# Efficient algorithms for finding the minimal polynomials and the inverses of level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrices

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*Abstract:* The level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrix over any field is introduced. The diagonalization and spectral decomposition of level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrices over any field are discussed. Algorithms for computing the minimal polynomial of this kind of matrices over any field are presented by means of the algorithm for the Gröbner basis of the ideal in the polynomial ring, and two algorithms for finding the inverses of such matrices are also presented. Finally, an algorithm for the inverse of partitioned matrix with level-$k$ FLS $(r_1, \ldots, r_k)$-circulant blocks over any field is given by using the Schur complement, which can be realized by CoCoA 4.0, an algebraic system, over the field of rational numbers or the field of residue classes of modulo prime number.

*Key–Words:* Level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrix, minimal polynomial, inverse, diagonalization, spectral decomposition, Gröbner basis.

## 1 Introduction

With the development of the mathematics research, multilevel circulant matrix had been defined. And it has been used on network engineering, approximate calculation and Image processing [1-4]. W. F. Trench [5, 6] considered properties of unilevel block circulates and multilevel block $\alpha$-circulates. S. Zhang, Z. Jiang and S. Liu [7] gave Algorithms for the minimal polynomial and the inverse of a level-$n(r_1, r_2, \cdots, r_n)$-block circulant matrix over any field are presented by means of the algorithm for the Gröbner basis for the ideal of the polynomial ring over the field. M. Morhac, V. Matousek [8] presents an efficient algorithm to solve a one-dimensional as well as $n$-dimensional circulant convolution system. M. Rezghi, L. Elden [9] defined tensors with diagonal and circulant structure, and developed framework for the analysis of such tensors. S. Georgiou and C. Koukouvinos [10] presented a new method for constructing multilevel supersaturated designs. Z. Jiang and S. Liu [11] introduced the level-$m$ scaled circulant factor matrix over the complex number field, and discussed its diagonalization and spectral decomposition and representation. A. J. H. Block [12] considered the property of circulates of level-k. J. Baker discussed the structure of multi-block circulates in [13]. More details on multilevel circulant matrix see [14, 15].

This paper is devoted to study the level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrix over any field, and it is organized as follows.

In Section 1, a level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrix over any field is introduced, and its algebraic properties are given. In addition, the diagonalization and spectral decomposition of level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrices over $\mathbb{F}$ are discussed.

In Section 2, we show that the ring of all level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrices over a field is isomorphic to a factor ring of a polynomial ring in $k$ variables over the same field, and then present an algorithm for the minimal polynomial of a level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrix by mean of the algorithm for the Gröbner basis for a kernel of a ring homomorphism.

In Section 3, we give a sufficient and necessary condition to determine whether a level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrix over a field is singular or not and then present an algorithm for finding the inverse of a level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrix over a field.

In Section 4, an algorithm for finding the inverse of partitioned matrix with level-$k$ FLS $(r_1, \ldots, r_k)$-circulant blocks over a field is presented by using the Schur complement and the Buchberger's algorithm.

We first introduce some terminologies and notation used in the sequel. Let $\mathbb{F}$ be a field and

$\mathbb{F}[x_1,\ldots,x_k]$ the polynomial ring in $k$ variables over a field $\mathbb{F}$. By Hilbert Basis Theorem, we know that every ideal $\mathbf{I}$ in $\mathbb{F}[x_1,\ldots,x_k]$ is finitely generated. Fixing a term order in $\mathbb{F}[x_1,\ldots,x_k]$, a set of non-zero polynomials $\mathbf{G}=\{g_1,\ldots,g_t\}$ in an ideal $\mathbf{I}$ is called a Gröbner basis for $\mathbf{I}$ if and only if for all non-zero $f\in\mathbf{I}$, there exists $i\in\{1,\ldots,t\}$ such that $lp(g_i)$ divides $lp(f)$, where $lp(g_i)$ and $lp(f)$ are the leading power products of $g_i$ and $f$, respectively. A Gröbner basis $\mathbf{G}=\{g_1,\ldots,g_t\}$ is called a reduced Gröbner basis if and only if, for all $i$, $lc(g_i)=1$ and $g_i$ is reduced with respect to $\mathbf{G}-\{g_i\}$, that is, for all $i$, no non-zero term in $g_i$ is divisible by any $lp(g_j)$ for any $j\neq i$, where $lc(g_i)$ is the leading coefficient of $g_i$.

In this paper, we set $A^0=I$ for any square matrix $A$, and $<f_1,\ldots,f_m>$ denotes an ideal of $\mathbb{F}[x_1,\ldots,x_k]$ generated by polynomials $f_1,\ldots,f_m$.

## 2 Diagonalization and spectral decomposition of level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrices

We define $\aleph_r$ as the basic FLS $r$-circulant matrix over $\mathbb{F}$, that is,

$$\aleph_r=\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0\\ 0 & 0 & 1 & \ldots & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & 0 & 1\\ r & 1 & 0 & \ldots & 0 & 0 \end{pmatrix}_{n\times n} \qquad (1)$$

It is easily verified that the polynomial $g(x)=x^n-x-r$ is both the minimal polynomial and the characteristic polynomial of the matrix $\aleph_r$ if $r\neq 0$ and $r^{n-1}\neq\frac{(1-n)^{n-1}}{n^n}$. In addition, $\aleph_r$ is nonsingular nonderogatory and

$$\aleph_r^n=rI_n+\aleph_r.$$

Let $\aleph_{r_i}$ be basic FLS $r_i$-circulant matrix over $\mathbb{F}$ and let $I_{n_i}$ be the $n_i\times n_i$ unit matrix for $i=1,2,\ldots,k$ and $N=n_1n_2\ldots n_k$. Set

$$\Pi_i=I_{n_1}\otimes\ldots\otimes I_{n_{i-1}}\otimes\aleph_{r_i}\otimes I_{n_{i+1}}\otimes\ldots\otimes I_{n_k},$$

$r_i\neq 0$ and $r_i^{n-1}\neq\frac{(1-n)^{n-1}}{n^n}$ for $i=1,2,\ldots,k$, where $\otimes$ is a Kronecker product of matrices.

**Definition 1** *An $N\times N$ matrix $A$ over $\mathbb{F}$ is called a $level\text{-}k\ FLS\ (r_1,\ldots,r_k)-circulant\ matrix$ if there exists a polynomial*

$$f(x_1,\ldots,x_k)=\sum_{i_1=0}^{n_1-1}\sum_{i_2=0}^{n_2-1}\ldots\sum_{i_k=0}^{n_k-1}a_{i_1\ldots i_k}x_1^{i_1}\ldots x_k^{i_k}\in\mathbb{F}[x_1,\ldots,x_k]$$

*such that*

$$A=f(\Pi_1,\ldots,\Pi_k)=\sum_{i_1=0}^{n_1-1}\sum_{i_2=0}^{n_2-1}\ldots\sum_{i_k=0}^{n_k-1}a_{i_1\ldots i_k}\Pi_1^{i_1}\ldots\Pi_k^{i_k} \qquad (2)$$

*where the polynomial $f(x_1,\ldots,x_k)$ will be called the representor of a level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrix $A$ and the coefficients $a_{i_1\ldots i_k}, i_j=1,2,\ldots,n_j, j=1,2,\ldots,k$ are just the entries of the first row of $A$.*

Obviously, if $k=1$, then we obtain the FLS $r$-circulant matrix [25].

By the property of the Kronecker product of matrices, the level-$k$ FLS $(r_1,\ldots,r_k)$- circulant matrix $A$ can be also expressed as

$$A=\sum_{i_1=0}^{n_1-1}\sum_{i_2=0}^{n_2-1}\ldots\sum_{i_k=0}^{n_k-1}a_{i_1\ldots i_k}\aleph_{r_1}^{i_1}\otimes\aleph_{r_2}^{i_2}\otimes\ldots\otimes\aleph_{r_k}^{i_k}.$$

For matrix $A$ over $\mathbb{F}$, if $\prod_{i=1}^k r_i\neq 0$, then $A$ is a level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrix if and only if $A$ commutes with the $\aleph_{r_1}\otimes\aleph_{r_2}\otimes\ldots\otimes\aleph_{r_k}$, that is,

$$A(\aleph_{r_1}\otimes\aleph_{r_2}\otimes\ldots\otimes\aleph_{r_k})=(\aleph_{r_1}\otimes\aleph_{r_2}\otimes\ldots\otimes\aleph_{r_k})A.$$

In addition to the algebraic properties that can be easily derived from the representation (2), we mention that level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrices have very nice structure. The product of two level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrices is also a level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrix. Furthermore, level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrices commute under multiplication and $A^{-1}$ is a level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrix, too.

