and [2], are particular cases of the Elliptic *Jes* and Elliptic *Mar* respectively.

For this study the following conditions are taken:

- $a = 1$
- $b > 0$ the height of the rectangle from the center.

4.1 Determination of the Elliptic *Jes* function

The Elliptical form in the figure 4 is written as the equation (4). Thus, given (5), the Elliptical *Jes* function can be determined using following method.
In fact:
$Cter(\alpha) = \frac{y}{x} = \frac{oc_y}{oc_x}$, it is significant to replace the equation $y = Cter(\alpha).x$ in that defined in (4).

$$\left(\frac{x}{a}\right)^2 + \left(Cter(\alpha)\frac{x}{b}\right)^2 = \left(\frac{x}{a}\right)^2\left(1 + \left(Cter(\alpha)\frac{x}{b}\right)^2\right) = 1 \Rightarrow$$

$$Ejes_b(\alpha) = \frac{\pm 1}{\sqrt{1+\left(\frac{a}{b}\mathrm{Cter}\,(\alpha)\right)^2}}$$

Therefore:

$$\bullet Ejes_b(\alpha) = \frac{+1}{\sqrt{1+\left(\frac{a}{b}\mathrm{Cter}\,(\alpha)\right)^2}} \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2};\ \frac{x}{a} \ge 0$$

$$\bullet Ejes_b(\alpha) = \frac{-1}{\sqrt{1+\left(\frac{a}{b}\mathrm{Cter}\,(\alpha)\right)^2}} \text{ for } \frac{\pi}{2} < x < \frac{3\pi}{2};\ \frac{x}{a} < 0$$

Thus the expression of the Elliptic *Jes* can be unified by using the angular function expression (1), therefore the expression becomes:

$$Ejes_b(\alpha) = \frac{ang_x(\alpha)}{\sqrt{1+\left(\frac{a}{b}\mathrm{Cter}\,(\alpha)\right)^2}} \Rightarrow$$

$$Ejes_b(x) = \frac{ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2}} \quad (28)$$

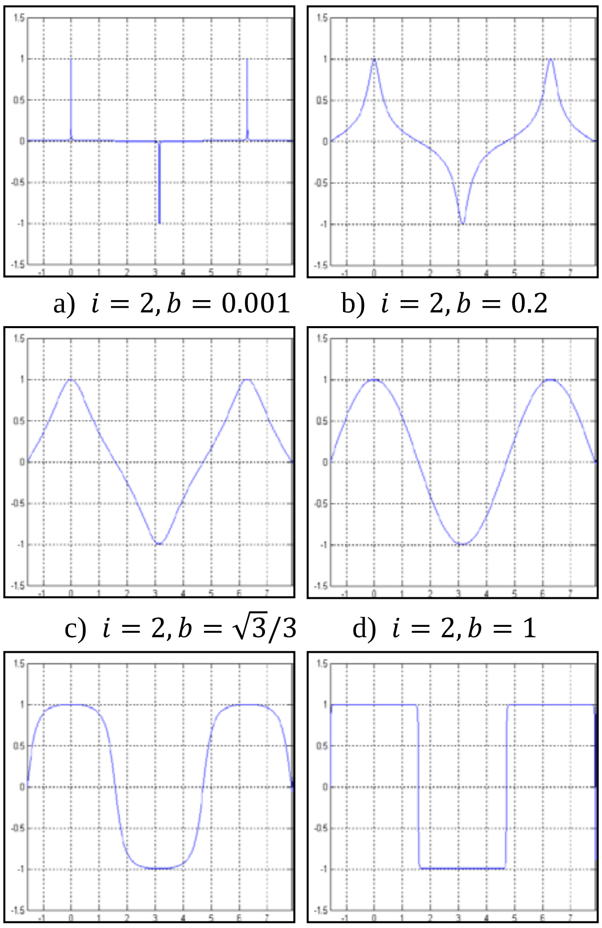
• Expression of the Absolute Elliptic *Jes*:

$$\bar{E}jes_{i,b}(x) = \frac{ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2}} \cdot \left(ang_x(x)\right)^i \ (29)$$

The Absolute Elliptic *Jes* is a powerful function that can produce more than 14 different signals by varying only two parameters i and b. Similar to the cosine function in the traditional trigonometry, the Absolute Elliptical *Jes* is more general than the precedent.

• Multi form signals made by $\bar{E}jes_{i,b}(x)$:

Figures 5 and 6 represent multi form signals obtained by varying two parameters (i and b). For the figures 5.a to 5.f the value of $i = 2$, for the figures 6.a to 6.f the value of $i = 1$.

