On the Explicit Determinants and Singularities of r-circulant and Left r-circulant Matrices with Some Famous Numbers

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Abstract: Let A be a r-circulant matrix and B be a left r-circulant matrix whose first rows are $(P_1, P_2, \ldots, P_n)$, $(Q_1, Q_2, \ldots, Q_n)$, $(J_1, J_2, \ldots, J_n)$ and $(j_1, j_2, \ldots, j_n)$ respectively, where P_n is the Pell number, Q_n is the Pell-Lucas number, J_n is the Jacobsthal number and j_n is the Jacobsthal-Lucas number. In this paper, by using the inverse factorization of polynomial of degree n, the explicit determinants of A and B whose first rows are $(P_1, P_2, \ldots, P_n)$ and $(Q_1, Q_2, \ldots, Q_n)$ are expressed by utilizing only Pell numbers, Pell-Lucas numbers and the parameter r, and the explicit determinants of A and B whose first rows are $(J_1, J_2, \ldots, J_n)$ and $(j_1, j_2, \ldots, j_n)$ are expressed by utilizing only Jacobsthal numbers, Jacobsthal-Lucas numbers and the parameter r. The results not only extend the original results, but also simpler in forms. Also, the singularities of those matrices are discussed. Furthermore, four identities of those famous numbers are given.

Key–Words: r-circulant matrix, Left r-circulant matrix, Determinant, Singularity, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers

1 Introduction

The Pell, Pell-Lucas sequences [1] and the Jacobsthal, Jacobsthal-Lucas sequences [2] are defined by the following recurrence relations, respectively:

$$\begin{aligned} &P_{n+1} = 2P_n + P_{n-1}, \text{ where } P_0 = 0, P_1 = 1, \\ &Q_{n+1} = 2Q_n + Q_{n-1}, \text{ where } Q_0 = 2, Q_1 = 2, \\ &J_{n+1} = J_n + 2J_{n-1}, \text{ where } J_0 = 0, J_1 = 1, \\ &j_{n+1} = j_n + 2j_{n-1}, \text{ where } j_0 = 2, j_1 = 1. \end{aligned}$$

The first few values of the sequences are given by the following table ($n \geq 0$):

n	0	1	2	3	4	5	6	7
P_n	0	1	2	5	12	29	70	169
Q_n	2	2	6	14	34	82	198	478
J_n	0	1	1	3	5	11	21	43
j_n	2	1	5	7	17	31	65	127

The sequences $\{P_n\}$, $\{Q_n\}$, $\{J_n\}$ and $\{j_n\}$ are given by the Binet formulae

$$P_n = \frac{\alpha_1^n - \beta_1^n}{\alpha_1 - \beta_1},$$

$$Q_n = \alpha_1^n + \beta_1^n,$$

$$J_n = \frac{\alpha_2^n - \beta_2^n}{\alpha_2 - \beta_2}$$

and

$$j_n = \alpha_2^n + \beta_2^n,$$

where α_1, β_1 are the roots of the characteristic equation $x^2 - 2x - 1 = 0$, and α_2,β_2 are the roots of the equation $x^2 - x - 2 = 0$.

Definition 1 ([3]). *A r-circulant matrix $M \in M_n$, denoted by $\mathrm{Circ}_r(a_1, a_2, \ldots, a_n)$, is a matrix of the form*

$$M := \begin{pmatrix} a_1 & a_2 & \ldots & a_{n-1} & a_n \\ ra_n & a_1 & a_2 & \ldots & a_{n-1} \\ \vdots & ra_n & a_1 & \ddots & \vdots \\ ra_3 & \vdots & \ddots & \ddots & a_2 \\ ra_2 & ra_3 & \ldots & ra_n & a_1 \end{pmatrix}_{n \times n}.$$

Note that the r-circulant matrix is a circulant matrix [4] when $r = 1$, and is a upper Toeplitz matrix when $r = 0$, and is a skew-circulant matrix [5] when $r = -1$.

