A Linearized Finite Difference Scheme with Non-uniform Meshes for Semi-linear Parabolic Equation

Jun Zhou
Yangtze Normal University
School of Mathematics and Computer Science
Chongqing 408100
China
flzjzklm@126.com

Abstract: In the present paper, a linearized difference scheme with non-uniform meshes for semi-linear parabolic equation is proposed. The scheme is constructed according to the change rule of the solution by travelling wave solution theory for partial differential equation. The existence and uniqueness of the numerical solution are derived by linear systems theory, and the convergence and stability of the difference scheme are proved by the discrete energy method. Numerical simulations verify the theoretical analysis, the results show that the numerical solution with non-uniform meshs is more accurate than that with uniform meshes in the sense of not costing much more computing time. It is concluded that our scheme is effective.

Key–Words: semi-linear parabolic equation, non-uniform meshes, difference schemes, convergence, stability

1 Introduction

The Semi-linear Parabolic Equations have wide applications in chemical reaction, neural conduction, biological competition and other fields. The studies on these equations have been a hot topic in past decades. It is of significance to explore theoretically and numerically the solutions to these equations. There are many available works contributed to investigation in this field for instance see [1-3]. Ames in [4] gave a large collection of physical problems having nonlinear parabolic equations as models. Also the survey lists various methods for exact, approximate and numerical solutions for those examples. Based on these equations, there had been some finite difference methods such as alternating direction iterative scheme, predictor-correctors methods and the linearized two or three level difference schemes [5-8]. Ramos in [9-11] compared various finite difference schemes that include explicit, implicit and linearized schemes. Besides these finite difference schemes, Tang in [12] studied finite element method of a nonlinear diffusion system. All methods mentioned above have not taken rule change of the analytical solution into account; the rule is that it changes quickly in some area, and slowly in other area. In fact, this rule can be deducted by travelling wave solution theory for partial differential equation. According to this rule, the traditional methods given above had a disadvantage. When the exactness of numerical solution is required, one has to refine grid by increasing grid points. This way causes increase of computing amount. To overcome the drawback, the finite difference schemes with non-uniform meshes have attracted great attention. Mattheij and Smooke, Samarskij and his co-operators investigated the stability and convergence of variable step (space and time) algorithms in the solution of the mixed initial-boundary problem of one-dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x} + f(x,t)$$

and two-dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x_1} + \frac{\partial^2 u}{\partial^2 x_2} + f(x_1,x_2,t)$$

in [13-15] respectively. For the generalized non-linear parabolic systems $u_t = A(x,t,u)u_{xx} + f(x,t,u,u_x)$, Zhou constructed the general finite difference scheme with non-uniform meshes and proved the existence and 1st order convergence in L_∞-norm of the discrete solutions for the difference scheme by the fixed point technique in [16]. Yuan proved the unique solvability and stability for the difference scheme constructed in [16] by the energy method in [17, 18]. Their work solved some unexpected phenomenon, but their proof is very complex .Meanwhile, they had no numerical experiments to justify their theoretical analysis. In order to solve the existed problem, Zhou and Hu constructed an implicit difference schemes with non-uniform meshes for the flame equation, and they prove the uniqueness, existence, convergence and stability of difference solution of the implicit scheme in [21, 22]. The the scheme with non-uniform meshes for space was constructed by a function transformation, but the meshs for time is still uniform. The numerical experiments were carried out to justify that the convergence of the solution is 1st-order for time. These results coincide with the previous theoretical analysis. However, in order to get the solution to the implicit scheme, the iteration method for non-linear equations needs to be applied, which costs a quantity of time for every time step. To overcome this drawback, we constructed a kind of linearized finite difference scheme on the base of implicit scheme in [21, 22]

In this paper, we will investigate a linearized difference scheme with non-uniform meshes approximating to the following Dirichlet problem of a semilinear parabolic equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \quad (1)$$

$$u(x,0) = \varphi(x), \ \ x \in [a,b], \quad (2)$$

$$u(a,t) = \alpha(t), \ \ u(b,t) = \beta(t), \ \ t \in [0,T], \quad (3)$$

Let $\bar{\Omega}_T = [a,b] \times [0,T]$.

In order to prove some properties of the finite difference scheme, we impose the following conditions:

(I) The analytical solution of problem (1) satisfies $u(x,t) \in C^{4,3}$,and there exists a positive constant C_0 satisfying $|u| + |u_t| + |u_x| + |u_{xx}| \leq C_0$.

(II) Let $f(s)$ be a two times continuous differentiable function for s, there exist two positive constants C_1 and P satisfying $\{|f(s)|, |f'(s)|, |f''(s)|\} \leq C_1$ when $|s| \leq C_0 + P$.

(III) Boundary value functions $\alpha(t), \beta(t)$ are continuous differentiable function for t, initial function $\varphi(x)$ is also continuous differentiable function with respect to x, and we have $\alpha(0) = \varphi(a), \beta(0) = \varphi(b)$.

2 Difference Scheme and Notations

Let us divide the rectangular domain $\bar{\Omega}_T = [a,b] \times [0,T]$ into the small rectangular grids

$$\Omega_h^\tau = \left\{ \begin{array}{l} a = x_0 < x_1, \ldots, < x_{I-1} < x_I = b, \\ 0 = t_0 < t_1 < \ldots, < t_K = T \end{array} \right\}.$$

The $i+1$th domain on space is $[x_i, x_{i+1}]$. The meshsteps of space is $h_{i+\frac{1}{2}} = x_{i+1} - x_i$ which are assumed to be unequal. The traditional uniform meshes are applied for time

$$0 = t_0 < t_1 < t_2 \ldots t_{K-1} < t_K = T, \ \tau = t_k - t_{k-1}.$$

We denote the maximum value of space meshsteps is

$$h = \max_{0 \leq i \leq I-1} \{h_{i+\frac{1}{2}}\},$$

the minimum of space meshsteps is

$$h_* = \min_{0 \leq i \leq I-1} \{h_{i+\frac{1}{2}}\},$$

the ratio of maximum and minimum of space meshsteps $Rh_* = \frac{h}{h_*}$. So we denote discrete functions

$$u_h = \{u_i^k \mid i = 0,1,2,\ldots,I, k = 0,1,2,\ldots,K\}$$

on Ω_T. The other notations are as follows:

$$\Delta_t u_i^k = \frac{u_i^{k+1} - u_i^k}{\tau}, \ \ \delta_x u_{i+\frac{1}{2}}^k = \frac{1}{h_{i+\frac{1}{2}}}(u_{i+1}^k - u_i^k),$$

$$\delta_x^2 u_i^k = \frac{1}{h_i^{(2)}}(\delta_x u_{i+\frac{1}{2}}^k - \delta_x u_{i-\frac{1}{2}}^k), \ \ h_i^{(2)} = \frac{h_{i+\frac{1}{2}} + h_{i-\frac{1}{2}}}{2},$$

$$\|u_h\|_\infty = \max_{0 \leq i \leq I} \{|u_i|\}, \ \ \|u_h\|_2^2 = \sum_{i=1}^{I-1} |u_i|^2 h_i^{(2)},$$

$$\|\delta_x u_h\|_2^2 = \sum_{i=0}^{I-1} |\delta_x u_{i+\frac{1}{2}}|^2 h_{i+\frac{1}{2}},$$

$$\|\delta_x u_h\|_\infty = \max_{0 \leq i \leq I} |\delta u_{i+\frac{1}{2}}|,$$

By Taylor expansion, we construct the linearized difference scheme:

$$\begin{array}{l} (1 - \tau f'(u_i^k)) \triangle_t u_i^k = \delta_x^2 u_i^{k+1} + f(u_i^k), \\ 1 \leq i \leq I-1, \ \ 0 \leq k \leq K-1, \end{array} \quad (4)$$

$$u_i^0 = \varphi(x_i), \quad 0 \leq i \leq I, \quad (5)$$

$$u_0^k = \alpha(t_k), \quad u_I^k = \beta(t_k), 0 \leq k \leq K, \quad (6)$$

3 Existence and Uniqueness

Theorem 1 *There exists unique difference solution u_i^k, $i = 1,2,\ldots,I-1, k = 1,2,\ldots,K+1$ satisfying the difference scheme (4)-(6)*