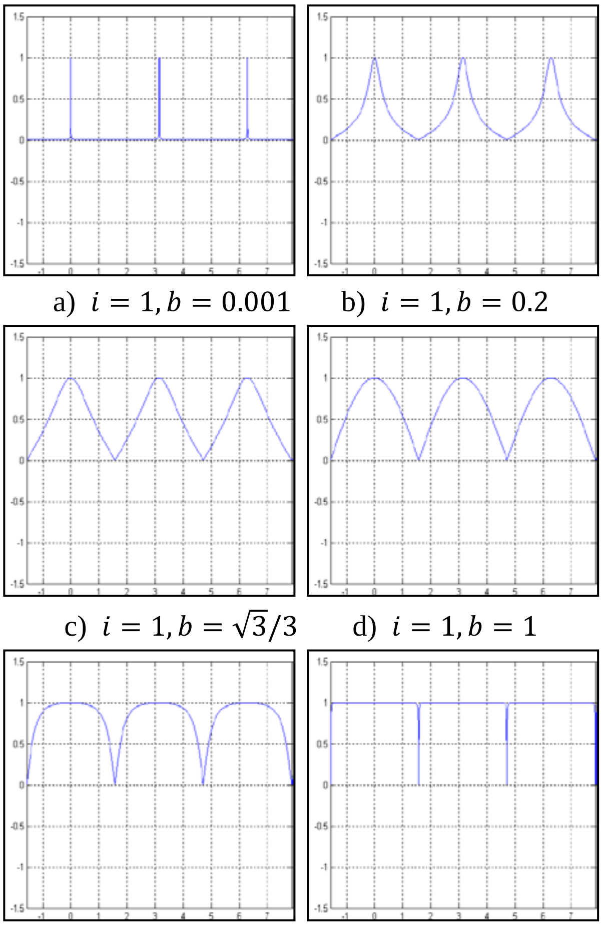


a) $i = 2, b = 0.001$ b) $i = 2, b = 0.2$

c) $i = 2, b = \sqrt{3}/3$ d) $i = 2, b = 1$

e) $i = 2, b = 3$ f) $i = 2, b = 90$

Fig. 5: multi form signals of the function $\bar{E}jes_{i,b}(x)$ for $i = 2$ and for different values of $b > 0$.



a) $i = 1, b = 0.001$ b) $i = 1, b = 0.2$

c) $i = 1, b = \sqrt{3}/3$ d) $i = 1, b = 1$

e) $i = 1, b = 3$ f) $i = 1, b = 90$

Fig. 6: multi form signals of the function $\bar{E}jes_{i,b}(x)$ for $i = 1$ and for different values of $b > 0$.

Important signals obtained using this function: Impulse train with positive and negative part, elliptic deflated, quasi-triangular, sinusoidal, elliptical swollen, square signal, rectangular signal, impulse train (positive part only), rectified elliptic deflated, saw signal, rectified elliptical swollen, continuous signal…

These types of signals are widely used in power electronics, electrical generator and in transmission of analog signals [18].

4.2 The Elliptic *Mar* function

The elliptical form in the figure 4 is written as the equation (4). Thus, given (8), the Elliptical *Mar* function can be determined using following method. In fact:

$Cter(\alpha) = \frac{y}{x} = \frac{oc_y}{oc_x} \Rightarrow x = \frac{y}{Cter(\alpha)}$, it is significant to replace the equation $x = \frac{y}{Cter(\alpha)}$ in that defined in (4). Thus, $(\frac{y}{a \cdot Cter(\alpha)})^2 + (y/b)^2 = 1 \Rightarrow$

$$Emar_b(\alpha) = \frac{y}{b} = \frac{\pm\frac{a}{b}\text{Cter}\,(\alpha)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}\,(\alpha)\right)^2}}$$

$$\Rightarrow Emar_b(x) = \frac{y}{b} = \frac{\pm\frac{a}{b}\text{Cter}\,(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}\,(x)\right)^2}}$$

Therefore:

- $Emar_b(x) = \frac{+\frac{a}{b}\text{Cter}\,(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}\,(x)\right)^2}}$ for $0 \leq x < \frac{\pi}{2}$
- $Emar_b(x) = \frac{-\frac{a}{b}\text{Cter}\,(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}\,(x)\right)^2}}$ for $\frac{\pi}{2} < x \leq \pi$
- $Emar_b(x) = \frac{-\frac{a}{b}\text{Cter}\,(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}\,(x)\right)^2}}$ for $\pi \leq x < 3\frac{\pi}{2}$
- $Emar_b(x) = \frac{+\frac{a}{b}\text{Cter}\,(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}\,(x)\right)^2}}$ for $3\frac{\pi}{2} \leq x \leq 2\pi$

Thus the expression of the Elliptic *Mar* can be unified by using the angular function expression (1), therefore the expression becomes:

$$Emar_b(x) = \frac{a}{b}\frac{\text{Cter}\,(x) ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}\,(x)\right)^2}} \quad (37)$$