Definition 2 ([6]). *A left r-circulant matrix $N \in M_n$, denoted by $\mathrm{LCirc}_r(a_1, a_2, \ldots, a_n)$, is a matrix of the* *form*

$$N := \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2 & a_3 & \dots & a_n & ra_1 \\ a_3 & \iddots & \iddots & \iddots & \vdots \\ \vdots & a_n & ra_1 & \dots & ra_{n-2} \\ a_n & ra_1 & \dots & ra_{n-2} & ra_{n-1} \end{pmatrix}_{n\times n}.$$

Lemma 3 ([6]). *If* $M = \mathrm{Circ}_r(a_1, a_2, \dots, a_n)$*, then*

$$\lambda_k = \sum_{j=1}^{n} a_j \varepsilon_k^{j-1}$$

and

$$\det M = \prod_{k=1}^{n} \lambda_k = \prod_{k=1}^{n} \sum_{j=1}^{n} a_j \varepsilon_k^{j-1},$$

where ε_k $(k = 1, 2, \dots, n)$ *are the roots of the equation*

$$x^n - r = 0. \tag{1}$$

Lemma 4 ([6]). *Let* $M = \mathrm{Circ}_r(a_1, a_2, \dots, a_n)$*. Then* M *is nonsingular if and only if*

$$(f(x), g(x)) = 1,$$

where $f(x) = \sum_{j=1}^{n} a_j x^{j-1}$ *and* $g(x) = x^n - r$ *for* $r \neq 0$.

The r-circulant matrix play an important role in various applications [7, 8, 9, 10]. Boman [11] presented a simple derivation of the Moore-Penrose pseudoinverse of an arbitrary square k-circulant matrix. Recently, there are many interests in properties and generalization of some special matrices involving famous numbers. Djordjević [12] presented a systematic investigation of the incompletes generalized Jacobsthal and Jacobsthal-Lucas numbers. Melham [1] gave some formulae involving Fibonacci and Pell numbers. Shen discussed the bounds of the norms of r-circulant matrix with some famous numbers in [3, 13]. The authors discussed some properties of special matrices involving Fibonacci or Lucas numbers in [14, 15, 16] and introduced certain of generalizations in [17, 18]. Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [19]. Lind presented the determinants of circulant and skew-circulant involving Fibonacci numbers in [20]. Lin gave the determinant of the Fibonacci-Lucas quasi-cyclic matrices [21]. In this paper, by using the inverse factorization of polynomial of degree n, the explicit determinants of the r-circulant and left r-circulant matrix involving Pell numbers, Pell-Lucas numbers are expressed by utilizing only Pell numbers, Pell-Lucas numbers and the parameter r, the explicit determinants of the r-circulant and left r-circulant matrix involving Jacobsthal numbers, Jacobsthal-Lucas numbers are expressed by utilizing only Jacobsthal numbers, Jacobsthal-Lucas numbers and the parameter r. Also, the singularities of those matrices are discussed by judging whether two given polynomials coprime. Furthermore, four identities of those famous numbers are given.

2 Determinants and singularities of r-circulant and left r-circulant matrix with Pell numbers

In this section, we first give a formula, and then give the explicit determinants of $\mathrm{Circ}_r(P_1, P_2, \dots, P_n)$ and $\mathrm{LCirc}_r(P_1, P_2, \dots, P_n)$, and then discuss the singularities of them.

Lemma 5.

$$\prod_{k=1}^{n} (y - \varepsilon_k z) = y^n - rz^n,$$

where ε_k $(k = 1, 2, \dots, n)$ *satisfies the equation* (1) *and* $y, z, r \in \mathbf{C}$.