Proof: Expanding the difference scheme(4)-(6), we get

$$-\frac{2\tau}{h_{i-\frac{1}{2}}(h_{i-\frac{1}{2}}+h_{i+\frac{1}{2}})}u_{i-1}^{k+1}$$

$$+(1+\frac{2\tau}{h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}}-\tau f'(u_i^k))u_i^{k+1}$$

$$-\frac{2\tau}{h_{i+\frac{1}{2}}(h_{i-\frac{1}{2}}+h_{i+\frac{1}{2}})}u_{i+1}^{k+1}$$

$$=(1-\tau f'(u_i^k))u_i^k+\tau f(u_i^k),$$

$$1\le i\le I-1,\ \ 0\le k\le K-1,$$

Obviously, it is triangle linear systems. Let

$$a_i=-\frac{2\tau}{h_{i-\frac{1}{2}}(h_{i-\frac{1}{2}}+h_{i+\frac{1}{2}})},$$

$$b_i=1+\frac{2\tau}{h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}}-\tau f'(u_i^k),$$

$$c_i=-\frac{2\tau}{h_{i+\frac{1}{2}}(h_{i-\frac{1}{2}}+h_{i+\frac{1}{2}})},$$

for $i=1,2,\ldots,I-1$. By $\|u^k\|_\infty\le C_0+P$(This result can be gotten by mathematical induction in section 3) and assumption (II),when time step $\tau<\frac{1}{C_1}$, we get $1-\tau f'(u_i^k)>1-C_1\tau>0$. Fetching the absolute value of b_1 and c_1,

$$|b_1|=1+\frac{2\tau}{h_{\frac{1}{2}}h_{\frac{3}{2}}}-\tau f'(u_i^k),$$

$$|c_1|=\frac{2\tau}{h_{\frac{3}{2}}(h_{\frac{1}{2}}+h_{\frac{3}{2}})}=\frac{2\tau}{h_{\frac{3}{2}}h_{\frac{1}{2}}+h_{\frac{3}{2}}^2},$$

we get $|b_1|>|c_1|$. Similarly we get $|b_{I-1}|>|a_{I-1}|$.

For $i=2,3,\ldots,I-2$, summing up the absolute value of a_i,c_i, we get

$$|a_i|+|c_i|=\frac{2\tau}{h_{i-\frac{1}{2}}(h_{i-\frac{1}{2}}+h_{i+\frac{1}{2}})}$$

$$+\frac{2\tau}{h_{i+\frac{1}{2}}(h_{i-\frac{1}{2}}+h_{i+\frac{1}{2}})}$$

$$=\frac{2\tau(h_{i+\frac{1}{2}}+h_{i-\frac{1}{2}})}{h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}(h_{i-\frac{1}{2}}+h_{i+\frac{1}{2}})}=\frac{2\tau}{h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}},$$

Comparing it with $|b_i|$ and by $1-\tau f'(u_i^k)>0$, we get

$$|b_i|>|a_i|+|c_i|.$$

It implies that the coefficient matrix is the strictly row diagonally dominant. Therefore, there are unique solution $u_i^k, i=1,2,\ldots,I-1, k=1,2,\ldots,K$ to the linear systems [23]. It implies that there exists a unique numerical solution to satisfy the difference scheme (4)-(6).

4 Convergence

In order to prove the convergence and stability of the solution to the difference scheme (4)-(6), we import four lemmas [16].

Lemma 2 *For any $u_h=\{u_i|i=0,1,2,\ldots,I\}$ and $v_h=\{v_i|i=0,1,2,\ldots,I\}$, there are*

$$\sum_{i=0}^{I-1}u_i(v_{i+1}-v_i)=-\sum_{i=1}^{I}v_i(u_i-u_{i-1})$$
$$-u_0v_0+u_Iv_I,$$

$$\sum_{i=1}^{I-1}u_i(\delta v_{i+\frac{1}{2}}-\delta v_{i-\frac{1}{2}})=-\sum_{i=0}^{I-1}\delta u_{i+\frac{1}{2}}\delta v_{i+\frac{1}{2}}h_{i+\frac{1}{2}}$$
$$-u_0\delta v_{\frac{1}{2}}+u_I\delta v_{I-\frac{1}{2}}.$$

Lemma 3 *For any $u_h=\{u_i|i=0,1,2,\ldots,I\}$ defined on the grid points $\{x_i|i=0,1,2,\ldots,I\}$ with unequal meshsteps, there are relations:*

$$\|u_h\|_\infty\le\frac{1}{\sqrt{h_*}}\|u_h\|_2,\ \ \|\delta u_h\|_\infty\le\frac{1}{\sqrt{h_*}}\|\delta u_h\|_2.$$

Lemma 4 *For any $u_h=\{u_i|i=0,1,2,\ldots,I\}$ defined on the grid points $\{x_i|i=0,1,2,\ldots,I\}$,there are relations:*

$$\|u_h\|_2^2\le l^2\|\delta u_h\|_2^2+2l|u_0|^2,$$

here $0=x_0<x_1<\ldots<x_{I-1}<x_I=l$.

Theorem 5 *For any $u_h=\{u_i|i=0,1,2,\ldots,I\}$ defined on the grid points$\{x_i|i=0,1,2,\ldots,I\}$,there are relations:*

$$\|u_h\|_2^2\le b^2\|\delta u_h\|_2^2+2(b-a)|u_0|^2,$$

here $a=x_0<x_1<\ldots<x_{I-1}<x_I=b$.

Remark: The proof of Theorem 5 is completely same as that of lemma 3 in [16].This result is only popularized from [0,l] to general domain $[a,b]$.