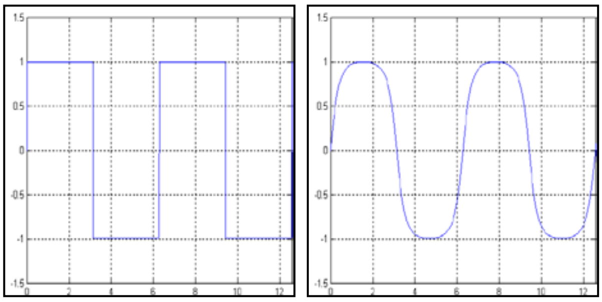
- Expression of the Absolute Elliptic *Mar*:

$$\bar{E}mar_{i,b}(x) = Emar_b(x) \cdot \left(ang_y(x)\right)^i \quad (38)$$

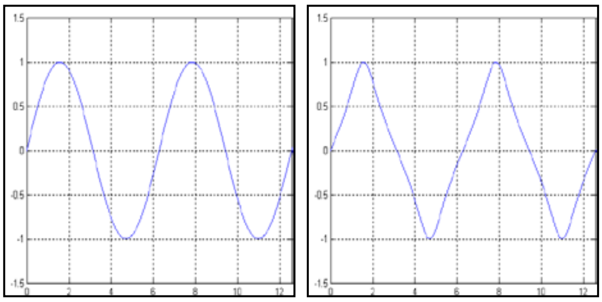
Similar to the Absolute Elliptic *Jes*, the Absolute Elliptic *Mar* is a powerful function that can produce more than 14 different signals by varying only two parameters i and b. Similar to the sine function in the traditional trigonometry, the Absolute Elliptical *Mar* is more general than the precedent.

- Multi form signals made by $\bar{E}mar_{i,b}(x)$:

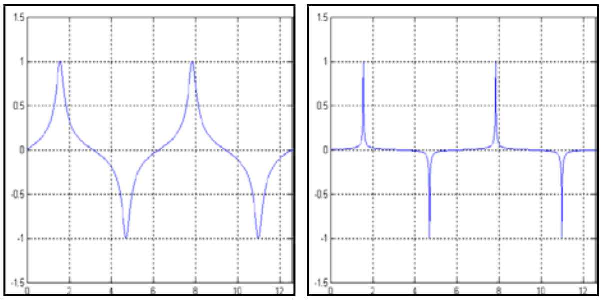
Figures 7 and 8 represent multi form signals obtained by varying two parameters (i and b). For the figures 7.a to 7.f the value of $i = 2$, for the figures 8.a to 8.f the value of $i = 1$.



a) $i = 2; b = 0.001$ b) $i = 2; b = 0.4$

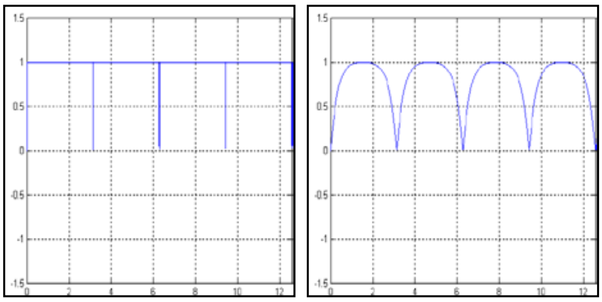


c) $i = 2; b = 1$ d) $i = 2; b = \sqrt{3}$

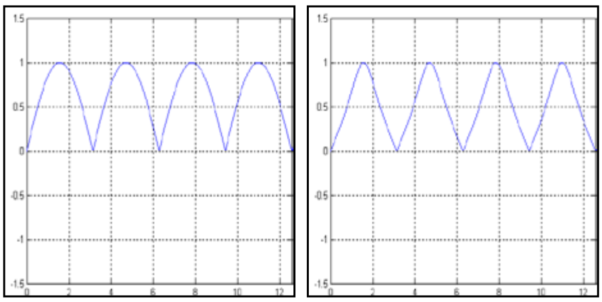


e) $i = 2; b = 6$ f) $i = 2; b = 100$

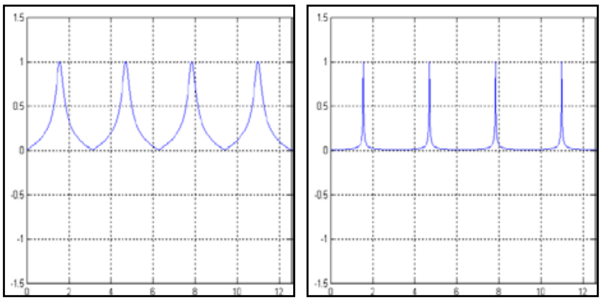
Fig. 7: multi form signals of the function $\bar{E}mar_{i,b}(x)$ for $i = 2$ and for different values of $b > 0$.



a) $i = 1; b = 0.001$ b) $i = 1; b = 0.4$



c) $i = 1; b = 1$ d) $i = 1; b = \sqrt{3}$



e) $i = 1; b = 6$ f) $i = 1; b = 100$

Fig. 8: multi form signals of the function $\bar{E}mar_{i,b}(x)$ for $i = 1$ and for different values of $b > 0$.