The following we consider diagonalization and spectral decomposition of level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrices over $\mathbb{F}$.

In this section, let $\varepsilon_{i,0},\varepsilon_{i,1},\ldots,\varepsilon_{i,n_i-1}$ be $n_i$ distinct roots of $g_i(x_i)=x_i^{n_i}-x_i-r_i$ in its splitting field over $\mathbb{F}$, and $V_{r_i}=V(\varepsilon_{i,0},\varepsilon_{i,1},\ldots,\varepsilon_{i,n_i-1})$ denotes the Vandermonde matrix of the $\varepsilon_{i,j}$'s. Then

$$V_{r_i}^{-1}\aleph_{r_i}V_{r_i}=D_{g_i}=\mathrm{diag}(\varepsilon_{i,0},\varepsilon_{i,1},\ldots,\varepsilon_{i,n_i-1}), i=1,2,\ldots,k \qquad (3)$$

**Theorem 2** *Let* $A = f(\Pi_1, \ldots, \Pi_k)$ *be a level-*$k$ *FLS* $(r_1, \ldots, r_k)$*-circulant matrix over* $\mathbb{F}$*, where*

$$f(x_1, \ldots, x_k) = \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} \ldots \sum_{i_k=0}^{n_k-1} a_{i_1 \ldots i_k} x_1^{i_1} \ldots x_k^{i_k} \in \mathbb{F}[x_1, \ldots, x_k].$$

*Then*

$$\begin{aligned}&(V_{r_1} \otimes V_{r_2} \otimes \ldots \otimes V_{r_k})^{-1} A (V_{r_1} \otimes V_{r_2} \otimes \ldots \otimes V_{r_k}) \\ &= f(D_{g_1}, D_{g_2}, \ldots, D_{g_k}) \\ &= diag(f(\varepsilon_{1,0}, \varepsilon_{2,0}, \ldots, \varepsilon_{k,0}), \ldots, \\ &\quad \ldots, f(\varepsilon_{1,n_1-1}, \varepsilon_{2,n_2-1}, \ldots, \varepsilon_{k,n_k-1})).\end{aligned}$$

**Proof:** By equations (2), (3) and the property of Kronecker product of matrices, we have

$$\begin{aligned}&(V_{r_1} \otimes V_{r_2} \otimes \ldots \otimes V_{r_k})^{-1} f(\Pi_1, \ldots, \Pi_k) \\ &(V_{r_1} \otimes V_{r_2} \otimes \ldots \otimes V_{r_k}) \\ &= (V_{r_1} \otimes V_{r_2} \otimes \ldots \otimes V_{r_k})^{-1} \\ &(\sum_{i_1=0}^{n_1-1} \ldots \sum_{i_k=0}^{n_k-1} a_{i_1 \ldots i_k} \Pi_1^{i_1} \ldots \Pi_k^{i_k}) \\ &\cdot (V_{r_1} \otimes V_{r_2} \otimes \ldots \otimes V_{r_k}) \\ &= \sum_{i_1=0}^{n_1-1} \ldots \sum_{i_k=0}^{n_k-1} a_{i_1 \ldots i_k} [(V_{r_1} \otimes \ldots \otimes V_{r_k})^{-1} \\ &\cdot (\aleph_{r_1}^{i_1} \otimes \ldots \otimes \aleph_{r_k}^{i_k})(V_{r_1} \otimes \ldots \otimes V_{r_k})] \\ &= \sum_{i_1=0}^{n_1-1} \ldots \sum_{i_k=0}^{n_k-1} a_{i_1 \ldots i_k} (V_{r_1}^{-1} \aleph_{r_1}^{i_1} V_{r_1} \otimes V_{r_2}^{-1} \aleph_{r_2}^{i_2} V_{r_2} \\ &\otimes \ldots \otimes V_{r_k}^{-1} \aleph_{r_k}^{i_k} V_{r_k}) \\ &= \sum_{i_1=0}^{n_1-1} \ldots \sum_{i_k=0}^{n_k-1} a_{i_1 \ldots i_k} (D_{g_1}^{i_1} \otimes D_{g_2}^{i_2} \otimes \ldots \otimes D_{g_k}^{i_k}) \\ &= f(D_{g_1}, D_{g_2}, \ldots, D_{g_k}) \\ &= \mathrm{diag}(f(\varepsilon_{1,0}, \varepsilon_{2,0}, \ldots, \varepsilon_{k,0}), \ldots, \\ &\quad f(\varepsilon_{1,n_1-1}, \varepsilon_{2,n_2-1}, \ldots, \varepsilon_{k,n_k-1})).\end{aligned}$$

□

**Corollary 3** *Let* $A = f(\Pi_1, \ldots, \Pi_k)$ *be a level-*$k$ *FLS* $(r_1, \ldots, r_k)$*-circulant matrix over* $\mathbb{F}$*. Then*

*(a)* $A$ *is a level-*$k$ *FLS* $(r_1, \ldots, r_k)$*-circulant matrix over* $\mathbb{F}$ *if and only if*

$$(V_{r_1} \otimes V_{r_2} \otimes \ldots \otimes V_{r_k})^{-1} A (V_{r_1} \otimes V_{r_2} \otimes \ldots \otimes V_{r_k})$$

*is a diagonal matrix.*

*(b) The eigenvalues of* $A$ *are given by*

$$\lambda_{j_1 j_2 \ldots j_k} = \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} \ldots \sum_{i_k=0}^{n_k-1} a_{i_1 \ldots i_k} \varepsilon_{1,j_1}^{i_1} \varepsilon_{2,j_2}^{i_2} \ldots \varepsilon_{k,j_k}^{i_k},$$

*where* $j_l = 0, 1, 2, \ldots, n_l - 1, l = 1, 2, \ldots, k$.

By equation (3), the basic FLS $r_i$-circulant matrix $\aleph_{r_i}$ over $\mathbb{F}$ is written as

$$\aleph_{r_i} = V_{r_i} D_{g_i} V_{r_i}^{-1}, i = 1, 2, \ldots, k.$$

We now let

$$\Omega_{j_i} = \mathrm{diag}[0, \ldots, 0, 1, 0, \ldots, 0],$$

$j_i = 0, 1, \ldots, n_i - 1, i = 1, 2, \ldots, k$, where 1 occupies the $(j_i + 1)$th entry. Then

$$\aleph_{r_i} = \sum_{j_i=0}^{n_i-1} \varepsilon_{i,j_i} B_{j_i}^{r_i}, i = 1, 2, \ldots, k,$$

where

$$B_{j_i}^{r_i} = V_{r_i} \Omega_{j_i} V_{r_i}^{-1}, j_i = 0, 1, \ldots, n_i - 1, i = 1, 2, \ldots, k,$$

and then $\{B_0^{r_i}, B_1^{r_i}, \ldots, B_{n_i-1}^{r_i}\}$ is the spectral basis, that is,

$$\sum_{j_i=0}^{n_i-1} B_{j_i}^{r_i} = I_{n_i}, B_{j_i}^{r_i} B_{l_i}^{r_i} = \delta_{j_i l_i} B_{j_i}^{r_i},$$

$j_i, l_i = 0, 1, \ldots, n_i - 1, i = 1, 2, \ldots, k$. Moreover, one can easily express the basis $\{I_{n_i}, \aleph_{r_i}, \ldots, \aleph_{r_i}^{n_i-1}\}$ in terms of the basis $\{B_0^{r_i}, B_1^{r_i}, \ldots, B_{n_i-1}^{r_i}\}$ by

$$\aleph_{r_i}^s = \sum_{j_i=0}^{n_i-1} \varepsilon_{i,j_i}^s B_{j_i}^{r_i}, \quad i = 1, 2, \ldots, k.$$