Lemma 6 *Suppose the discrete function $u^\tau=\{u^k|k=0,1,2,\ldots,K\}$ defined on the grid* *points* $\{t^k | k = 0, 1, 2, \ldots, K\}$ *with unequal mesh-steps* $\tau = \{\tau^k = t^{k+1} - t^k > 0 | k = 0, 1, 2, \ldots, K-1\}$ *satisfies recurring relation*

$$u^{k+1} - u^k \leq A\tau^k(u^{k+1} + u^k) + C\tau^k$$

then there is $u^k \leq e^{3At^k}u^0 + 2Ct^k e^{3At^k}$, *where mesh-steps* $0 = t^0 < t^1 < \ldots < t^{K-1} < t^K = T$ *are sufficiently small that* $2A\tau \leq 1$, *A and C are constants.*

The convergence theorem and its proof are as follows:

Theorem 7 *Suppose that the initial boundary problem of partial difference equations (1)-(3) satisfy assumptions (I), (II) and (III), the meshsteps* $h, \tau, \frac{\tau}{\sqrt{h_*}}$ *be sufficiently small and* Rh_* *be boundary. We denote the error of discrete solution* $e_h = \{e_i^k = U_i^k - u_i^k | i = 0, 1, 2, \ldots, I; k = 0, 1, 2, \ldots, K\}$,*then there are estimates*

$$\max_{0\leq k\leq K} \|e_h^k\|_2, \ \max_{0\leq k\leq K} \|e_h^k\|_\infty, \ \max_{0\leq k\leq K} \|\delta_x e_h^k\|_2,$$

$$\Big(\sum_{i=0}^{K-1} \|\delta_x^2 e_h^{k+1}\|_2^2 \tau\Big)^{\frac{1}{2}}, \ \left(\sum_{i=0}^{K-1} \|\tfrac{e_h^{k+1} - e_h^k}{\tau}\|_2^2 \tau\right)^{\frac{1}{2}}$$

$$= O(\tau + h),$$

and $\max_{0\leq k\leq K} \|\delta_x e_h^k\|_\infty = O(\frac{\tau}{\sqrt{h_*}}, h^{\frac{1}{2}})$, *where* $U_i^k = u(x_i, t_k)$.

Proof: By Taylor expansion at point (x_i, t_k),we get

$$\begin{aligned} u(x_i, t_{k+1}) &= u(x_i, t_k) + \frac{\partial u}{\partial t}(x_i, t_k)\tau \\ &+ \frac{\partial^2 u}{\partial t^2}(x_i, \eta_{ik}^{(1)})\tau^2, \end{aligned} \tag{7}$$

where $\eta_{ik}^{(1)}$ is between t_k and t_{k+1}.

It implies that

$$\begin{aligned} \frac{\partial u}{\partial t}(x_i, t_k) &= \frac{u(x_i, t_{k+1}) - u(x_i, t_k)}{\tau} \\ &- \frac{\partial^2 u}{\partial t^2}(x_i, \eta_{ik}^{(1)})\tau, \end{aligned} \tag{8}$$

From the notation, we get

$$\frac{\partial u}{\partial t}(x_i, t_k) = \Delta_t U_i^k + R_{ik}^{(1)}, \tag{9}$$

where

$$R_{ik}^{(1)} = -\frac{\partial^2 u}{\partial t^2}(x_i, \eta_{ik}^{(1)})\tau.$$

For the diffusion part, we have

$$\begin{aligned} u(x_{i+1}, t_{k+1}) &= u(x_i, t_{k+1}) + \frac{\partial u}{\partial x}(x_i, t_{k+1})h_{i+\frac{1}{2}} \\ &+ \frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1})h_{i+\frac{1}{2}}^2 + \frac{\partial^3 u}{\partial x^3}(\zeta_{ik}^{(1)}, t_{k+1})h_{i+\frac{1}{2}}^3, \end{aligned} \tag{10}$$

$$\begin{aligned} u(x_{i-1}, t_{k+1}) &= u(x_i, t_{k+1}) - \frac{\partial u}{\partial x}(x_i, t_{k+1})h_{i-\frac{1}{2}} \\ &+ \frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1})h_{i-\frac{1}{2}}^2 - \frac{\partial^3 u}{\partial x^3}(\zeta_{ik}^{(2)}, t_{k+1})h_{i-\frac{1}{2}}^3, \end{aligned} \tag{11}$$

where $\zeta_{ik}^{(1)}$ is between x_i and x_{i+1}, $\zeta_{ik}^{(2)}$ is between x_i and x_{i-1}.

Multiplying (10), (11) by $h_{i-\frac{1}{2}}$, $h_{i+\frac{1}{2}}$ respectively, we get

$$\begin{aligned} h_{i-\frac{1}{2}}u(x_{i+1}, t_{k+1}) &= h_{i-\frac{1}{2}}u(x_i, t_{k+1}) \\ &+ h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}\frac{\partial u}{\partial x}(x_i, t_{k+1}) + h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}^2\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1}) \\ &+ h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}^3\frac{\partial^3 u}{\partial x^3}(\zeta_{ik}^{(1)}, t_{k+1}), \end{aligned} \tag{12}$$

$$\begin{aligned} h_{i+\frac{1}{2}}u(x_{i-1}, t_{k+1}) &= h_{i+\frac{1}{2}}u(x_i, t_{k+1}) \\ &+ h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}\frac{\partial u}{\partial x}(x_i, t_{k+1}) + h_{i+\frac{1}{2}}h_{i-\frac{1}{2}}^2\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1}) \\ &+ h_{i+\frac{1}{2}}h_{i-\frac{1}{2}}^3\frac{\partial^3 u}{\partial x^3}(\zeta_{ik}^{(2)}, t_{k+1}), \end{aligned} \tag{13}$$

Adding (12) to (13), then dividing by $h_{i+\frac{1}{2}}h_{i-\frac{1}{2}}(h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}})$, we get

$$\begin{aligned} &\frac{1}{h_{i-\frac{1}{2}}(h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}})}u(x_{i-1}, t_{k+1}) \\ &- \frac{1}{h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}}u(x_i, t_{k+1}) \\ &+ \frac{1}{h_{i+\frac{1}{2}}(h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}})}u(x_{i+1}, t_{k+1}) \\ &= \frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1}) + \frac{h_{i+\frac{1}{2}}^2}{h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}}}\frac{\partial^3 u}{\partial x^3}(\zeta_{ik}^{(1)}, t_{k+1}) \\ &- \frac{h_{i-\frac{1}{2}}^2}{h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}}}\frac{\partial^3 u}{\partial x^3}(\zeta_{ik}^{(2)}, t_{k+1}), \end{aligned} \tag{14}$$

From the notation, we get

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1}) = \delta_x^2 U_i^{k+1} + R_{ik}^{(2)}, \tag{15}$$

where

$$R_{ik}^{(2)} = -\frac{h_{i+\frac{1}{2}}^2}{h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}}} \frac{\partial^3 u}{\partial x^3}(\zeta_{ik}^{(1)}, t_{k+1})$$

$$+\frac{h_{i-\frac{1}{2}}^2}{h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}}} \frac{\partial^3 u}{\partial x^3}(\zeta_{ik}^{(2)}, t_{k+1}).$$

For the reaction part, similarly by Taylor expansion, we have

$$f(u(x_i, t_{k+1}) = f(u(x_i, t_k)) + \tau f'(u(x_i, t_k))$$

$$\times \frac{\partial u}{\partial t}(x_i, t_k) + \frac{\tau^2}{2}\Big\{f''(u(x_i, \eta_{ik}^{(2)}))\Big(\frac{\partial u}{\partial t}(x_i, \eta_{ik}^{(2)})\Big)^2$$

$$+f'(u(x_i, \eta_{ik}^{(2)}))\frac{\partial^2 u}{\partial t^2}(x_i, \eta_{ik}^{(2)})\Big\}, \tag{16}$$

where $\eta_{ik}^{(2)}$ is between t_k and t_{k+1}.
From the notation,we get

$$f(u(x_i, t_{k+1}) = f(U_i^k) + \tau f'(U_i^k)\Delta_t U_i^k + R_{ik}^{(3)}, \tag{17}$$

where

$$R_{ik}^{(3)} = \frac{\tau^2}{2}\Big\{f''(u(x_i, \eta_{ik}^{(2)}))\Big(\frac{\partial u}{\partial t}(x_i, \eta_{ik}^{(2)})\Big)^2$$

$$+f'(u(x_i, \eta_{ik}^{(2)}))\frac{\partial^2 u}{\partial t^2}(x_i, \eta_{ik}^{(2)})\Big\}.$$

Substituting (9),(15) and (17) into initial problem of Eqs. (1)-(3), we get

$$(1 - \tau f'(U_i^k))\triangle_t U_i^k = \delta_x^2 U_i^{k+1} + f(U_i^k) + R_{ik},$$
$$1 \le i \le I-1,\ \ 0 \le k \le K-1, \tag{18}$$

$$U_i^0 = \varphi(x_i), \quad 0 \le i \le I, \tag{19}$$

$$U_0^k = \alpha(t_k), \quad U_I^k = \beta(t_k), \quad 0 \le k \le K, \tag{20}$$

where $R_{ik} = R_{ik}^{(1)} + R_{ik}^{(2)} + R_{ik}^{(3)} = O(\tau + h)$, so we get the difference scheme with non-uniform meshes (4)-(6).