4.5 Original formulae of the Elliptical Trigonometry

In this sub-section, a brief review on some remarkable formulae formed using the elliptical trigonometric functions.

- $\left(Ejes_b(x)\right)^2 + \left(Emar_b(x)\right)^2 = 1$ (43)

In fact:

$$\left(Ejes_b(x)\right)^2 + \left(Emar_b(x)\right)^2 = \left(\frac{ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2}}\right)^2 + \left(\frac{a}{b}\frac{\mathrm{Cter}\,(x) ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2}}\right)^2 = \frac{1}{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2} + \left(\frac{a}{b}\right)^2 \frac{\mathrm{Cter}\,(x)^2}{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2} = \frac{1+\left(\frac{a}{b}\right)^2 \mathrm{Cter}\,(x)^2}{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2} = \frac{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2}{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2} = 1$$

- $\frac{1}{Ejes_x^2(x)+Emar_x^2(x)} + \frac{1}{Ejes_y^2(x)+Emar_y^2(x)} = 1$ (44)

In fact:

$$Ejes_x^2(x) + Emar_x^2(x) = \left(\frac{Ejes(x)}{Cjes(x)}\right)^2 + \left(\frac{Emar(x)}{Cjes(x)}\right)^2 = \frac{1}{\left(Cjes(x)\right)^2} \Rightarrow \frac{1}{Ejes_x^2(x)+Emar_x^2(x)} = \left(Cjes(x)\right)^2$$

and

$$Ejes_y^2(x) + Emar_y^2(x) = \left(\frac{Ejes(x)}{Cmar(x)}\right)^2 + \left(\frac{Emar(x)}{Cmar(x)}\right)^2 = \frac{1}{\left(Cmar(x)\right)^2} \Rightarrow \frac{1}{Ejes_y^2(x)+Emar_y^2(x)} = \left(Cmar(x)\right)^2$$

Therefore:

$$\frac{1}{Ejes_x^2(x)+Emar_x^2(x)} + \frac{1}{Ejes_y^2(x)+Emar_y^2(x)} = Cjes^2(x) + Cmar^2(x) = \cos^2(x) + \sin^2(x) = 1$$

5 Elliptical Trigonometry Series

Considering a periodic function $x(t)$ with a period T that verifies the conditions of Dirichlet:

• $x(t)$ has an integrable module on the period T:

$$\int_{-\frac{T}{2}+t_0}^{\frac{T}{2}+t_0} |x(t)|dt \text{ exist.}$$

• $x(t)$ has a boundary condition and its discontinuities are limited in number on a period.

Therefore the general equation of the *Elliptical Trigonometry Series* can be written as the following:

$$x(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\alpha_n \bar{E}jes_{i_n,b_n}(x)\left(\frac{2\pi nt}{T} - \varphi_n\right) + \beta_n \bar{E}mar_{j_n,c_n}\left(\frac{2\pi nt}{T} - \varphi'_n\right)\right) \quad (45)$$

with

$$\alpha_0 = \frac{2}{T}\int_{-\frac{T}{2}+t_0}^{\frac{T}{2}+t_0} x(t)dt$$

If and only if the value of i_n and j_n are even.

In general, α_0, α_n and β_n don't have a specific form because the equation (45) is general. We can obtain α_n and β_n for particular cases. For example: If we have i_n and j_n even and

• $\alpha_n = \frac{2}{T}\int_{-\frac{T}{2}+t_0}^{\frac{T}{2}+t_0} x(t)\bar{E}jes_{i_n,b_n}\left(\frac{2\pi nt}{T} - \varphi_n\right)dt$

$$= \frac{2}{T}\int_{-\frac{T}{2}+t_0}^{\frac{T}{2}+t_0} x(t)cos\left(\frac{2\pi nt}{T} - \varphi_n\right)dt$$

with $a = b = 1$ and $\varphi_n = \varphi'_n$.

• $\beta_n = \frac{2}{T}\int_{-\frac{T}{2}+t_0}^{\frac{T}{2}+t_0} x(t)\bar{E}mar_{j_n,c_n}\left(\frac{2\pi nt}{T} - \varphi'_n\right)dt$

$$= \frac{2}{T}\int_{-\frac{T}{2}+t_0}^{\frac{T}{2}+t_0} x(t)sin\left(\frac{2\pi nt}{T} - \varphi'_n\right)dt$$

with $a = b = 1$ and $\varphi_n = \varphi'_n$

We remark that the particular case can be compared to the Fourier series. We conclude that the Fourier Series is a particular case of the *Elliptical Trigonometry Series* when the values of $a = b = 1$ and $\varphi_n = \varphi'_n = 0$, i_n and j_n are even.