Proof. When $z = 0$, the result is obvious.
When $z \neq 0$, we deduce that

$$\prod_{k=1}^{n} (y - \varepsilon_k z) = z^n \prod_{k=1}^{n} \left(\frac{y}{z} - \varepsilon_k \right).$$

Since ε_k $(k = 1, 2, \dots, n)$ satisfies the equation (1), so we must have

$$x^n - r = \prod_{k=1}^{n} (x - \varepsilon_k). \tag{2}$$

By using the inverse factorization of polynomial (2), we obtain

$$\prod_{k=1}^{n} (y - \varepsilon_k z) = z^n \left[\left(\frac{y}{z} \right)^n - r \right] = y^n - rz^n.$$

□

Theorem 6. *Let* $A = \mathrm{Circ}_r(P_1, P_2, \dots, P_n)$*. Then*

$$\det A = \frac{(1 - rP_{n+1})^n - r^{n+1} P_n^n}{1 + (-1)^n r^2 - rQ_n}.$$

Furthermore, A is singular if and only if

$$1 - rP_{n+1} - r\rho\omega_k P_n = 0$$

and

$$(1-\alpha_1\rho\omega_k)(1-\beta_1\rho\omega_k) \neq 0,$$

where $\rho = |r|^{\frac{1}{n}}$, $\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$.

Proof. We give the explicit determinant of the matrix A firstly.

By Lemma 3 and ε_k $(k = 1, 2, \ldots, n)$ satisfies the equation (1), we can get

$$\begin{aligned} \lambda_k &= P_1 + P_2\varepsilon_k + \cdots + P_n\varepsilon_k^{n-1} \\ &= \frac{\alpha_1-\beta_1}{\alpha_1-\beta_1} + \frac{\alpha_1^2-\beta_1^2}{\alpha_1-\beta_1}\varepsilon_k + \cdots + \frac{\alpha_1^n-\beta_1^n}{\alpha_1-\beta_1}\varepsilon_k^{n-1} \\ &= \frac{1}{\alpha_1-\beta_1}\Big[\alpha_1\left(1+\alpha_1\varepsilon_k+\cdots+\alpha_1^{n-1}\varepsilon_k^{n-1}\right) \\ &\quad -\beta_1\left(1+\beta_1\varepsilon_k+\cdots+\beta_1^{n-1}\varepsilon_k^{n-1}\right)\Big] \\ &= \frac{1}{\alpha_1-\beta_1}\left(\alpha_1\frac{1-r\alpha_1^n}{1-\alpha_1\varepsilon_k} - \beta_1\frac{1-r\beta_1^n}{1-\beta_1\varepsilon_k}\right) \\ &= \frac{1-rP_{n+1}-\varepsilon_k rP_n}{(1-\alpha_1\varepsilon_k)(1-\beta_1\varepsilon_k)} \end{aligned}$$

and

$$\begin{aligned} \det A &= \prod_{k=1}^{n}\lambda_k \\ &= \prod_{k=1}^{n}\frac{1-rP_{n+1}-\varepsilon_k rP_n}{(1-\alpha_1\varepsilon_k)(1-\beta_1\varepsilon_k)}. \end{aligned}$$

According to Lemma 5, we have

$$\begin{aligned} \det A &= \frac{(1-rP_{n+1})^n - r\,(rP_n)^n}{(1-r\alpha_1^n)(1-r\beta_1^n)} \\ &= \frac{(1-rP_{n+1})^n - r^{n+1}P_n^n}{1+(-1)^n r^2 - rQ_n}. \end{aligned}$$

Next, we discuss the singularity of the matrix A.

If $r = 0$, then all the eigenvalues of the matrix A are 1, and A is nonsingular.