Furthermore, $\{B_{j_1 j_2 \cdots j_k}^{r_1, r_2, \ldots, r_k}, j_i = 0, 1, \ldots, n_i - 1, i = 1, 2, \ldots, k\}$ is the spectral basis, that is,

$$\sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \ldots \sum_{j_k=0}^{n_k-1} B_{j_1 j_2 \cdots j_k}^{r_1, r_2, \ldots, r_k} = I_{n_1 n_2 \ldots n_k},$$

$$B_{j_1 j_2 \cdots j_k}^{r_1, r_2, \ldots, r_k} B_{l_1 l_2 \ldots l_k}^{r_1, r_2, \ldots, r_k} = \delta_{j_1 j_2 \ldots j_k l_1 l_2 \ldots l_k} B_{j_1 j_2 \cdots j_k}^{r_1, r_2, \ldots, r_k},$$

$$j_i, l_i = 0, 1, \ldots, n_i - 1, i = 1, 2, \ldots, k,$$

where

$$B_{j_1 j_2 \cdots j_k}^{r_1, r_2, \ldots, r_k} = B_{j_1}^{r_1} \otimes B_{j_2}^{r_2} \otimes \ldots \otimes B_{j_k}^{r_k}$$

$$= V_{r_1 r_2 \ldots r_k} (\Omega_{j_1} \otimes \Omega_{j_2} \otimes \ldots \otimes \Omega_{j_k}) V_{r_1 r_2 \ldots r_k}^{-1}$$

and

$$V_{r_1 r_2 \ldots r_k} = (V_{r_1} \otimes V_{r_2} \otimes \ldots \otimes V_{r_k}).$$

We summarize our discussion in the following.

**Theorem 4** *A level-k FLS $(r_1,\dots,r_k)$-circulant matrix over $\mathbb{F}$ can be represented as*

$$\begin{aligned} A &= \sum_{i_1=0}^{n_1-1}\sum_{i_2=0}^{n_2-1}\cdots\sum_{i_k=0}^{n_k-1} a_{i_1\dots i_k}\Pi_1^{i_1}\dots\Pi_k^{i_k} \\ &= \sum_{i_1=0}^{n_1-1}\sum_{i_2=0}^{n_2-1}\cdots\sum_{i_k=0}^{n_k-1} a_{i_1\dots i_k}\aleph_{r_1}^{i_1}\otimes\aleph_{r_2}^{i_2}\otimes\dots\otimes\aleph_{r_k}^{i_k} \\ &= V_{r_1r_2\dots r_k}^{-1} f(D_{g_1},D_{g_2},\dots,D_{g_k})V_{r_1r_2\dots r_k} \\ &= \sum_{j_1=0}^{n_1-1}\sum_{j_2=0}^{n_2-1}\cdots\sum_{j_k=0}^{n_k-1} \lambda_{j_1j_2\dots j_k}B_{j_1j_2\dots j_k}^{r_1,r_2,\dots,r_k}, \end{aligned}$$

*where*

$$V_{r_1r_2\dots r_k} = (V_{r_1}\otimes V_{r_2}\otimes\dots\otimes V_{r_k}),$$

$$\lambda_{j_1j_2\dots j_k} = \sum_{i_1=0}^{n_1-1}\sum_{i_2=0}^{n_2-1}\cdots\sum_{i_k=0}^{n_k-1} a_{i_1\dots i_k}\varepsilon_{1,j_1}^{i_1}\varepsilon_{2,j_2}^{i_2}\dots\varepsilon_{k,j_k}^{i_k},$$

*and*

$$B_{j_1j_2\dots j_k}^{r_1,r_2,\dots,r_k} = B_{j_1}^{r_1}\otimes B_{j_2}^{r_2}\otimes\dots\otimes B_{j_k}^{r_k}$$

$$= V_{r_1r_2\dots r_k}(\Omega_{j_1}\otimes\Omega_{j_2}\otimes\dots\otimes\Omega_{j_k})V_{r_1r_2\dots r_k}^{-1}.$$

## 3 Efficient algorithms for finding the minimal polynomials of level-$k$ FLS $(r_1,\dots,r_k)$-circulant matrices

Let

$$\mathbb{F}[\Pi_1,\dots,\Pi_k] = \{A|A = f(\Pi_1,\dots,\Pi_k),$$

$$f(x_1,\dots,x_k)\in\mathbb{F}[x_1,\dots,x_k]\}.$$

It is a routine to prove that $\mathbb{F}[\Pi_1,\dots,\Pi_k]$ is a commutative ring with the matrix addition and multiplication.

**Theorem 5** $\mathbb{F}[x_1,\dots,x_k]/\langle x_1^{n_1}-x_1-r_1,\dots,x_k^{n_k}-x_k-r_k\rangle \cong \mathbb{F}[\Pi_1,\dots,\Pi_k]$.

**Proof:** Consider the following $F$-algebra homomorphism

$$\begin{aligned} \varphi:\mathbb{F}[x_1,\dots,x_k] &\to \mathbb{F}[\Pi_1,\dots,\Pi_k] \\ f(x_1,\dots,x_k) &\mapsto A = f(\Pi_1,\dots,\Pi_k) \end{aligned}$$

for

$$f(x_1,\dots,x_k)\in\mathbb{F}[x_1,\dots,x_k].$$

It is clear that $\varphi$ is an $F$-algebra epimorphism. So we have

$$\mathbb{F}[x_1,\dots,x_k]/\langle\ker\varphi\rangle \cong \mathbb{F}[\Pi_1,\dots,\Pi_k].$$

We can prove that

$$\ker\varphi = \langle x_1^{n_1}-x_1-r_1,\dots,x_k^{n_k}-x_k-r_k\rangle.$$

In fact, for $i=1,2,\dots,k$,

$$x_i^{n_i}-x_i-r_i\in\ker\varphi,$$

because

$$\Pi_i^{n_i}-\Pi_i-r_i=0.$$

Hence

$$\ker\varphi \supseteq \langle x_1^{n_1}-x_1-r_1,\dots,x_k^{n_k}-x_k-r_k\rangle.$$

Conversely, for any $f(x_1,\dots,x_k)\in\ker\varphi$, we have

$$A = f(\Pi_1,\dots,\Pi_k) = 0.$$

Fix the lexicographical order on $\mathbb{F}[x_1,\dots,x_k]$ with $x_1 > x_2 > \dots > x_k$. $x_1^{n_1}-x_1-r_1$ dividing $f(x_1,\dots,x_k)$, there exist

$$u_1(x_1,\dots,x_k), v_1(x_1,\dots,x_k)\in\mathbb{F}[x_1,\dots,x_k]$$

such that

$$f(x_1,\dots,x_k) = u_1(x_1,\dots,x_k)(x_1^{n_1}-x_1-r_1)$$
$$+v_1(x_1,\dots,x_k),$$

where $v_1(x_1,\dots,x_k)=0$ or the largest degree of $x_1$ in $v_1(x_1,\dots,x_k)$ is less than $n_1$. If $v_1(x_1,\dots,x_k)=0$, then

$$f(x_1,\dots,x_k)\in\langle x_1^{n_1}-x_1-r_1,\dots,x_k^{n_k}-x_k-r_k\rangle.$$

Otherwise, $x_2^{n_2}-x_2-r_2$ dividing $v_1(x_1,\dots,x_k)$, there exist

$$u_2(x_1,\dots,x_k), v_2(x_1,\dots,x_k)\in\mathbb{F}[x_1,\dots,x_k]$$

such that

$$v_1(x_1,\dots,x_k) = u_2(x_1,\dots,x_k)(x_2^{n_2}-x_2-r_2)$$
$$+v_2(x_1,\dots,x_k),$$

where $v_2(x_1,\dots,x_k)=0$ or the largest degree of $x_2$ in $v_2(x_1,\dots,x_k)$ is less than $n_2$. If $v_2(x_1,\dots,x_k)=0$, then

$$f(x_1,\dots,x_k)\in\langle x_1^{n_1}-x_1-r_1,\dots,x_k^{n_k}-x_k-r_k\rangle.$$

Otherwise, if the largest degree of $x_1$ in $v_2(x_1,\dots,x_k)$ is less than $n_1$ because $x_1$ does not appear in $x_2^{n_2} - x_2 - r_2$. Continuing this procedure, there exist