Subtracting (4)-(6) from (18)-(20), we arrive at the error equations:

$$(1 - \tau f'(U_i^k))\Delta_t e_i^k = \delta_x^2 e_i^{k+1} + f(U_i^k) - f(u_i^k) +$$

$$\tau[f'(U_i^k) - f'(u_i^k)]\Delta_t u_i^k + R_{ik},$$

$$1 \le i \le I-1,\ \ 0 \le k \le K-1, \tag{21}$$

$$e_i^0 = 0, \quad 0 \le i \le I, \tag{22}$$

$$e_0^k = 0,\ \ e_I^k = 0,\ \ 0 \le k \le K, \tag{23}$$

By the differentiability of f and the differential mean value theorem, the second and the third term of (21) are changed into

$$f(U_i^k) - f(u_i^k) = f'(\xi_{ik}^{(1)})e_i^k, \tag{24}$$

where $\xi_{ik}^{(1)}$ is between U_i^k and u_i^k.

The fourth term of (21) is changed to

$$\tau[f'(U_i^k) - f'(u_i^k)]\Delta_t u_i^k = \tau f''(\xi_{ik}^{(2)})e_i^k\Delta_t u_i^k$$

$$-\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_t U_i^k + \tau f''(\xi_{ik}^{(2)})e_i^k\Delta_t U_i^k$$

$$= \tau f''(\xi_{ik}^{(2)})e_i^k\Delta_t U_i^k - \tau f''(\xi_{ik}^{(2)})e_i^k\Delta_t e_i^k, \tag{25}$$

where $\xi_{ik}^{(2)}$ is between U_i^k and u_i^k.

Substituting (13)-(14) to (10),we get

$$(1 - \tau f'(U_i^k))\Delta_t e_i^k = \delta_x^2 e_i^{k+1} + f'(\xi_{ik}^{(1)})e_i^k$$

$$+\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_t U_i^k - \tau f''(\xi_{ik}^{(2)})e_i^k\Delta_t e_i^k + R_{ik},$$

$$1 \le i \le I-1,\ \ 0 \le k \le K-1. \tag{26}$$

By assumption (II), we know $|f'(U_i^k)| \le C_1$. It implies

$$1 - \tau C_1 \le 1 - \tau f'(U_i^k) \le 1 + \tau C_1,$$

when $\tau < \frac{1}{C_1}$,

$$\frac{1}{1 + \tau C_1} \le \frac{1}{1 - \tau f'(U_i^k)} \le \frac{1}{1 - \tau C_1}.$$

Set

$$C_2 = \frac{1}{1 - \tau C_1},\ \ C_3 = \frac{1}{1 + \tau C_1},$$

we have

$$C_3 \le \frac{1}{1 - \tau f'(U_i^k)} \le C_2. \tag{27}$$

Therefore by (26), we get

$$\Delta_t e_i^k = \frac{1}{(1 - \tau f'(U_i^k))}\Big(\delta_x^2 e_i^{k+1} + f'(\xi_{ik}^{(1)})e_i^k +$$

$$\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_t U_i^k - \tau f''(\xi_{ik}^{(2)})e_i^k\Delta_t e_i^k + R_{ik}\Big). \tag{28}$$

Multiplying (28) by $\delta_i^2 e_i^{k+1} h_i^{(2)} \tau$, and summing up from 1 to $I-1$, we get

$$\sum_{i=1}^{I-1} \delta_x^2 e_i^{k+1}(e_i^{k+1} - e_i^k) h_i^{(2)}$$
$$= \sum_{i=1}^{I-1} \frac{\tau}{1-\tau f'(U_i^k)} \delta_x^2 e_i^{k+1} \Big(\delta_x^2 e_i^{k+1} + f'(\xi_{ik}^{(1)}) e_i^k$$
$$+ \tau f''(\xi_{ik}^{(2)}) e_i^k \Delta_t U_i^k - \tau f''(\xi_{ik}^{(2)}) e_i^k \Delta_t e_i^k + R_{ik} \Big) h_i^{(2)}, \tag{29}$$

By lemma 2 and the definition of norm, the left hand of (29) can be written as

$$\sum_{i=1}^{I-1} \delta_x^2 e_i^{k+1}(e_i^{k+1} - e_i^k) h_i^{(2)} = -\frac{1}{2}\|\delta_x e_h^{k+1}\|_2^2$$
$$+ \frac{1}{2}\|\delta_x e_h^k\|_2^2 - \frac{1}{2}\|\delta_x(e_h^{k+1} - e_h^k)\|_2^2. \tag{30}$$

In fact, the left hand of (29) is as follows

$$\sum_{i=1}^{I-1} \Big(\delta_x^2 e_i^{k+1} - \delta_x^2 e_i^k + \delta_x^2 e_i^k \Big)\Big(e_i^{k+1} - e_i^k \Big) h_i^{(2)}$$
$$= \sum_{i=1}^{I-1} \delta_x^2 (e_i^{k+1} - e_i^k)(e_i^{k+1} - e_i^k) h_i^{(2)}$$
$$+ \sum_{i=1}^{I-1} \delta_x^2 e_i^k (e_i^{k+1} - e_i^k) h_i^{(2)}$$
$$= I + II,$$

By the definition of the second order difference quotient, lemma 2 and the definition of the first order difference quotient in 2-norm,we get

$$I = -\sum_{i=1}^{I-1} \delta_x(e_{i+\frac{1}{2}}^{k+1} - e_{i+\frac{1}{2}}^k)\delta_x(e_{i+\frac{1}{2}}^{k+1} - e_{i+\frac{1}{2}}^k) h_{i+\frac{1}{2}}$$
$$= -\|\delta_x(e_h^{k+1} - e_h^k)\|_2^2,$$