If $r \neq 0$, then the roots of polynomial $g(x) = x^n - r$ are $\rho\omega_k$ $(k = 1, 2, \ldots, n)$, where

$$\rho = |r|^{\frac{1}{n}}, \quad \omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}.$$

So we have

$$\begin{aligned} f(\rho\omega_k) &= P_1 + P_2\rho\omega_k + \cdots + P_n(\rho\omega_k)^{n-1} \\ &= \frac{\alpha_1-\beta_1}{\alpha_1-\beta_1} + \frac{\alpha_1^2-\beta_1^2}{\alpha_1-\beta_1}\rho\omega_k + \cdots + \\ &\quad \frac{\alpha_1^n-\beta_1^n}{\alpha_1-\beta_1}(\rho\omega_k)^{n-1} \\ &= \frac{1}{\alpha_1-\beta_1}\left[\alpha_1\frac{1-r\alpha_1^n}{1-\alpha_1\rho\omega_k} - \beta_1\frac{1-r\beta_1^n}{1-\beta_1\rho\omega_k}\right] \\ &= \frac{1-rP_{n+1}-r\rho\omega_k P_n}{(1-\alpha_1\rho\omega_k)(1-\beta_1\rho\omega_k)}. \end{aligned}$$

By Lemma 4, the matrix A is nonsingular if and only if $f(\rho\omega_k) \neq 0$. That is when

$$(1-\alpha_1\rho\omega_k)(1-\beta_1\rho\omega_k) \neq 0,$$

A is nonsingular if and only if

$$1 - rP_{n+1} - r\rho\omega_k P_n \neq 0$$

for any $r \in \mathbf{C}$.

When $(1-\alpha_1\rho\omega_k)(1-\beta_1\rho\omega_k) = 0$, we have $\rho\omega_k = \frac{1}{\alpha_1}$ or $\rho\omega_k = \frac{1}{\beta_1}$.

If $\rho\omega_k = \frac{1}{\alpha_1}$, then the eigenvalue of A is

$$\lambda_k = \frac{\frac{1}{\alpha_1-\beta_1}\left[\left(\frac{\beta_1}{\alpha_1}\right)^n - 1\right]}{1-\frac{\beta_1}{\alpha_1}} = \frac{P_n}{\alpha_1^{n-1}(\beta_1-\alpha_1)} \neq 0,$$

for $\alpha_1 = 1+\sqrt{2}, \beta_1 = 1-\sqrt{2}, n \in \mathbf{N}_+$, $k = 1, 2, \ldots, n$.

If $\rho\omega_k = \frac{1}{\beta_1}$, then the eigenvalue of A is

$$\lambda_k = \frac{\frac{1}{\alpha_1-\beta_1}\left(1-\frac{\alpha_1^n}{\beta_1^n}\right)}{1-\frac{\alpha_1}{\beta_1}} = \frac{P_n}{\beta_1^{n-1}(\alpha_1-\beta_1)} \neq 0,$$

for $\alpha_1 = 1+\sqrt{2}, \beta_1 = 1-\sqrt{2}, n \in \mathbf{N}_+$, $k = 1, 2, \ldots, n$.

So A is nonsingular for $(1-\alpha_1\rho\omega_k)(1-\beta_1\rho\omega_k) = 0$.

Thus, the proof is completed. □

Theorem 7. *Let* $B = \mathrm{LCirc}_r(P_1, P_2, \ldots, P_n)$. *Then*

$$\det B = \frac{r(r-P_{n+1})^n - P_n^n}{r^2+(-1)^n - rQ_n}(-1)^{\frac{(n-1)(n-2)}{2}}.$$

Furthermore, B is singular if and only if

$$r\rho - \rho P_{n+1} - \omega_k P_n = 0$$

and

$$(\rho-\alpha_1\omega_k)(\rho-\beta_1\omega_k) \neq 0,$$

where $\rho = |r|^{\frac{1}{n}}$, $\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$.

Proof. We give the explicit determinant of the matrix B firstly.