$$u_1(x_1,\ldots,x_k), u_2(x_1,\ldots,x_k),\ldots,u_k(x_1,\ldots,x_k),$$
$$v_k(x_1,\ldots,x_k)\in\mathbb{F}[x_1,\ldots,x_k]$$

such that

$$f(x_1,\ldots,x_k)=u_1(x_1,\ldots,x_k)(x_1^{n_1}-x_1-r_1)$$
$$+\ldots+u_k(x_1,\ldots,x_k)(x_k^{n_k}-x_k-r_k)+v_k(x_1,\ldots,x_k),$$

where $v_k(x_1,\ldots,x_k) = 0$ or the degrees of $x_1,x_2,\ldots,x_k$ in $v_k(x_1,\ldots,x_k)$ are less than $n_1,n_2,\ldots,n_k$, respectively. Since

$$f(\Pi_1,\ldots,\Pi_k)=0$$

and

$$\Pi_i^{n_i}-\Pi_i-r_i=0$$

for all $i=1,2,\ldots,k, u_k(\Pi_1,\ldots,\Pi_k)=0$. The coefficients of all terms in $v_k(x_1,\ldots,x_k)$ are the entries of the matrix $v_k(\Pi_1,\ldots,\Pi_k)$ because the degrees of $x_1,x_2,\ldots,x_k$ in $v_k(x_1,\ldots,x_k)$ are less than $n_1,n_2,\ldots,n_k$, respectively. Therefore, the coefficient of each term in $v_k(x_1,\ldots,x_k)$ is 0, i.e., $v_k(x_1,\ldots,x_k)=0$. Thus

$$f(x_1,\ldots,x_k)\in\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k\rangle.$$

**Definition 6** *Let* $\mathbf{I}$ *be a non-zero ideal of the polynomial ring* $\mathbb{F}[y_1,\ldots,y_t]$*. Then* $\mathbf{I}$ *is called an annihilation ideal of square matrices* $A_1,\ldots,A_t$*, denoted by* $\mathbf{I}(A_1,\ldots,A_t)$*, if* $f(A_1,\ldots,A_t)=0$ *for all* $f(y_1,\ldots,y_t)\in\mathbf{I}$.

**Definition 7** *Suppose that* $A_1,\ldots,A_t \in \mathbb{F}[\Pi_1,\ldots,\Pi_k]$ *are not all zero matrices. The unique monic polynomial* $g(x)$ *of minimum degree that simultaneously annihilates* $A_1,\ldots,A_t$ *is called the common minimal polynomial of* $A_1,\ldots,A_t$.

We give the special case of [16, Theorem 2.4.10] here for the convenience of applications.

**Lemma 8** *Let* $\mathbf{I}$ *be an ideal of* $\mathbb{F}[x_1,\ldots,x_k]$*. Given* $f_1,\ldots,f_m\in\mathbb{F}[x_1,\ldots,x_k]$*, consider the following* $F$*- algebra homomorphism*

$$\begin{array}{rcl}\varphi:\mathbb{F}[y_1,\ldots,y_m] & \to & \mathbb{F}[x_1,\ldots,x_k]/\mathbf{I}\\ y_1 & \mapsto & f_1+\mathbf{I}\\ \cdots & \cdots & \cdots\\ y_m & \mapsto & f_m+\mathbf{I}\end{array}$$

*Let* $\mathbf{K}=\langle\mathbf{I},y_1-f_1,\ldots,y_m-f_m\rangle$ *be an ideal of* $\mathbb{F}[x_1,\ldots,x_k,y_1,\ldots,y_m]$ *generated by* $\mathbf{I},y_1-f_1,\ldots,y_m-f_m$*. Then*

$$ker\,\varphi=\mathbf{K}\cap\mathbb{F}[y_1,\ldots,y_m].$$

**Lemma 9** [19] *Let* $A$ *be a non-zero matrix over* $\mathbb{F}$*, if the minimal polynomial of* $A$ *is:*

$$p(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n$$

*and* $a_n\neq 0$*, then*

$$A^{-1}=\frac{1}{a_n}(-a_0A^{n-1}-a_1A^{n-2}-\ldots-a_{n-1}I).$$

The following Lemma is the Exercise 2.38 of [16].

**Lemma 10** *Let* $\mathbf{L}_1,\mathbf{L}_2,\ldots,\mathbf{L}_m$ *be ideals of* $\mathbb{F}[x_1,x_2,\ldots,x_k]$ *and let*

$$\mathbf{J}=\langle 1-\sum_{i=1}^{m}w_i, w_1\mathbf{L}_1, w_2\mathbf{L}_2,\ldots,w_m\mathbf{L}_m\rangle$$

*be an ideal of* $\mathbb{F}[x_1,x_2,\ldots,x_k,w_1,\ldots,w_m]$ *generated by* $1-\sum_{i=1}^{m}w_i, w_1\mathbf{L}_1, w_2\mathbf{L}_2,\ldots,w_m\mathbf{L}_m$*. Then*

$$\bigcap_{i=1}^{m}\mathbf{L}_i=\mathbf{J}\bigcap\mathbb{F}[x_1,x_2,\ldots,x_k].$$

By the Theorem 5 and the Lemma 8, we can prove the following theorem.

**Theorem 11** *The minimal polynomial of a level-*$k$ *FLS* $(r_1,\ldots,r_k)$*-circulant matrix* $A\in\mathbb{F}[\Pi_1,\ldots,\Pi_k]$ *is the monic polynomial that generates the ideal*

$$\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k,y-f(x_1,\ldots,x_k)\rangle$$

$\cap\mathbb{F}[y]$*, where the polynomial* $f(x_1,\ldots,x_k)$ *is the representer of* $A$.

**Proof:** Consider the following $F$- algebra homomorphism

$$\phi:\mathbb{F}[y]\to\mathbb{F}[x_1,\ldots,x_k]/\langle x_1^{n_1}-x_1-r_1,$$
$$\ldots,x_k^{n_k}-x_k-r_k\rangle\longrightarrow\mathbb{F}[\Pi_1,\ldots,\Pi_k]$$
$$y\mapsto f(x_1,\ldots,x_k)$$
$$+\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k\rangle\longmapsto$$
$$A=f(\Pi_1,\ldots,\Pi_k).$$

It is clear that $q(y)\in\ker\ \phi$ if and only if $q(A)=0$. By Lemma 8, we have

$$\ker\phi=\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k,$$
$$y-f(x_1,\ldots,x_k)\rangle\cap\mathbb{F}[y].$$

By Theorem 11 and Lemma 9, we know that the minimal polynomial and the inverse of a level-$k$

FLS $(r_1,\ldots,r_k)$-circulant matrix $A\in\mathbb{F}[\Pi_1,\ldots,\Pi_k]$ is calculated by a Gröbner basis for a kernel of an $F$-algebra homomorphism. Therefore, we have the following algorithm to calculate the minimal polynomial and the inverse of a level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrix $A=f(\Pi_1,\ldots,\Pi_k)$:

**Step 1** Calculate the reduced Gröbner basis **G** for the ideal

$$\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k,y$$
$$-f(x_1,\ldots,x_k)\rangle\cap\mathbb{F}[y]$$

by CoCoA 4.0, using an elimination order with $x_1>x_2>\ldots>x_k>y$.

**Step 2** Find the polynomial in **G** in which the variables $x_1,x_2,\ldots,x_k$ are not appear. This polynomial $p(x)$ is the minimal polynomial of $A$.