Similarly, we get

$$II = -\sum_{i=1}^{I-1} \delta_x e_{i+\frac{1}{2}}^k \delta_x(e_{i+\frac{1}{2}}^{k+1} - e_{i+\frac{1}{2}}^k) h_{i+\frac{1}{2}}$$
$$= -\sum_{i=1}^{I-1} \delta_x e_{i+\frac{1}{2}}^k \delta_x e_{i+\frac{1}{2}}^{k+1} h_{i+\frac{1}{2}} + \|\delta_x e_h^k\|_2^2$$
$$= \frac{1}{2}\sum_{i=1}^{I-1} \Big[-2\delta_x e_{i+\frac{1}{2}}^k \delta_x e_{i+\frac{1}{2}}^{k+1} + \Big(\delta_x e_{i+\frac{1}{2}}^k\Big)^2$$
$$- \Big(\delta_x e_{i+\frac{1}{2}}^k\Big)^2 + \Big(\delta_x e_{i+\frac{1}{2}}^{k+1}\Big)^2 - \Big(\delta_x e_{i+\frac{1}{2}}^{k+1}\Big)^2 \Big] h_{i+\frac{1}{2}}$$
$$+ \|\delta_x e_h^k\|_2^2$$
$$= \frac{1}{2}\sum_{i=1}^{I-1} \Big[-2\delta_x e_{i+\frac{1}{2}}^k \delta_x e_{i+\frac{1}{2}}^{k+1} + \Big(\delta_x e_{i+\frac{1}{2}}^k\Big)^2$$
$$+ \Big(\delta_x e_{i+\frac{1}{2}}^{k+1}\Big)^2 \Big] h_{i+\frac{1}{2}} - \frac{1}{2}\sum_{i=1}^{I-1} \Big[\Big(\delta_x e_{i+\frac{1}{2}}^k\Big)^2$$
$$+ \Big(\delta_x e_{i+\frac{1}{2}}^{k+1}\Big)^2 \Big] h_{i+\frac{1}{2}} + \|\delta_x e_h^k\|_2^2$$
$$= \frac{1}{2}\sum_{i=1}^{I-1} \Big(\delta_x e_{i+\frac{1}{2}}^{k+1} - \delta_x e_{i+\frac{1}{2}}^k \Big)^2 h_{i+\frac{1}{2}}$$
$$+ \frac{1}{2}\|\delta_x e^k\|_2^2 - \frac{1}{2}\|\delta_x e^{k+1}\|_2^2$$
$$= \frac{1}{2}\|\delta_x(e_h^{k+1} - e_h^k)\|_2^2 + \frac{1}{2}\|\delta_x e_h^k\|_2^2 - \frac{1}{2}\|\delta_x e_h^{k+1}\|_2^2,$$

By 2-norm's definition of two order difference divided and inequality (27), the first term of right hand in (28) satisfies

$$\tau \sum_{i=1}^{I-1} \frac{1}{1-\tau f'(U_i^k)} \delta_x^2 e_i^{k+1} \delta_x^2 e_i^{k+1} h_i^{(2)},$$
$$\geq C_3 \tau \|\delta_x^2 e_h^{k+1}\|_2^2, \tag{31}$$

By (29),(30) and (31), we get

$$C_3\tau\|\delta_x^2 e^{k+1}\|_2^2 + \sum_{i=1}^{I-1} \frac{\tau}{1-\tau f'(U_i^k)} \delta_x^2 e_i^{k+1} \Big(f'(\xi_{ik}^{(1)}) e_i^k$$
$$+ \tau f''(\xi_{ik}^{(2)}) e_i^k \Delta_t U_i^k - \tau f''(\xi_{ik}^{(2)}) e_i^k \Delta_t e_i^k + R_{ik} \Big) h_i^{(2)}$$
$$\leq -\frac{1}{2}\|\delta_x e_h^{k+1}\|_2^2 + \frac{1}{2}\|\delta_x e_h^k\|_2^2 - \frac{1}{2}\|\delta_x(e_h^{k+1} - e_h^k)\|_2^2$$
$$\leq -\frac{1}{2}\|\delta_x e_h^{k+1}\|_2^2 + \frac{1}{2}\|\delta_x e_h^k\|_2^2.$$

By transposing, we get

$$\frac{1}{2}\|\delta_x e_h^{k+1}\|_2^2 - \frac{1}{2}\|\delta_x e_h^k\|_2^2 + C_3\tau\|\delta_x^2 e^{k+1}\|_2^2$$
$$\leq -\tau \sum_{i=1}^{I-1} \frac{1}{1-\tau f'(U_i^k)} \delta_x^2 e_i^{k+1} \Big(f'(\xi_{ik}^{(1)}) e_i^k +$$
$$\tau f''(\xi_{ik}^{(2)}) e_i^k \Delta_t U_i^k - \tau f"(\xi_{ik}^{(2)}) e_i^k \Delta_t e_i^k + R_{ik} \Big) h_i^{(2)}, \tag{32}$$

Using Young's inequality and inequality (27), the right hand of (32) is changed to

$$-\tau\sum_{i=1}^{I-1}\frac{1}{1-\tau f'(U_i^k)}\delta_x^2 e_i^{k+1}\Big(f'(\xi_{ik}^{(1)})e_i^k$$

$$+\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_t U_i^k-\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_t e_i^k+R_{ik}\Big)h_i^{(2)}$$

$$\leq\frac{\varepsilon^2C_3\tau}{2}\|\delta_x^2e_h^{k+1}\|_2^2+\frac{2C_2\tau}{\varepsilon^2C_3}\sum_{i=1}^{I-1}|f'(\xi_{ik}^{(1)})e_i^k$$

$$+\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_tU_i^k-\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_te_i^k+R_{ik}|^2h_i^{(2)},\tag{33}$$

From the second term of the right hand in (33), we get:

$$\frac{2C_2\tau}{\varepsilon^2C_3}\sum_{i=1}^{I-1}|f'(\xi_{ik}^{(1)})e_i^k+\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_tU_i^k$$

$$-\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_te_i^k+R_{ik}|h_i^{(2)}$$

$$\leq\frac{8C_2\tau}{\varepsilon^2C_3}\Bigg\{\sum_{i=1}^{I-1}\Big(f'(\xi_{ik}^{(1)})e_i^k\Big)^2h_i^{(2)}$$

$$+\sum_{i=1}^{I-1}\Big(\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_tU_i^k\Big)^2h_i^{(2)}$$

$$+\sum_{i=1}^{I-1}\Big(\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_te_i^k\Big)^2h_i^{(2)}+\sum_{i=1}^{I-1}R_{ik}^2h_i^{(2)}\Bigg\}$$

$$=Q_1+Q_2+Q_3+Q_4,\tag{34}$$

where

$$Q_1=\frac{8C_2\tau}{\varepsilon^2C_3}\sum_{i=1}^{I-1}\Big(f'(\xi_{ik}^{(1)})e_i^k\Big)^2h_i^{(2)},$$
$$Q_2=\frac{8C_2\tau}{\varepsilon^2C_3}\sum_{i=1}^{I-1}\Big(\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_tU_i^k\Big)^2h_i^{(2)},$$
$$Q_3=\frac{8C_2\tau}{\varepsilon^2C_3}\sum_{i=1}^{I-1}\Big(\tau f''(\xi_{ik}^{(2)})e_i^k\Delta_te_i^k\Big)^2h_i^{(2)},$$
$$Q_4=\frac{8C_2\tau}{\varepsilon^2C_3}\sum_{i=1}^{I-1}R_{ik}^2h_i^{(2)}.$$

We now estimate $Q_i(i=1,2,3,4)$ as follows.