We use the *Elliptical Trigonometry Series* in order to describe periodic signals which are impossible to be described using the Fourier series, and it is used to reduce the number of parameters which imply to reduce the size of the circuit and the cost. For example a rectangular signal is obtained from the Fourier series with infinite summations, infinite multiplications and an infinite number of parameters. But the case will be different for the *Elliptical Trigonometry Series* in which only one function with one parameter can give the rectangular signal (refer to figure 5.f and 12.a), the same for a triangular signal which is obtained from Fourier series using infinite summations, multiplications and an infinite number of parameters, this form of signal can be obtained using only one parameter for the *Elliptical Trigonometry Series* (refer to figure 5.c and 12.d).

In general, a Fourier series can't describe the general form of the *Elliptical Trigonometry Series* but the inverse is correct because the *Elliptical Trigonometry Series* is the general form of the Fourier Series. So any periodic function needs an infinite number of parameters to be described using the Fourier series but it needs less number of parameters by using the *Elliptical Trigonometry Series.*

We have demonstrated that every periodic signal can be decomposed into *AEjes* and *AEmar* functions. The series can be finite or infinite. In electronics we are interested in the finite number of parameters in order to design the correct circuit which is not the case of the Fourier Series.

So in electronics, the design circuit of Fourier series is impossible to be done with an infinite number of parameters, in reality they choose the first sinusoidal signal and many of its harmonics in a limited number, but the disadvantage is that this circuit will not give the exact original signal, the produced signal is distorted with a certain degree or percentage. But this is not the case of the *Elliptical Trigonometry Series*, this series is more practical than the Fourier series because the number of parameters is reduced and the circuit can be simplified at minimum cost.

6 Example using the Elliptical Trigonometry Series

Let's consider the equation (45) and we take α_n and β_n as the following example

$$x(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}\left(\alpha_n \bar{E}jes_{i_n,b_n}(x)\left(\frac{2\pi nt}{T} - \varphi_n\right) + \beta_n \bar{E}mar_{j_n,c_n}\left(\frac{2\pi nt}{T} - \varphi'_n\right)\right) \quad (45)$$

We consider that:

$$i_n = 2, j_n = 2$$

$$\alpha_0 = 0; \alpha_1 = 1 \text{ and } \alpha_n = 0 \text{ for } n \neq 1$$

$$\beta_0 = 0; \beta_1 = 1 \text{ and } \beta_n = 0 \text{ for } n \neq 1$$

$$\varphi_n = \varphi'_n = 0$$

$$\Rightarrow x(t) = \bar{E}jes_{2,b_1}\left(\frac{2\pi t}{T}\right) + \bar{E}mar_{2,c_1}\left(\frac{2\pi t}{T}\right) \quad (46)$$

Therefore the signals formed by varying the parameters "b" and "c" are as following:

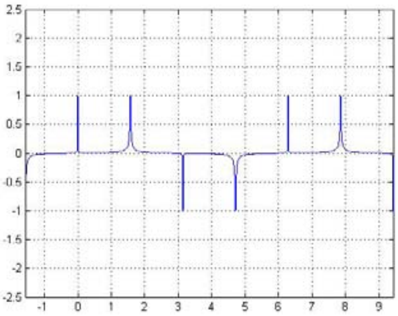


Fig. 9.1: $b = 0.001; c = 100$

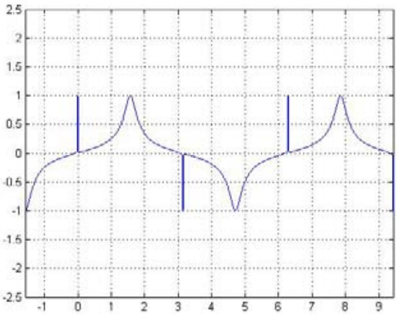


Fig. 9.2: $b = 0.001; c = 6$

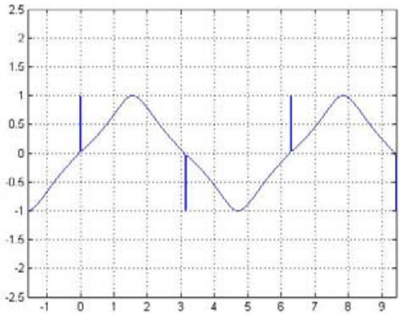


Fig. 9.3: $b = 0.001; c = \sqrt{3}$

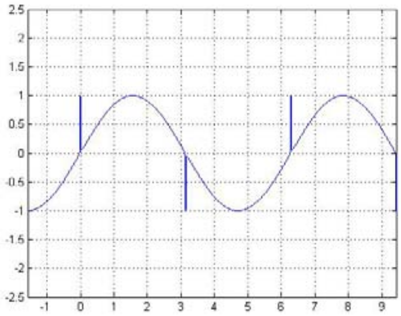


Fig. 9.4: $b = 0.001; c = 1$

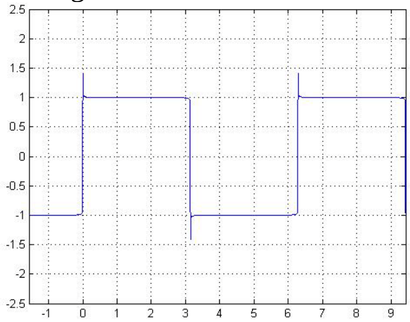


Fig. 9.5: $b = 0.001; c = 1$

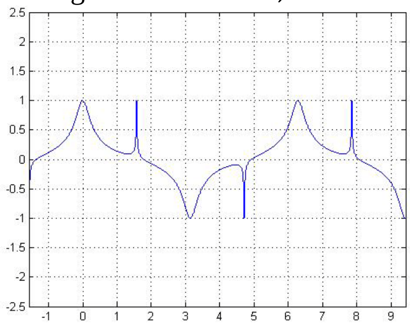


Fig. 9.6: $b = 0.2; c = 100$

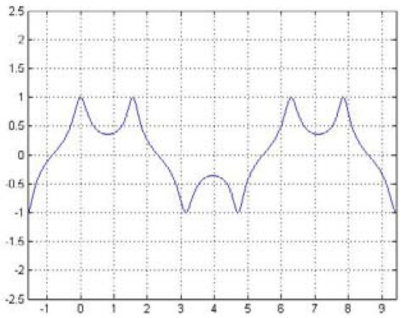


Fig. 9.7: $b = 0.2; c = 6$

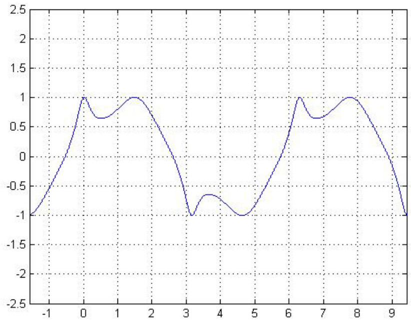


Fig. 9.8: $b = 0.2; c = \sqrt{3}$

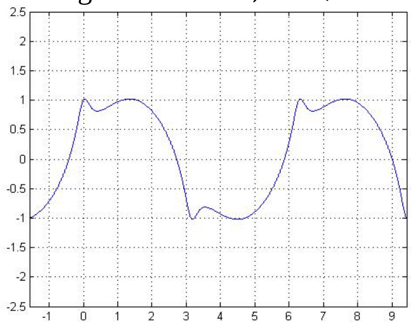


Fig. 9.9: $b = 0.2; c = 1$

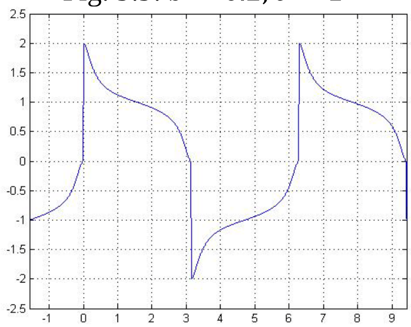


Fig. 9.10: $b = 0.2; c = 0.001$

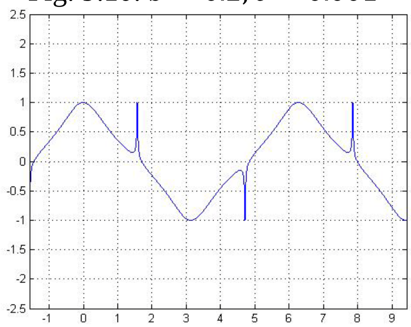


Fig. 9.11: $b = \sqrt{3}/3; c = 100$

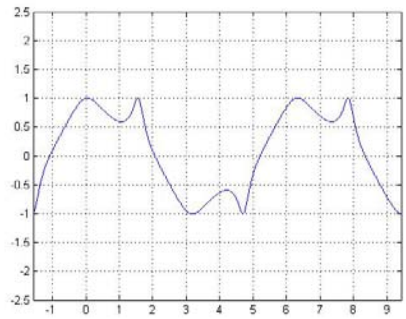


Fig. 9.12: $b = \sqrt{3}/3; c = 6$

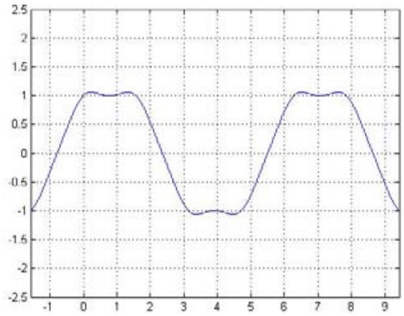


Fig. 9.13: $b = \sqrt{3}/3; c = \sqrt{3}$

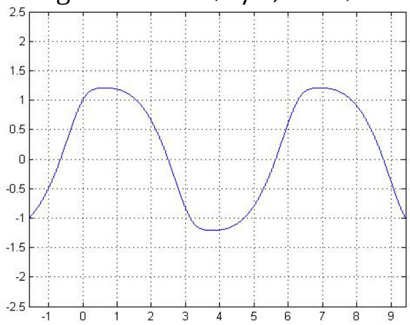


Fig. 9.14: $b = \sqrt{3}/3; c = 1$

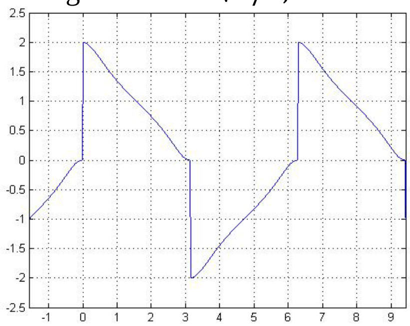


Fig. 9.15: $b = \sqrt{3}/3; c = 0.001$

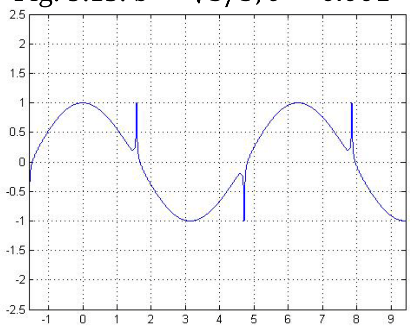


Fig. 9.16: $b = 1; c = 100$

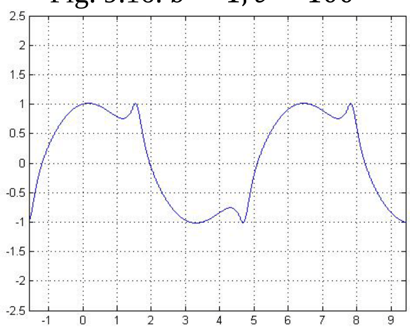


Fig. 9.17: $b = 1; c = 6$

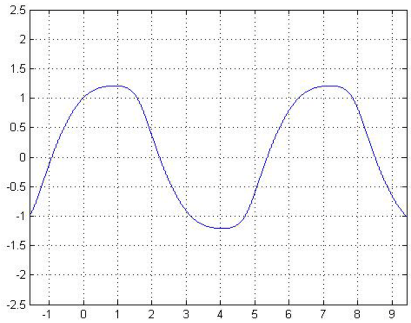


Fig. 9.18: $b = 1; c = \sqrt{3}$

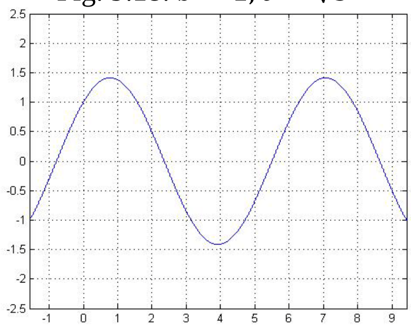


Fig. 9.19: $b = 1; c = 1$

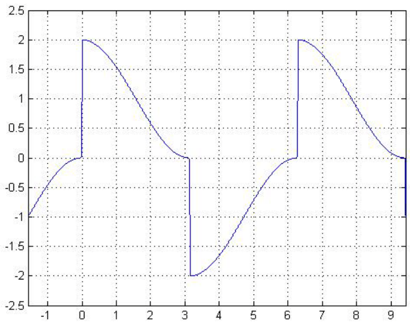


Fig. 9.20: $b = 1; c = 0.001$

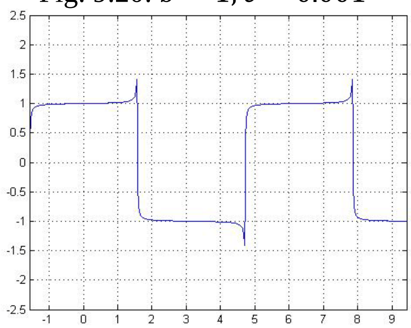


Fig. 9.21: $b = 100; c = 100$

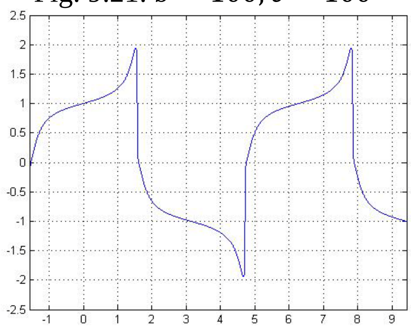


Fig. 9.22: $b = 100; c = 6$

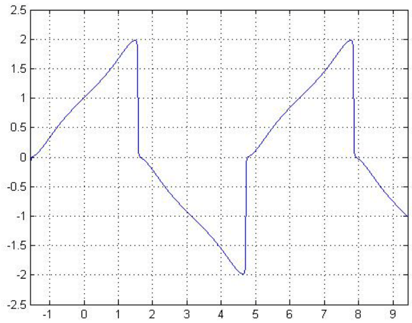


Fig. 9.23: $b = 100; c = \sqrt{3}$

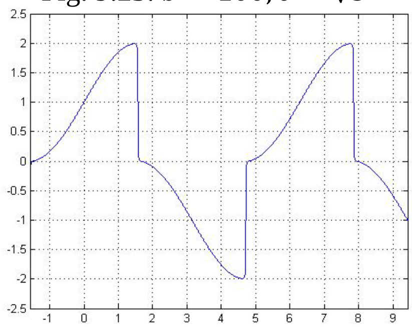


Fig. 9.24: $b = 100; c = 1$