**Step 3** By step 2, if $a_n$ in the minimal polynomial of $A$

$$p(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n$$

is zero, stop. Otherwise, calculate

$$A^{-1}=\frac{1}{a_n}(-a_0A^{n-1}-a_1A^{n-2}-\ldots-a_{n-1}I).$$

**Example 12** *Let* $A=f(\Pi_1,\Pi_2)$ *be a level-2 FLS* $(2,5)$*-circulant matrix, where*

$$f(x,y)=x^3y^2+3x^3y+x^2y^2+7x^3$$
$$+x^2y+2x^2+3xy^2+4y^2+5xy+2x+3y+2.$$

*and* $\Pi_1=\aleph_2\otimes I_3,\Pi_2=I_4\otimes\aleph_5$ *and*

$$\aleph_2=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\2&1&0&0\end{pmatrix},\aleph_5=\begin{pmatrix}0&1&0\\0&0&1\\5&1&0\end{pmatrix},$$

$$I_3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},I_4=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

We can now calculate the minimal polynomial and the inverse of $A$ with coefficients in the field $\mathbf{Z}_{11}$ as following:

In fact, the reduced Gröbner basis for the ideal $\langle x^4-x-2,y^3-y-5,z-f(x,y)\rangle$:

$$\mathbf{G}=\{z^{12}-4z^{11}+3z^{10}-z^9-5z^8+4z^6$$
$$+z^5+3z^4+5z^3+3z^2-3z+1,x+z^{11}+3z^9$$
$$-5z^8-4z^7+5z^6-5z^5+4z^3-z^2+3z+1,$$
$$y-z^{11}+z^{10}-3z^9-2z^7-2z^6-z^5$$
$$-4z^4-2z^3-2z^2\}.$$

So the minimal polynomial of $A$ is

$$z^{12}-4z^{11}+3z^{10}-z^9-5z^8+4z^6+z^5$$
$$+3z^4+5z^3+3z^2-3z+1,$$

and the inverse of $A$ is

$$A^{-1}=-A^{11}+4A^{10}-3A^9+A^8+5A^7-4A^5$$
$$-A^4-3A^3-5A^2-3A+3I.$$

**Theorem 13** *The annihilation ideal of level-k FLS* $(r_1,\ldots,r_k)$*-circulant matrices* $A_1,\ldots,A_t\in\mathbb{F}[\Pi_1,\ldots,\Pi_k]$ *is*

$$\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k,y_1-f_1(x_1,\ldots,x_k),$$
$$\ldots,y_t-f_t(x_1,\ldots,x_k)\rangle\cap\mathbb{F}[y_1,\ldots,y_t],$$

*where the polynomial* $f_i(x_1,\ldots,x_k)$ *is the representer of* $A_i,i=1,2,\ldots,t$.

**Proof** Consider the following $F$- algebra homomorphism

$$\varphi:\mathbb{F}[y_1,\ldots,y_t]\to\mathbb{F}[x_1,\ldots,x_k]/\langle x_1^{n_1}-x_1-r_1,$$
$$\ldots,x_k^{n_k}-x_k-r_k\rangle\longrightarrow\mathbb{F}[\Pi_1,\ldots,\Pi_k]$$
$$y_1\mapsto f_1(x_1,\ldots,x_k)+\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}$$
$$-x_k-r_k\rangle\longmapsto A_1=f_1(\Pi_1,\ldots,\Pi_k)$$
$$\cdots$$
$$\cdots$$
$$y_t\mapsto f_t(x_1,\ldots,x_k)+\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}$$
$$-x_k-r_k\rangle\longmapsto A_t=f_t(\Pi_1,\ldots,\Pi_k).$$

It is clear that $\varphi(g(y_1,\ldots,y_t))=0$ if and only if $g(A_1,\ldots,A_t)=0$. Hence, by Lemma 8,

$$\mathbf{I}(A_1,\ldots,A_t)=\ker\varphi=\mathbf{J}\cap\mathbb{F}[y_1,\ldots,y_t].$$

□

According to Theorem 13, we give the following algorithm for the annihilation ideal of level-$k$ FLS $(r_1,\ldots,r_k)$-circulate matrices $A_1,\ldots,A_t\in\mathbb{F}[\Pi_1,\ldots,\Pi_k]$.

**Step 1** Calculate the reduced Gröbner basis $\mathbf{G}$ for the ideal

$$\langle x_1^{n_1} - x_1 - r_1, \ldots, x_k^{n_k} - x_k - r_k, y_1$$

$$-f_1(x_1, \ldots, x_k), \ldots, y_t - f_t(x_1, \ldots, x_k)\rangle$$

by CoCoA 4.0, using an elimination order with $x_1 > \ldots > x_k > y_1 > \ldots > y_k$.

**Step 2** Find the polynomial in $\mathbf{G}$ in which the variables $x_1, x_2, \ldots, x_k$ are not appear. Then the ideal generated by these polynomials is the annihilation ideal of $A_1, \ldots, A_t$.

To calculate the common minimal polynomial of $A_1, \ldots, A_t$, we first give the following Lemma.

**Lemma 14** *Let $h(x)$ be the least common multiple of $p_1(x), p_2(x), \ldots, p_k(x)$. Then*

$$\bigcap_{i=1}^{k} \langle p_i(x)\rangle = \langle h(x)\rangle.$$

**Proof:** For any

$$f(x) \in \bigcap_{i=1}^{k} \langle p_i(x)\rangle,$$

we have

$$p_i(x) \mid f(x)$$

for $i = 1, 2, \ldots, k$. Since $h(x)$ is the least common multiple of $p_1(x), p_2(x), \ldots, p_k(x), h(x) \mid f(x)$. So

$$f(x) \in \langle h(x)\rangle.$$

Hence

$$\bigcap_{i=1}^{k} \langle p_i(x)\rangle \subseteq \langle h(x)\rangle.$$

Conversely, $p_i(x) \mid h(x)$ for $i = 1, 2, \ldots, k$, because $h(x)$ is the least common multiple of $p_1(x), p_2(x), \ldots, p_k(x)$. Therefore

$$\bigcap_{i=1}^{k} \langle p_i(x)\rangle \supseteq \langle h(x)\rangle.$$

By Lemma 14 and Lemma 10, If the minimal polynomial of $A_i$ is $p_i(x)$ for $i = 1, 2, \ldots, t$, then the common minimal polynomial of $A_1, \ldots, A_t$ is the least common multiple of $p_1(x), p_2(x), \ldots, p_t(x)$. So we have the following algorithm for the common minimal polynomial of level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrices $A_i = f_i(\Pi_1, \ldots, \Pi_k)$ for $i = 1, 2, \ldots, t$:

**Step 1** Calculate the Gröbner basis $\mathbf{G}_i$ for the ideal $\langle x_1^{n_1} - x_1 - r_1, \ldots, x_k^{n_k} - x_k - r_k, y - f_i(x_1, \ldots, x_k)\rangle$ by CoCoA 4.0 for each $i = 1, 2, \ldots, t$, using an elimination order with $x_1 > \ldots > x_k > y$.
**Step 2** Find out the polynomial $g_i(y)$ in $\mathbf{G}_i$ in which the variables $x_i, \ldots, x_k$ do not appear for each $i = 1, 2, \ldots, t$.
**Step 3** Calculate the Gröbner basis $\mathbf{G}$ for the ideal $\langle 1 - \sum_{i=1}^{t} w_i, w_1 g_1(y), \ldots, w_t g_t(y)\rangle$ by CoCoA 4.0, using elimination with $w_1 > \ldots > w_t > y$.
**Step 4** Find out the polynomial $g(y)$ in $\mathbf{G}$ in which the variables $w_1, \ldots, w_t$ do not appear. Then the polynomial $g(y)$ is the common minimal polynomial of $A_i = f_i(\Pi_1, \ldots, \Pi_k)$ for $i = 1, 2, \ldots, t$.