By (22), since

$$\|e^0\|_\infty=0,\|u^0\|_\infty\leq\max_{a\leq x\leq b}|\phi(x)|\leq C_0,$$

by induction hypothesis, there exist positive constant P satisfying

$$\max_{0\leq l\leq k}\|e^l\|_\infty\leq P.$$

Therefore, we have $\|e^l\|_\infty\leq P,\ \ \|u^l\|_\infty\leq\|U\|_\infty+P\leq C_0+P$[7] when $l=0,1,2,\dots,k$. Using assumption (II), we get

$$Q_1\leq\frac{8C_1^2C_2\tau}{\varepsilon^2C_3}\sum_{i=1}^{I-1}(e_i^k)^2h_i^{(2)}=\frac{8C_1^2C_2\tau}{\varepsilon^2C_3}\|e_h^k\|_2^2,$$

$$Q_2=\frac{8C_2\tau}{\varepsilon^2C_3}\sum_{i=1}^{I-1}\Big(\tau f''(\xi_{ik}^{(2)})e_i^k\frac{U_i^{k+1}-U_i^k}{\tau}\Big)^2h_i^{(2)}$$

$$\leq\frac{8C_1^2C_2\tau}{\varepsilon^2C_3}\sum_{i=1}^{I-1}(e_i^k)^2\Big(|U_i^{k+1}|+|U_i^k|\Big)^2h_i^{(2)}$$

$$\leq\frac{32C_0^2C_1^2C_2\tau}{\varepsilon^2C_3}\sum_{i=1}^{I-1}(e_i^k)^2h_i^{(2)}$$

$$=\frac{32C_0^2C_1^2C_2\tau}{\varepsilon^2C_3}\|e_h^k\|_2^2,$$

$$Q_3=\frac{8C_2\tau}{\varepsilon^2C_3}\sum_{i=1}^{I-1}\Big(\tau f''(\xi_{ik}^{(2)})e_i^k\frac{e_i^{k+1}-e_i^k}{\tau}\Big)^2h_i^{(2)}$$

$$\leq\frac{16C_1^2C_2P^2\tau}{\varepsilon^2C_3}\sum_{i=1}^{I-1}\Big((e_i^{k+1})^2+\Big((e_i^k)^2\Big)h_i^{(2)}$$

$$\leq\frac{16C_1^2C_2P^2\tau}{\varepsilon^2C_3}\Big(\|e_h^{k+1}\|_2^2+\|e_h^k\|_2^2\Big),$$

and

$$Q_4\leq C_4\tau O(\tau+h)^2,$$

where C_4 depends upon C_1 and ratio constant Rh_* of meshsteps.

Let $\varepsilon=1$. From (32), (34) and the inequalities about Q_1,Q_2,Q_3,Q_4, we get

$$\|\delta_xe_h^{k+1}\|_2^2-\|\delta_xe_h^k\|_2^2+C_3\tau\|\delta_x^2e^{k+1}\|_2^2$$

$$\leq C_5\tau\Big(\|e_h^{k+1}\|_2^2+\|e_h^k\|_2^2+O(\tau+h)^2\Big)$$

$$\leq C_5\tau\Big(\|e_h^{k+1}\|_2^2+\|e_h^k\|_2^2+\|\delta_xe_h^{k+1}\|_2^2+\|\delta_xe_h^k\|_2^2$$

$$+O(\tau+h)^2\Big),\tag{35}$$

By lemma 4, formula (35) can be written as

$$\|\delta_x e_h^{k+1}\|_2^2 - \|\delta_x e_h^k\|_2^2 \le C_6\tau\Big(\|e_i^{k+1}\|_2^2 + \|e_i^k\|_2^2$$

$$+O(\tau+h)^2\Big).$$

By lemma 5, we get

$$\max_{0\le k\le I} |\delta_x e_h^k\|_2 \le C_7(\tau+h),$$

where C_7 depends on C_1 and the ratio constant Rh_* of meshsteps. Therefore,

$$\max_{0\le k\le K}\|e_h^k\|_2,\ \max_{0\le k\le K}\|e_h^k\|_\infty,\ \max_{0\le k\le K}\|\delta_x e_h^k\|_2,$$

$$\Big(\sum_{i=0}^{K-1}\|\delta_x^2 e_h^{k+1}\|_2^2\tau\Big)^{\frac{1}{2}},\ \left(\sum_{i=0}^{K-1}\|\frac{e_h^{k+1}-e_h^k}{\tau}\|_2^2\tau\right)^{\frac{1}{2}}$$

$$\le O(\tau+h),$$

So, we have

$$\max_{0\le k\le K 0\le i\le I}|e_i^k|_2 \le O(\tau+h).$$

By lemma 3,we have

$$\max_{0\le k\le K\ \ 0\le i\le I}|\delta_x e_{i+\frac{1}{2}}^k| \le \tfrac{1}{\sqrt{h_*}}\|\delta_x e^k\|_2$$

$$\le C_7\Big(\tfrac{\tau}{\sqrt{h_*}} + h^{\frac{1}{2}}\sqrt{Rh_*}\Big).$$

When $\frac{\tau}{\sqrt{h_*}}$ is sufficiently small and Rh_* is boundary, we have

$$\max_{0\le k\le K}\|\delta_x e_h^k\|_\infty = O(\frac{\tau}{\sqrt{h_*}}, h^{\frac{1}{2}}).$$

5 Stability

In order to prove stability of the difference scheme, we import the initial boundary problem

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f(v) + \omega(x,t), \tag{36}$$

$$v(x,0) = \varphi(x) + \psi(x),\ \ x\in[a,b], \tag{37}$$

$$v(a,t) = \alpha(t),\ \ v(b,t) = \beta(t),\ \ t\in[0,T], \tag{38}$$

where $\omega(x,t), \psi(x)$ is smooth enough.

Problem (36)-(38) have unique solution $v(x,t)$, which satisfy the assumptions (I),(II) and (III). Suppose that $v_i^k, i = 0,1,\dots,I, k = 0,1,\dots,K+1$ satisfy the following difference scheme:

$$(1-\tau f'(v_i^k))\triangle_t v_i^k = \delta_x^2 v_i^{k+1} + f(v_i^k) + \omega_i^k,$$
$$1\le i\le I-1,\ \ 0\le k\le K-1, \tag{39}$$

$$v_i^0 = \varphi(x_i) + \psi_i, 0\le i\le I, \tag{40}$$

$$v_0^k = \alpha(t_k),\ \ v_I^k = \beta(t_k), 0\le k\le K, \tag{41}$$

where $\omega_i^k = \omega(x_i,t_k), \psi_i = \psi(x_i)$,so we have the following stability theorem.

Theorem 8 *Suppose u_i^k is the numerical solution of the difference scheme (4)-(6),v_i^k is the numerical solution of the difference scheme (36)-(38),denote $z_i^k = v_i^k - u_i^k$, then when h,τ is sufficiently small, $\frac{\tau}{\sqrt{h_*}}$ is sufficiently small too. Then*

$$\|z_h^{k+1}\|_2^2 + \sum_{k=0}^{K-1}\|\delta z_h^{k+1}\|_2^2\tau \le C\Big(\|\psi\|_2^2 + \sum_{k=0}^{K-1}\|\omega^k\|_2^2\tau\Big),$$

where C doesn't depend on h and τ which is a constant.

Proof: Subtracting (4)-(6) from (39)-(41) and by mathematical treatment, we get

$$(1-\tau f'(v_i^k))\triangle_t z_i^k = \delta_x^2 z_i^{k+1} + f(v_i^k) - f(u_i^k)$$

$$+\tau(f'(v_i^k) - f'(u_i^k))\triangle_t u_i^k + \omega_i^k,$$

$$1\le i\le I-1,\ \ 0\le k\le K-1, \tag{42}$$

$$z_i^0 = \psi_i, 0\le i\le I, \tag{43}$$

$$z_0^k = 0,\ \ z_I^k = 0, 0\le k\le K, \tag{44}$$

By the differentiability of f and the differential mean value theorem, the second and the third term of (42) are changed to

$$f(v_i^{k+1}) - f(u_i^{k+1}) = f'(\xi_{ik}^{(3)})z_i^k, \tag{45}$$

where $\xi_{ik}^{(3)}$ is between v_i^k and u_i^k.