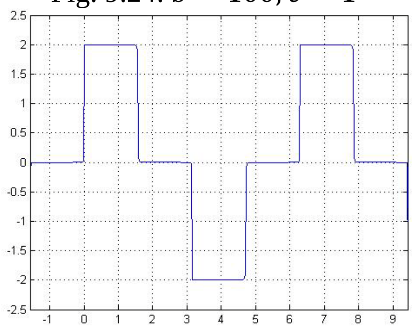


Fig. 9.25: $b = 100; c = 0.001$

We can obtain an infinite number of signals using the equation (46) by varying only two parameters b and c. In this paper, the study is limited to certain number of waveforms in order to give a small idea about the importance of the *Elliptical Trigonometry Series*.

$$x(t) = \bar{E}jes_{2,b_1}\left(\frac{2\pi t}{T}\right) + \bar{E}mar_{2,c_1}\left(\frac{2\pi t}{T}\right) \qquad (46)$$

In fact, if we apply the Fourier series for the equation (46) for a particular case when we have *AEjes*=cosine and *AEmar*=Sine, we obtain the following equation:

$$x(t) = cos\left(\frac{2\pi t}{T}\right) + sin\left(\frac{2\pi t}{T}\right) \qquad (47)$$

And we obtain only the figure 9.19 as following:

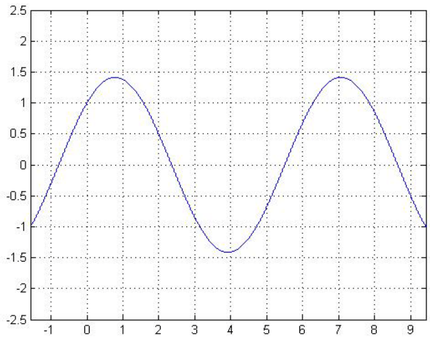


Fig. 9.19: $b = 1; c = 1$

We conclude that the Fourier Series is a particular case of the *Elliptical Trigonometry Series* when we have $b = c = 1$ therefore *AEjes*=cosine and *AEmar*=Sine.

As we see, by using one equation of the *Elliptical Trigonometry Series* we obtain an infinite number of periodic signals (by varying only two parameters) which are very important in electronics and in signal processing. We can imagine if we use more harmonics or we change more variables, then we can describe more important signals by varying limited number of parameters. This is not the case of the Fourier series.

Practically, the Fourier Series is not applicable in electronics when we use an infinite number of parameters to describe a single function. In fact, we choose only the first harmonics to describe the desired signal and this signal has a unique form. But the problem here is that we do not obtain the original signal but we obtain a signal similar to the original one with some distortion because we didn't take all the harmonics of the signal.