**Example 15** *Let $A_1 = f_1(\Pi_1, \Pi_2)$ and $A_2 = f_2(\Pi_1, \Pi_2)$ be both level-2 (11,14)-circulant matrices, where $\Pi_1 = \aleph_{11} \otimes I_4, \Pi_2 = I_4 \otimes \aleph_{14}$,*

$$f_1(x, y) = x^3y^3 + 2x^3y^2 + x^3y + 3x^3 + 7x^2y^3 + 4x^2y^2$$

$$+3x^2y + 2x^2 + xy^3 + 7xy^2 + xy + 6x + 2y^3 + 3y^2 + 2y + 5,$$

$$f_2(x, y) = x^3y^3 + x^3y^2 + 3x^3y + 2x^3 + 6x^2y^3 + 5x^2y^2$$

$$+7x^2y + x^2 + 4xy^3 + 3xy^2 + xy + 4x + 6y^3 + 3y^2 + y + 4,$$

*and*

$$\aleph_{11} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 11 & 1 & 0 & 0 \end{pmatrix},$$

$$\aleph_{14} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 14 & 1 & 0 & 0 \end{pmatrix}, I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We calculate the common minimal polynomial of $A_1$ and $A_2$ in the field $\mathbf{Z}_{11}$ as following:

By CoCoa 4.0, we obtain reduced Gröbner basis for the ideal
$\langle x^4 - x - 11, y^4 - y - 14, z - f_1(x, y)\rangle$ is

$$\begin{aligned}
\mathbf{G_1} = \{&z^{12} + 2z^{11} + 5z^{10} - 2z^9 + 5z^8 - 4z^7 + 4z^6 \\
&+3z^5 + 2z^4 - 3z^3 + 2z^2 + 5z + 1, \\
&x^2 - x - yz^2 - 4yz - 4y - 5z^{11} - 2z^{10} + z^9 + z^8 + \\
&4z^7 + 4z^6 - z^5 + z^4 - z^3 + 4z^2 + 5z + 2, \\
&xy + 5x + 5yz^2 + y + 3z^{11} - 3z^{10} + 3z^9 - 2z^8 \\
&-5z^7 - 2z^6 + 5z^5 + 4z^4 + z^3 - 3z^2 + 5z + 5, \\
&xz + 4x - yz^2 + 3yz - y + z^{11} + z^{10} + 5z^9 - 5z^8 \\
&-5z^7 - z^6 - 3z^5 - 4z^4 - 3z^3 + 4z^2 + 5z, \\
&y^2 + 2yz^2 + 5yz - y + 5z^{11} + z^{10} - z^9 - 5z^8 \\
&-5z^7 - z^6 + 2z^5 + 2z^3 + 3z^2 + 2z - 5, \\
&yz^3 - yz^2 + 3yz - 4y + 3z^{11} + 5z^{10} - z^9 - 5z^7 \\
&-2z^6 - 5z^5 + 4z^4 + z^3 + 5z + 3\}.
\end{aligned}$$

So the minimal polynomial $p_1(z)$ of $A_1$ is

$$z^{12}+2z^{11}+5z^{10}-2z^9+5z^8-4z^7$$

$$+4z^6+3z^5+2z^4-3z^3+2z^2+5z+1.$$

By CoCoa 4.0, we get the reduced Gröbner basis for the ideal
$\langle x^4-x-11, y^4-y-14, z-f_2(x,y)\rangle$ is

$$\begin{aligned}\mathbf{G_2}=\{&z^{15}-z^{14}-2z^{13}-3z^{12}+2z^{11}-2z^{10}\\&+z^9-z^8+2z^7+3z^6+2z^5+5z^2-5z,\\&x^2+x-2z^{14}-z^{13}-z^{12}+z^{11}-5z^{10}+4z^9\\&+4z^8-z^7-4z^5+2z^4+4z^3+5z^2+2z,\\&xz-3x+2z^{14}+3z^{13}+2z^{12}+5z^{11}+2z^{10}+3z^9\\&-4z^8+5z^7+2z^6+z^5+3z^4-z^3-4z^2+5z,\\&y+3z^{14}-3z^{12}+3z^{11}+2z^{10}-3z^9+4z^8-3z^7\\&-3z^6-5z^5-z^4-5z^3-z^2+3z-2\}.\end{aligned}$$

So the minimal polynomial $p_2(z)$ of $A_2$ is

$$z^{15}-z^{14}-2z^{13}-3z^{12}+2z^{11}-2z^{10}$$

$$+z^9-z^8+2z^7+3z^6+2z^5+5z^2-5z.$$

By CoCoa 4.0, we obtain the reduced Gröbner basis for the ideal $\langle 1-u-v, up_1(z), vp_2(z)\rangle$ is

$$\begin{aligned}\mathbf{G}=\{&u+v-1,\\&vz^4-4vz^3-5vz^2-4vz+3v-5z^{22}-3z^{21}+z^{20}\\&-2z^{19}+5z^{18}+5z^{17}+z^{16}+5z^{15}-4z^{14}\\&+5z^{13}-2z^{12}+z^{11}+z^{10}-z^9-4z^8-2z^7+3z^6\\&+4z^5+3z^4-5z^3+5z^2+z-3,\\&z^{23}+5z^{22}+4z^{21}-2z^{20}-5z^{18}+4z^{17}-2z^{16}\\&+3z^{15}+3z^{14}-4z^{13}+2z^{12}+z^{11}+2z^{10}\\&+3z^9-z^8-z^7-5z^6+z^4+3z^3-4z^2+2z\}.\end{aligned}$$

So the common minimal polynomial $p(z)$ of $A_1$ and $A_2$ is

$$\begin{aligned}&z^{23}+5z^{22}+4z^{21}-2z^{20}-5z^{18}+4z^{17}-2z^{16}\\&+3z^{15}+3z^{14}-4z^{13}+2z^{12}+z^{11}+2z^{10}\\&+3z^9-z^8-z^7-5z^6+z^4+3z^3-4z^2+2z.\end{aligned}$$

# 4 Efficient algorithms for finding the inverses of level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrices

In this section, we discuss the singularity and the inverse of a level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrix.

**Theorem 16** *Let $A\in\mathbb{F}[\Pi_1,\ldots,\Pi_k]$ be an $N\times N$ level-k FLS $(r_1,\ldots,r_k)$-circulant matrix. Then $A$ is nonsingular if and only if*

$$1\in\langle f(x_1,\ldots,x_k), x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k\rangle,$$

*where the polynomial $f(x_1,\ldots,x_k)$ is the representer of $A$.*

**Proof.** $A$ is nonsingular if and only if

$$f(x_1,\ldots,x_k)+\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k\rangle$$

is an invertible element in

$$\mathbb{F}(x_1,\ldots,x_k)/\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k\rangle.$$

By Theorem 5, if and only if there exists

$$h(x_1,\ldots,x_k)+\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k\rangle$$

$$\in\mathbb{F}[x_1,\ldots,x_k]/\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k\rangle$$

such that $h(x_1,\ldots,x_k)f(x_1,\ldots,x_k)$

$$+\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k\rangle$$

$$=1+\langle x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k\rangle$$

if and only if there exist

$$h(x_1,\ldots,x_k), u_1,\ldots,u_k\in\mathbb{F}[x_1,\ldots,x_k]$$

such that

$$h(x_1,\ldots,x_k)f(x_1,\ldots,x_k)+u_1(x_1^{n_1}-x_1-r_1)$$

$$+\ldots+u_k(x_k^{n_k}-x_k-r_k)=1$$

if and only if

$$1\in\langle f(x_1,\ldots,x_k), x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k\rangle.$$

□

Let $A\in\mathbb{F}[\Pi_1,\ldots,\Pi_k]$ be an $N\times N$ level-$k$ FLS $(r_1,\ldots,r_k)$-circulant matrix, by Theorem 16, we have the following algorithm which can find the inverse of the matrix $A$:

**Step 1** Calculate the reduced Gröbner basis **G** for the ideal

$$\langle f(x_1,\ldots,x_k), x_1^{n_1}-x_1-r_1,\ldots,x_k^{n_k}-x_k-r_k\rangle,$$

where the polynomial $f(x_1,\ldots,x_k)$ is the representer of $A$, by CoCoA 4.0, using a given term order with $x_1>\ldots>x_k$. If $\mathbf{G}\neq\{1\}$, then $A$ is singular. Stop. Otherwise, go to step 2.