Similarly the fourth term of(42) is changed to

$$\tau(f'(v_i^k) - f'(u_i^k))\triangle_t u_i^k = f''(\xi_{ik}^{(4)})z_i^k\triangle_t u_i^k, \tag{46}$$

where $\xi_{ik}^{(4)}$ is between v_i^k and u_i^k. Thus formula (42) becomes into

$$(1-\tau f'(v_i^k))\triangle_t z_i^k = \delta_x^2 z_i^{k+1} + f'(\xi_{ik}^{(3)})z_i^k$$

$$+\tau f''(\xi_{ik}^{(4)})z_i^k\Delta_t u_i^k + \omega_i^k,$$

$$1\le i\le I-1,\ \ 0\le k\le K-1, \tag{47}$$

Multiplying (47) by $z_i^{k+1}h_i^{(2)}\tau$, and summing up from 1 to $I-1$, we get

$$\sum_{i=1}^{I-1}(1-\tau f'(v_i^k))z_i^{k+1}(z_i^{k+1}-z_i^k)h_i^{(2)}$$

$$=\tau\sum_{i=1}^{I-1}\delta_x^2 z_i^{k+1}z_i^{k+1}h_i^{(2)}+\tau\sum_{i=1}^{I-1}f'(\xi_{ik}^{(3)})z_i^k z_i^{k+1}h_i^{(2)}$$

$$+\tau\sum_{i=1}^{I-1}f'(\xi_{ik}^{(4)})z_i^k z_i^{k+1}h_i^{(2)}+\tau\sum_{i=1}^{I-1}\omega_i^k z_i^{k+1}h_i^{(2)},$$
$$0\le k\le K-1, \tag{48}$$

By proper deformation, (48) is changed to

$$\sum_{i=1}^{I-1}z_i^{k+1}(z_i^{k+1}-z_i^k)h_i^{(2)}=\tau\sum_{i=1}^{I-1}\delta_x^2 z_i^{k+1}z_i^{k+1}h_i^{(2)}$$

$$+\tau\sum_{i=1}^{I-1}f'(\xi_{ik}^{(3)})z_i^k z_i^{k+1}h_i^{(2)}+\tau\sum_{i=1}^{I-1}f'(\xi_{ik}^{(4)})z_i^k z_i^{k+1}h_i^{(2)}$$

$$+\tau\sum_{i=1}^{I-1}f'(v_i^k)(z_i^{k+1})^2h_i^{(2)}-\tau\sum_{i=1}^{I-1}f'(v_i^k)z_i^k z_i^{k+1}h_i^{(2)}$$

$$+\sum_{i=1}^{I-1}\omega_i^k z_i^{k+1}h_i^{(2)}\tau,$$
$$0\le k\le K-1, \tag{49}$$

By the method in reference [21, 22], the left hand in (49) is written as

$$\sum_{i=1}^{I-1}z_i^{k+1}(z_i^{k+1}-z_i^k)h_i^{(2)}$$

$$=\sum_{i=1}^{I-1}((z_i^{k+1})^2-z_i^{k+1}z_i^k)h_i^{(2)}$$

$$=\|z_h^{k+1}\|_2^2+\tfrac{1}{2}\sum_{i=1}^{I-1}(-2z_i^{k+1}z_i^k+(z_i^{k+1})^2$$

$$+(z_i^k)^2)h_i^{(2)}-\frac{1}{2}\|z_h^{k+1}\|_2^2-\frac{1}{2}\|z_h^k\|_2^2$$

$$=\frac{1}{2}(\|z_h^{k+1}\|_2^2-\|z_h^k\|_2^2)+\frac{1}{2}\|z_h^{k+1}-z_h^k\|_2^2, \tag{50}$$

By lemma 2 and the definition of 2-norm, the first term of the right hand in (49) is written into

$$\tau\sum_{i=1}^{I-1}\delta_x^2 z_i^{k+1}z_i^{k+1}h_i^{(2)}=-\tau\sum_{i=1}^{I-1}(\delta_x z_i^{k+1})^2h_{i+\frac{1}{2}}$$

$$=-\tau\|\delta z_h^{k+1}\|_2^2, \tag{51}$$

By assumption (II) and the definition of 2-norm, the fourth term of the right hand in (49) has estimation

$$\tau\sum_{i=1}^{I-1}f'(v_i^k)(z_i^{k+1})^2h_i^{(2)}$$

$$\le C_1\tau\sum_{i=1}^{I-1}(z_i^{k+1})^2h_i^{(2)}=C_1\tau\|z_h^{k+1}\|_2^2, \tag{52}$$

Using the mean inequality and the definition of 2-norm, the second term, the third term, the fifth term and the sixth term have estimations as follows

$$\tau\sum_{i=1}^{I-1}f'(\xi_{ik}^{(3)})z_i^k z_i^{k+1}h_i^{(2)}\le\frac{1}{2}C_1\tau\sum_{i=1}^{I-1}(z_i^k)^2+$$

$$\frac{1}{2}C_1\tau\sum_{i=1}^{I-1}(z_i^{k+1})^2h_i^{(2)}=\frac{1}{2}C_1\tau\|z_h^k\|_2^2$$

$$+\frac{1}{2}C_1\tau\|z_h^{k+1}\|_2^2, \tag{53}$$

$$\tau\sum_{i=1}^{I-1}f'(\xi_{ik}^{(4)})z_i^k z_i^{k+1}h_i^{(2)}\le\frac{1}{2}C_1\tau\|z_h^k\|_2^2$$
$$+\frac{1}{2}C_1\tau\|z_h^{k+1}\|_2^2, \tag{54}$$

$$-\tau\sum_{i=1}^{I-1}f'(v_i^k)z_i^k z_i^{k+1}h_i^{(2)}\le\frac{1}{2}C_1\tau\|z_h^k\|_2^2$$
$$+\frac{1}{2}C_1\tau\|z_h^{k+1}\|_2^2, \tag{55}$$

$$\sum_{i=1}^{I-1}\omega_i^k z_i^{k+1}h_i^{(2)}\tau\le\frac{1}{2}\tau\sum_{i=1}^{I-1}(\omega_i^k)^2h_i^{(2)}$$

$$+\frac{1}{2}\tau\sum_{i=1}^{I-1}(z_i^{k+1})^2h_i^{(2)}=\frac{1}{2}\tau\|\omega_h^k\|_2^2+\frac{1}{2}\tau\|z_h^{k+1}\|_2^2, \tag{56}$$

Combining (50)-(56), we have

$$\|z_h^{k+1}\|_2^2-\|z_h^k\|_2^2+\|\delta z_h^{k+1}\|_2^2\tau$$

$$\le C_8\tau(\|z_h^{k+1}\|_2^2+\|z_h^k\|_2^2+\|\omega_h^k\|_2^2), \tag{57}$$

where C_8 is dependent on C_1, but independent of h and τ. By discrete Gronwall's inequality and lemma 2, we have

$$\|z_h^{k+1}\|_2^2+\sum_{k=0}^{K-1}\|\delta z_h^{k+1}\|_2^2\tau$$

$$\le C_8(\|\psi_h\|_2^2+\sum_{k=0}^{K-1}\|\omega_h^k\|_2^2\tau). \tag{58}$$

Therefore, the Theorem 8 is proved. □

6 Numerical Experiments and Conclusion

Numerical example We apply the difference scheme proposed in this paper to the following initial boundary problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2(1-u),$$

$$u(x,0) = \frac{1}{1+e^{\frac{\sqrt{2}}{2}x}}, \quad x \in [-50, 50],$$

$$u(-50,t) = \frac{1}{1+e^{-25\sqrt{2}-\frac{1}{2}t}},$$

$$u(50,t) = \frac{1}{1+e^{25\sqrt{2}-\frac{1}{2}t}}, t \in [0, 10],$$

the classical solution is $u(x,t) = \dfrac{1}{1+e^{\frac{\sqrt{2}}{2}x-\frac{1}{2}t}}$.