When $r \neq 0$, the matrix B can be written as

$$B = \begin{pmatrix} P_1 & P_2 & \dots & P_n \\ P_2 & \dots & P_n & rP_1 \\ \vdots & \iddots & \iddots & \vdots \\ P_n & rP_1 & \dots & rP_{n-1} \end{pmatrix} = \Gamma^{-1} A_1.$$

where

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{r} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 0 & \frac{1}{r} & \dots & 0 \\ 0 & \frac{1}{r} & 0 & \dots & 0 \end{pmatrix},$$

and

$$A_1 = \begin{pmatrix} P_1 & \dots & P_{n-1} & P_n \\ \frac{1}{r}P_n & P_1 & \dots & P_{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{r}P_2 & \dots & \frac{1}{r}P_n & P_1 \end{pmatrix}.$$

Hence, we have

$$\det B = \det \Gamma^{-1} \det A_1,$$

where A_1 is a $\frac{1}{r}$-circulant matrix and its determinant can be gotten from Theorem 6 by replacing r with $\frac{1}{r}$,

$$\det A_1 = \frac{(1 - \frac{1}{r}P_{n+1})^n - \left(\frac{1}{r}\right)^{n+1} P_n^n}{1 + \frac{(-1)^n}{r^2} - \frac{1}{r}Q_n},$$

and

$$\det \Gamma = (-1)^{\frac{(n-1)(n-2)}{2}} \left(\frac{1}{r}\right)^{n-1}.$$

So

$$\begin{aligned} & \det B = \det A_1 \det \Gamma^{-1} \\ = & \frac{(1 - \frac{1}{r}P_{n+1})^n - \left(\frac{1}{r}\right)^{n+1} P_n^n}{1 + \frac{(-1)^n}{r^2} - \frac{1}{r}Q_n} \times \\ & r^{n-1}(-1)^{\frac{(n-1)(n-2)}{2}} \\ = & \frac{r(r - P_{n+1})^n - P_n^n}{r^2 + (-1)^n - rQ_n}(-1)^{\frac{(n-1)(n-2)}{2}}. \end{aligned}$$

When $r = 0$,

$$\begin{aligned} & \det B = (-1)^{\frac{n(n-1)}{2}} P_n^n \\ = & \frac{r(r - P_{n+1})^n - P_n^n}{r^2 + (-1)^n - rQ_n}(-1)^{\frac{(n-1)(n-2)}{2}}. \end{aligned}$$

Next, we discuss the singularity of the matrix B.

If $r = 0$, then

$$\det B = (-1)^{\frac{n(1-n)}{2}} P_n^n \neq 0$$

for any $n \in \mathbf{N}_+$, and B is nonsingular.

If $r \neq 0$, A_1 is singular if and only if

$$1 - \frac{1}{r}P_{n+1} - \frac{1}{r}|\frac{1}{r}|^{\frac{1}{n}}\omega_k P_n = 0$$

and

$$\left(1 - \alpha_1|\frac{1}{r}|^{\frac{1}{n}}\omega_k\right)\left(1 - \beta_1|\frac{1}{r}|^{\frac{1}{n}}\omega_k\right) \neq 0$$

by Theorem 6. That is

$$r\rho - \rho P_{n+1} - \omega_k P_n = 0$$

and

$$(\rho - \alpha_1\omega_k)(\rho - \beta_1\omega_k) \neq 0.$$

Furthermore, the matrix Γ is nonsingular with $r \neq 0$.

This completes the proof. □

3 Determinants and singularities of r-circulant and left r-circulant matrix with Pell-Lucas numbers

In this section, we first give the explicit determinants of $\mathrm{Circ}_r(Q_1, Q_2, \dots, Q_n)$ and $\mathrm{LCirc}_r(Q_1, Q_2, \dots, Q_n)$, and then discuss the singularities of them.

Theorem 8. *Let $A = \mathrm{Circ}_r(Q_1, Q_2, \dots, Q_n)$. Then*

$$\det A = \frac{(2 - rQ_{n+1})^n - r(rQ_n - 2)^n}{1 + (-1)^n r^2 - rQ_n}.$$

Furthermore, A is singular if and only if

$$2 - rQ_{n+1} + (2 - rQ_n)\rho\omega_k = 0$$

and

$$(1 - \alpha_1\rho\omega_k)(1 - \beta_1\rho\omega_k) \neq 0,$$

where $\rho = |r|^{\frac{1}{n}}$, $\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$.