The case will be different for the *Elliptical Trigonometry Series* when we can describe the same signal with a limited number of parameters, therefore we can put all these parameters in an electronic circuit, so the original signal is kept as it is, without distortion. And moreover, the signal is variable, so we can obtain more periodic signals by varying some parameters as shown in the previous figures.

7 Conclusion

In this paper, an original study in mathematics is introduced. The *Elliptical Trigonometry Series* is the application of the Elliptical Trigonometry is signal theory and in signal processing. In fact the *Elliptical Trigonometry Series* represents the general case of the Fourier series using two functions *AEjes* instead of cosine and *AEmar* instead of sine. The new series has many advantages ahead the Fourier series. So the main advantages are: the number of parameters is reduced, the electronic circuit is reduced and it becomes more efficient, the number of harmonics is also reduced, one circuit has the capability to describe an infinite number of signals by varying the value of some parameters etc…

As conclusion, as the Elliptical trigonometry is much more complicated than the traditional trigonometry therefore the *Elliptical Trigonometry Series* is also much more complicated than the Fourier series. But this complication gives the new series a huge advantages ahead the Fourier series as we have seen. Many studies will follow this paper in order to find more applications of the new series. Many complicated circuits will be replaced by simplified circuits, and many difficult equations will be replaced by simplified equations.

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