**Step 2** By Buchberger's algorithm for computing Gröbner bases, keeping track of linear combinations that give rise to the new polynomials in the generating set, we get $h(x_1,\dots,x_k), u_1,\dots,u_k \in \mathbb{F}[x_1,\dots,x_k]$ such that

$$h(x_1,\dots,x_k)f(x_1,\dots,x_k)+u_1(x_1^{n_1}-x_1-r_1) + \dots + u_k(x_k^{n_k}-x_k-r_k)=1 \quad (4)$$

**Step 3** The variables $x_1,\dots,x_k$ in the above formula (4) are replaced by $\Pi_1,\dots,\Pi_k$, respectively, we have

$$A^{-1}=h(\Pi_1,\dots,\Pi_k).$$

# 5 Inverse of partitioned matrix with level-$k$ FLS $(r_1,\dots,r_k)$-circulant matrix blocks

Let $A_1, A_2, A_3, A_4$ be level-$k$ FLS $(r_1,\dots,r_k)$-circulant matrices with the representer $f_1(x_1,\dots,x_k)$, $f_2(x_1,\dots,x_k)$, $f_3(x_1,\dots,x_k)$, $f_4(x_1,\dots,x_k)$, respectively. If $A_1$ is nonsingular, let

$$\Sigma=\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

$$\Gamma_1=\begin{pmatrix} I & 0 \\ -A_3A_1^{-1} & I \end{pmatrix},\ \Gamma_2=\begin{pmatrix} I & -A_1^{-1}A_2 \\ 0 & I \end{pmatrix},$$

then

$$\Gamma_1\Sigma\Gamma_2=\begin{pmatrix} A_1 & 0 \\ 0 & A_4-A_3A_1^{-1}A_2 \end{pmatrix}. \quad (5)$$

So $\Sigma$ is nonsingular if and only if $A_4-A_3A_1^{-1}A_2$ is nonsingular. Since $A_1,A_2,A_3,A_4$ are all level-$k$ FLS $(r_1,\dots,r_k)$-circulant matrices, then the $A_i$ commutes with the $A_j$ if $i\neq j$. Thus

$$A_1(A_4-A_3A_1^{-1}A_2)=A_1A_4-A_2A_3. \quad (6)$$

By the equation (6), we conclude that $\Sigma$ is nonsingular if and only if $A_1A_4-A_2A_3$ is nonsingular. Since

$$f_1(x_1,\dots,x_k)f_4(x_1,\dots,x_k) - f_2(x_1,\dots,x_k)f_3(x_1,\dots,x_k)$$

is the representer of $A_1A_4-A_2A_3$, then $\Sigma$ is nonsingular if and only if

$$1\in\langle f_1(x_1,\dots,x_k)f_4(x_1,\dots,x_k) - f_2(x_1,\dots,x_k)f_3(x_1,\dots,x_k), x_1^{n_1}-x_1-r_1,\dots,x_k^{n_k}-x_k-r_k\rangle.$$

In addition, if $\Sigma$ is nonsingular, by the equation (5), we have

$$\Sigma^{-1}=\begin{pmatrix} I & -A_1^{-1}A_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1^{-1} & 0 \\ 0 & (A_4-A_3A_1^{-1}A_2)^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -A_3A_1^{-1} & I \end{pmatrix} =\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

where

$$T_1=A_1^{-1}+(A_1A_4-A_2A_3)^{-1}A_2A_3A_1^{-1},$$
$$T_2=-(A_1A_4-A_2A_3)^{-1}A_2,$$
$$T_3=-(A_1A_4-A_2A_3)^{-1}A_3,$$
$$T_4=(A_1A_4-A_2A_3)^{-1}A_1.$$

Therefore, we have the following result.

**Theorem 17** *Let*

$$\Sigma=\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

*where $A_1,A_2,A_3$ and $A_4$ are all level-k FLS $(r_1,\dots,r_k)$-circulant matrices with the representer $f_1(x_1,\dots,x_k), f_2(x_1,\dots,x_k), f_3(x_1,\dots,x_k)$, $f_4(x_1,\dots,x_k)$, respectively. If $A_1$ is nonsingular, then $\Sigma$ is nonsingular if and only if*

$$1\in\langle f_1(x_1,\dots,x_k)f_4(x_1,\dots,x_k) - f_2(x_1,\dots,x_k)f_3(x_1,\dots,x_k), x_1^{n_1}-x_1-r_1,\dots,x_k^{n_k}-x_k-r_k\rangle.$$

*Moreover, if $\Sigma$ is nonsingular, then*

$$\Sigma^{-1}=\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}. \quad (7)$$

*where*

$$T_1=A_1^{-1}+(A_1A_4-A_2A_3)^{-1}A_2A_3A_1^{-1},$$
$$T_2=-(A_1A_4-A_2A_3)^{-1}A_2,$$
$$T_3=-(A_1A_4-A_2A_3)^{-1}A_3,$$
$$T_4=(A_1A_4-A_2A_3)^{-1}A_1.$$

**Theorem 18** *Let*

$$\Sigma=\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

*where $A_1,A_2,A_3,A_4$ are all level-k FLS $(r_1,\dots,r_k)$-circulant matrices with the representer $f_1(x_1,\dots,x_k), f_2(x_1,\dots,x_k), f_3(x_1,\dots,x_k)$, $f_4(x_1,\dots,x_k)$, respectively. If $A_4$ is nonsingular, then $\Sigma$ is nonsingular if and only if*

$$1\in\langle f_1(x_1,\dots,x_k)f_4(x_1,\dots,x_k) - f_2(x_1,\dots,x_k)f_3(x_1,\dots,x_k), x_1^{n_1}-x_1-r_1,\dots,x_k^{n_k}-x_k-r_k\rangle.$$

*Moreover, if $\Sigma$ is nonsingular, then*

$$\Sigma^{-1} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \tag{8}$$

*where*
$T_1 = (A_1A_4 - A_2A_3)^{-1}A_4,$
$T_2 = -(A_1A_4 - A_2A_3)^{-1}A_2,$
$T_3 = -(A_1A_4 - A_2A_3)^{-1}A_3,$
$T_4 = A_4^{-1} + (A_1A_4 - A_2A_3)^{-1}A_2A_3A_4^{-1}.$

**Proof.** Since $A_4$ is nonsingular, then

$$\begin{pmatrix} I & -A_2A_4^{-1} \\ 0 & I \end{pmatrix} \Sigma \begin{pmatrix} I & 0 \\ -A_4^{-1}A_3 & I \end{pmatrix}$$

$$= \begin{pmatrix} A_1 - A_2A_4^{-1}A_3 & 0 \\ 0 & A_4 \end{pmatrix}. \tag{9}$$

So $\Sigma$ is nonsingular if and only if $A_1 - A_2A_4^{-1}A_3$ is nonsingular. Since $A_1, A_2, A_3, A_4$ are all level-$k$ FLS $(r_1, \ldots, r_k)$-circulant matrices, then the $A_i$ commutes with the $A_j$ if $i \neq j$. Thus

$$A_4(A_1 - A_2A_4^{-1}A_3) = A_1A_4 - A_2A_3. \tag{10}$$

By the equation (10), we conclude that $\Sigma$ is nonsingular if and only if $A_1A_4 - A_2A_3$ is nonsingular. Since $f_1(x_1, \ldots, x_k)f_4(x_1, \ldots, x_k) - f_2(x_1, \ldots, x_k)f_3(x_1, \ldots, x_k)$ is the representer of $A_1A_4 - A_2A_3$, then $\Sigma$ is nonsingular if and only if

$$\begin{aligned} 1 \in \langle & f_1(x_1, \ldots, x_k)f_4(x_1, \ldots, x_k) \\ & -f_2(x_1, \ldots, x_k)f_3(x_1, \ldots, x_k), \\ & x_1^{n_1} - x_1 - r_1, \ldots, x_k^{n_k} - x_k - r_k\rangle. \end{aligned}$$

In addition, if $\Sigma$ is nonsingular, by the equation (9), we have

$$\begin{aligned} \Sigma^{-1} = & \\ & \begin{pmatrix} I & 0 \\ -A_4^{-1}A_3 & I \end{pmatrix} \\ & \begin{pmatrix} (A_1 - A_2A_4^{-1}A_3)^{-1} & 0 \\ 0 & A_4^{-1} \end{pmatrix} \begin{pmatrix} I & -A_2A_4^{-1} \\ 0 & I \end{pmatrix} \\ = & \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \end{aligned}$$

where
$T_1 = (A_1A_4 - A_2A_3)^{-1}A_4,$
$T_2 = -(A_1A_4 - A_2A_3)^{-1}A_2,$
$T_3 = -(A_1A_4 - A_2A_3)^{-1}A_3,$
$T_4 = A_4^{-1} + (A_1A_4 - A_2A_3)^{-1}A_2A_3A_4^{-1}.$

We have the following algorithm for determining the nonsingularity and computing the inverse of $\Sigma$ if it is nonsingular.