Proof. We give the explicit determinant of the matrix A firstly. By Lemma 3, we obtain

$$\begin{aligned} \lambda_k &= Q_1 + Q_2\varepsilon_k + \dots + Q_n\varepsilon_k^{n-1} \\ &= (\alpha_1 + \beta_1) + \left(\alpha_1^2 + \beta_1^2\right)\varepsilon_k + \dots + \\ & \quad (\alpha_1^n + \beta_1^n)\varepsilon_k^{n-1} \\ &= \alpha_1\left(1 + \alpha_1\varepsilon_k + \dots + \alpha_1^{n-1}\varepsilon_k^{n-1}\right) + \\ & \quad \beta_1\left(1 + \beta_1\varepsilon_k + \dots + \beta_1^{n-1}\varepsilon_k^{n-1}\right) \\ &= \alpha_1\frac{1 - r\alpha_1^n}{1 - \alpha_1\varepsilon_k} + \beta_1\frac{1 - r\beta_1^n}{1 - \beta_1\varepsilon_k} \\ &= \frac{(2 - rQ_{n+1}) + (2 - rQ_n)\varepsilon_k}{(1 - \alpha_1\varepsilon_k)(1 - \beta_1\varepsilon_k)}, \end{aligned}$$

and

$$\det A = \prod_{k=1}^{n} \lambda_k = \prod_{k=1}^{n} \frac{(2 - rQ_{n+1}) + (2 - rQ_n)\varepsilon_k}{(1 - \alpha_1\varepsilon_k)(1 - \beta_1\varepsilon_k)}.$$

According to Lemma 5, we have

$$\det A = \frac{(2 - rQ_{n+1})^n - r(rQ_n - 2)^n}{(1 - r\alpha_1^n)(1 - r\beta_1^n)} = \frac{(2 - rQ_{n+1})^n - r(rQ_n - 2)^n}{1 + (-1)^n r^2 - rQ_n}.$$

Next, we discuss the singularity of A.

If $r = 0$, then all the eigenvalues of A are 2, and A is nonsingular.

If $r \neq 0$, then the roots of polynomial $g(x) = x^n - r$ are $\rho\omega_k$ $(k = 1, 2, \ldots, n)$, where

$$\rho = |r|^{\frac{1}{n}}, \omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}.$$

Thus we have

$$\begin{aligned} & f(\rho\omega_k) = Q_1 + Q_2\rho\omega_k + \cdots + Q_n(\rho\omega_k)^{n-1} \\ = & \ \alpha_1\left(1 + \alpha_1\rho\omega_k + \cdots + \alpha_1^{n-1}(\rho\omega_k)^{n-1}\right) \\ & + \beta_1\left(1 + \beta_1\rho\omega_k + \cdots + \beta_1^{n-1}(\rho\omega_k)^{n-1}\right) \\ = & \ \alpha_1\frac{1 - r\alpha_1^n}{1 - \alpha_1\rho\omega_k} + \beta_1\frac{1 - r\beta_1^n}{1 - \beta_1\rho\omega_k} \\ = & \ \frac{2 - rQ_{n+1} + (2 - rQ_n)\rho\omega_k}{(1 - \alpha_1\rho\omega_k)(1 - \beta_1\rho\omega_k)}. \end{aligned}$$

By Lemma 4, the matrix A is nonsingular if and only if $f(\rho\omega_k) \neq 0$. That is when

$$(1 - \alpha_1\rho\omega_k)(1 - \beta_1\rho\omega_k) \neq 0,$$

A is nonsingular if and only if

$$2 - rQ_{n+1} + (2 - rQ_n)\rho\omega_k \neq 0$$

for any $r \in \mathbf{C}$.

When

$$(1 - \alpha_1\rho\omega_k)(1 - \beta_1\rho\omega_k) = 0,$$

we have $\rho\omega_k = \frac{1}{\alpha_1}$ or $\rho\omega_k = \frac{1}{\beta_1}$.