Firstly, we introduce the generation method of non-uniform meshs. From the curve of initial function (see in Figure 1), we can see that the curve vary quickly near $x = 0$, but it changes gently near the two endpoints. Using the transformation $x = \frac{50\sinh(\alpha\xi)}{\sinh(\alpha)}$ as used in [21], we transform the uniform grid nodes ξ_i in $[-1, 1]$ to non-uniform grid nodes x_i in $[-50, 50]$. From Figure 2, we see that the grid nodes are centralized near $x = 0$, the grid nodes are relative sparse on the interval endpoints, the bigger the transformation parameter α is, the more the grid nodes is centralized.

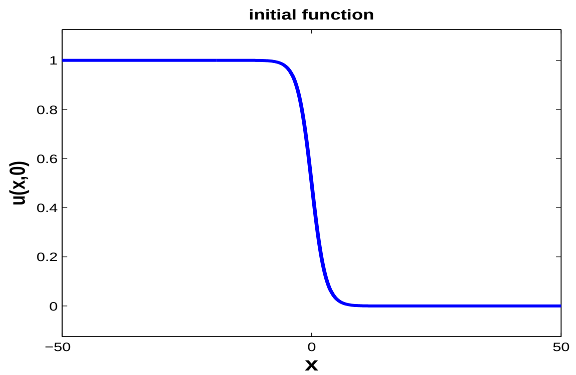

Figure 1: the curve of the initial function

Secondly, we search the optimized transformation parameter α for different grid partition. Here the optimal parameter is the parameter that makes the numerical solution's error attains it minimum. In Figure 3, numerical solution's error is in the sense of L_2 norm,

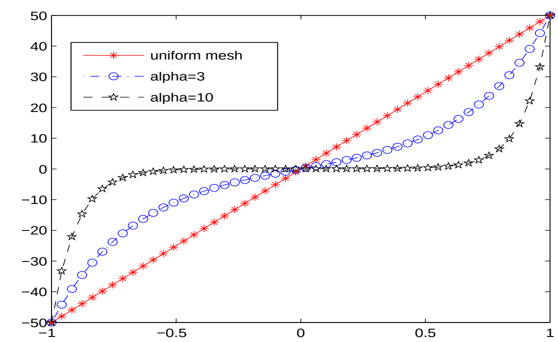

Figure 2: nonuniform grid nodes changed under transformation with parameter α

when $I = 100$, $\tau = 0.25$. This figure only is an example. The similar results for other grid partition can be obtained by the same method.

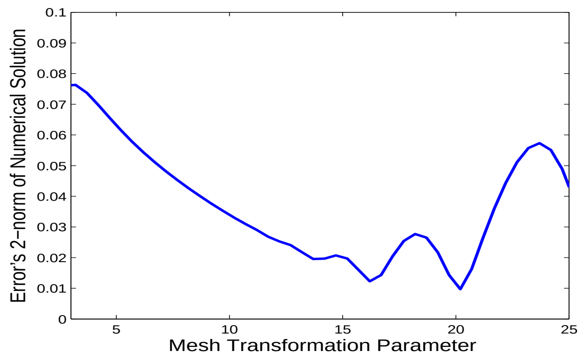

Figure 3: Variation of Error of the Numerical Solution in L_2 Norm

From Figure 3, we see that the error decays with the increasing of the transformation parameter α. It implies that the more the grid nodes are centralized, the less the error is. But this grid centralization cannot be unlimited, this is because Rh_* and $\frac{\tau}{\sqrt{h_*}}$ may be very big when the grid nodes are centralized to a certain extent. As a result, they do not satisfy the condition of Theorem 7 and Theorem 8 in which these two values are the boundary. The numerical oscillation appearing in the right hand of the curve can prove this point. So the centralization parameter α must be chosen exactly, that develops the exactness of the difference solution and ensures the stability and convergence of the numerical solution. By this method, we get the optimal transformation parameter $\alpha = 13.8$. Similarly, we can get the optimal transformation parameter of the other grid partition.

Table 1: Numerical results

			linearized uniform		linearized non-uniform		implicit non-uniform	
I	τ	α	time	$\|E\|_\infty$	time	$\|E\|_\infty$	time	$\|E\|_\infty$
100	1	11.7	0.0110	0.1270	0.0081	0.1231	0.2701	0.0649
100	1/2	11.7	0.0116	0.0556	0.0111	0.0435	0.5161	0.0438
100	1/4	11.7	0.0126	0.0224	0.0139	0.0184	1.0231	0.0170
100	1/8	11.7	0.0143	0.0067	0.0142	0.0070	2.0114	0.0067
200	1	15.9	0.0114	0.1326	0.0125	0.1288	2.6974	0.0696
200	1/2	15.9	0.0120	0.0619	0.0141	0.0578	5.3530	0.0521
200	1/4	15.9	0.0142	0.0289	0.0134	0.0244	10.643	0.0237
200	1/8	15.9	0.0157	0.0131	0.0198	0.0093	21.193	0.0090
400	1	20	0.0127	0.1346	0.0115	0.1318	20.758	0.0771
400	1/2	20	0.0139	0.0638	0.0375	0.0614	41.459	0.0559
400	1/4	20	0.0159	0.0307	0.0153	0.0284	80.521	0.0277
400	1/8	20	0.0194	0.0148	0.0200	0.0125	157.73	0.0123

The third is to get the minimum parameter of the same space freedom degree and different time steps. Although the transformation parameter α of different grid partition can be applied to solve the numerical solution and the errors are minimum, the convergence order cannot be tested because of the different parameters. In order to test the convergence order for τ, we get the minimum parameter by comparing the different optimized parameters, when space freedom degrees are same, time steps are in half in turn. The parameters α are applied to numerical solving which can justify the convergence order for τ. The value of α and the computing results are listed in Table 1.

The L_∞ norm of the errors for the chosen different α are listed in Table 1 when $T = 10s, I = 100, I = 200$ and $I = 400$. From Table 1, we can see that the linearized difference scheme with non-uniform meshes put forward in this paper is more accurate than that with uniform meshes. Meanwhile, it costs less computing time than implicit difference scheme with non-uniform meshes of [21,22] on condition that the exactness of the numerical solution has little difference. In addition, It is known that the convergence order of the numerical solution is 1st order and stable which is proved in Theorem 7 and Theorem 8. Summarily, the linearized difference scheme studied in this paper is effective.

Acknowledgements: The research was supported by Science and Technology Research Projects of Chongqing City Education Commission (No. KJ111308).

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