**Step 1** Calculate the Gröbner bases $\mathbf{G_1}, \mathbf{G_4}$ for the ideals

$$\langle f_1(x_1, \ldots, x_k), x_1^{n_1} - x_1 - r_1, \ldots, x_k^{n_k} - x_k - r_k\rangle,$$
$$\langle f_4(x_1, \ldots, x_k), x_1^{n_1} - x_1 - r_1, \ldots, x_k^{n_k} - x_k - r_k\rangle,$$

respectively. If $\mathbf{G_1} \neq \{1\}, \mathbf{G_4} \neq \{1\}$, Stop. Otherwise, go to step 2.

**Step 2.** If $\mathbf{G_1} = \{1\}$, find $u_1, \ldots, u_k$, $h_1(x_1, \ldots, x_k) \in \mathbb{F}[x_1, \ldots, x_k]$ such that

$$h_1(x_1, \ldots, x_k)f_1(x_1, \ldots, x_k) + u_1(x_1^{n_1} - x_1 - r_1)$$
$$+ \ldots + u_k(x_k^{n_k} - x_k - r_k) = 1.$$

Then $h_1(x_1, \ldots, x_k)$ is the representer of $A_1^{-1}$, go to step 4. Otherwise, go to step 3.

**Step 3.** If $\mathbf{G_4} = \{1\}$, find $u_1', \ldots, u_k'$, $h_4(x_1, \ldots, x_k) \in \mathbb{F}[x_1, \ldots, x_k]$ such that

$$h_4(x_1, \ldots, x_k)f_4(x_1, \ldots, x_k) + u_1'(x_1^{n_1}$$
$$-x_1 - r_1) + \ldots + u_k'(x_k^{n_k} - x_k - r_k) = 1.$$

Then $h_4(x_1, \ldots, x_k)$ is the representer of $A_4^{-1}$, go to step 4.

**Step 4.** Calculate the Gröbner bases $\mathbf{G}$ for the ideals

$$\begin{aligned} \langle & f_1(x_1, \ldots, x_k)f_4(x_1, \ldots, x_k) \\ & -f_2(x_1, \ldots, x_k)f_3(x_1, \ldots, x_k), \\ & x_1^{n_1} - x_1 - r_1, \ldots, x_k^{n_k} - x_k - r_k\rangle. \end{aligned}$$

If $\mathbf{G} \neq \{1\}$, then $A_1A_4 - A_2A_3$ is singular, Stop. Otherwise, go to step 5.

**Step 5.** Find $v_1, \ldots, v_k$, $h(x_1, \ldots, x_k) \in \mathbb{F}[x_1, \ldots, x_k]$ such that

$$h(x_1, \ldots, x_k)[f_1(x_1, \ldots, x_k)f_4(x_1, \ldots, x_k)$$

$$\begin{aligned} & -f_2(x_1, \ldots, x_k)f_3(x_1, \ldots, x_k)] \\ & +v_1(x_1^{n_1} - x_1 - r_1) + \ldots + v_k(x_k^{n_k} - x_k - r_k) = 1. \end{aligned}$$

Then $h(x_1, \ldots, x_k)$ is the representer of $(A_1A_4 - A_2A_3)^{-1}$. Then we obtain

If $A_1$ is nonsingular, then

$$\Sigma^{-1} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix};$$

if $A_4$ is nonsingular, then

$$\Sigma^{-1} = \begin{pmatrix} T_5 & T_2 \\ T_3 & T_6 \end{pmatrix},$$

where

$$
\begin{aligned}
&T_1 = h_1(\Pi_1, \ldots, \Pi_k)[I + h(\Pi_1, \ldots, \Pi_k)A_2A_3],\\
&T_2 = -h(\Pi_1, \ldots, \Pi_k)A_2,\\
&T_3 = -h(\Pi_1, \ldots, \Pi_k)A_3,\\
&T_4 = h(\Pi_1, \ldots, \Pi_k)A_1,\\
&T_5 = h(\Pi_1, \ldots, \Pi_k)A_4,\\
&T_6 = h_4(\Pi_1, \ldots, \Pi_k)[I + h(\Pi_1, \ldots, \Pi_k)A_2A_3].
\end{aligned}
$$

□

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*References:*


[1] S. S. Capizzano and E. Tyrtyshnikov, Any circulant-like perconditioner for multilevel matrices is not superlinear, *SIAM J. Matrix. Anal. Appl.* 21, 1999, pp. 431–439.

[2] S. S. Capizzano and E. Tyrtyshnikov, Multigrid methods for multilevel circulant matrices, *SIAM J. Sci. Comput.* 26, 2004, pp. 55–85.

[3] R. M. Mallery, Reverse Engineering Gene Networks with Microarray Data. http://www.caam.rice.edu/ cox/genenets.pdf.

[4] G. Songa and Y. Xua, Approximation of high-dimensional kernel matrices by multilevel circulant matrices, *Journal of Complexity* 26, 2010, pp. 375–405.

[5] W. F. Trench, Properties of unilevel block circulants, *Linear Algebra and its Appl.* 430, 2009, pp. 2012–2025.

[6] W. F. Trench, Properties of multilevel block $\alpha$-circulants, *Linear Algebra and its Appl.* 431, 2009, pp. 1833–1847.

[7] S. G. Zhang, Z. L. Jiang and S. Y. Liu, An application of the Gröbner basis in computation for the minimal polynomials and inverses of block circulant matrices, *Linear Algebra and its Appl.* 347, 2002, pp. 101–114.

[8] M. Morhac and V. Matousek, Exact algorithm of multidimensional circulant deconvolution, *Appl. Math. and Comput.* 164, 2005, pp. 155–166.

[9] M. Rezghi and L. Elden, Diagonalization of tensors with circulant structure, *Linear Algebra and its Appl.* 435, 2011, pp. 422–447.

[10] S. Georgiou and C. Koukouvinos, Multi-level k-circulant supersaturated designs, *Metrika* 64, 2006, pp. 209–220.

[11] Z. L. Jiang and S. Y. Liu, Level- $m$ scaled circulant factor matrices over the complex number field and the quaternion division algebra, *J. Appl. Math. Computing* 14, 2004, pp. 81–96.

[12] A. J. H, Block circulants of level-k. Division of applied Mathematics. Brown University, providence, R. I. 1976.

[13] J. Baker, The structure of multi-block circulants, *Kyungpook Math. J.* 25, 1985, pp. 71–75.

[14] R. L. Smith, Moore-Penrose inverses of block circulant and block k-circulant matrices, *Linear Algebra and its Appl.* 16, 1977, pp. 237–245.

[15] S. Rjasanow, Effective algorithms with circulant-block matrices, *Linear Algebra and its Appl.* 202, 1994, pp. 55–69.

[16] W. W. Adams and P. Loustaunau, *An introduction to Gröbner bases*, American Mathematical Society, 1994.

[17] B. Mishra, *Algorithmic Algebra*, Springer-Verlag, 2001.

[18] D. G. Northcott, Injective envelopes and inverse polynomials, *J. London Math. Soc.* 8, 1974, pp. 190–196.

[19] D. Greenspan, Methods of matrix inversion, *Amer. Math. Monthly.* 62, 1955, pp. 303–308.

[20] H. M. and T. Mora, New constructive methods in classical ideal theory, *J. Algebra* 100, 1986, pp. 138–178.

[21] P. J. Davis, *Circulant Matrices*, Wiley, New York, 1979.

[22] R. A. Horn and C. R. Johnson, *Matrix Analysis*, New York: Cambridge University Press, 1985.

[23] I. Gohberg, P. Lancaster and L. Rodman, *Matrix Polynomials*, Academic Press, New York, 1982.

[24] Z. L. Jiang, Z. X. Zhou, *Circulant Matrices*, Chengdu Technology University Publishing Company, Chengdu, 1999.

[25] Z. L. Jiang, Z. B. Xu, Eifficient algorithm for finding the inverse and group inverse of FLS $r$-circulant matrix, *J. Appl. Math. Comput.* 18, 2005, pp. 45–13.