If

$$\rho\omega_k = \frac{1}{\alpha_1},$$

then the eigenvalue of the matrix A is

$$\lambda_k = \frac{\beta_1^n}{\alpha_1^{n-1}(\beta_1 - \alpha_1)} \neq 0,$$

for $\alpha_1 = 1 + \sqrt{2}, \beta_1 = 1 - \sqrt{2}, k = 1, 2, \ldots, n.$

If

$$\rho\omega_k = \frac{1}{\beta_1},$$

then the eigenvalue of the matrix A is

$$\lambda_k = \frac{\alpha_1^n}{\beta_1^{n-1}(\alpha_1 - \beta_1)} \neq 0,$$

for $\alpha_1 = 1 + \sqrt{2}, \beta_1 = 1 - \sqrt{2}, k = 1, 2, \ldots, n.$

So the matrix A is nonsingular for $(1 - \alpha_1\rho\omega_k)(1 - \beta_1\rho\omega_k) = 0$.

Hence, the proof is completed. □

Theorem 9. *Let* $B = \mathrm{LCirc}_r(Q_1, Q_2, \ldots, Q_n)$. *Then*

$$\det B = \frac{r(2r - Q_{n+1})^n - (Q_n - 2r)^n}{r^2 + (-1)^n - rQ_n}(-1)^{\frac{(n-1)(n-2)}{2}}.$$

Furthermore, B *is singular if and only if*

$$2r\rho - \rho Q_{n+1} + (2r - Q_n)\omega_k = 0$$

and

$$(\rho - \alpha_1\omega_k)(\rho - \beta_1\omega_k) \neq 0,$$

where $\rho = |r|^{\frac{1}{n}}$, $\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$.

Proof. We give the explicit determinant of the matrix B firstly.

When $r \neq 0$, the matrix B can be written as

$$B = \begin{pmatrix} Q_1 & Q_2 & \ldots & Q_n \\ Q_2 & \ldots & Q_n & rQ_1 \\ \vdots & \iddots & \iddots & \vdots \\ Q_n & rQ_1 & \ldots & rQ_{n-1} \end{pmatrix} = \Gamma^{-1} A_2.$$

where

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \frac{1}{r} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 0 & \frac{1}{r} & \ldots & 0 \\ 0 & \frac{1}{r} & 0 & \ldots & 0 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} Q_1 & \ldots & Q_{n-1} & Q_n \\ \frac{1}{r}Q_n & Q_1 & \cdots & Q_{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{r}Q_2 & \ldots & \frac{1}{r}Q_n & Q_1 \end{pmatrix}.$$

Thus, we have

$$\det B = \det \Gamma^{-1} \det A_2,$$

where A_2 is a $\frac{1}{r}$-circulant matrix and its determinant can be obtained from Theorem 8 by replacing r with $\frac{1}{r}$,

$$\det A_2 = \frac{\left(2 - \frac{1}{r}Q_{n+1}\right)^n - \frac{1}{r}\left(\frac{1}{r}Q_n - 2\right)^n}{1 + \frac{(-1)^n}{r^2} - \frac{1}{r}Q_n},$$

where

$$\det \Gamma = (-1)^{\frac{(n-1)(n-2)}{2}} \left(\frac{1}{r}\right)^{n-1}.$$

So

$$\begin{aligned} &\det B = \det A_2 \det \Gamma^{-1} \\ = &\frac{\left(2 - \frac{1}{r}Q_{n+1}\right)^n - \frac{1}{r}\left(\frac{1}{r}Q_n - 2\right)^n}{1 + \frac{(-1)^n}{r^2} - \frac{1}{r}Q_n} \times \\ &r^{n-1}(-1)^{\frac{(n-1)(n-2)}{2}} \\ = &\frac{r(2r - Q_{n+1})^n - (Q_n - 2r)^n}{r^2 + (-1)^n - rQ_n} \times \\ &(-1)^{\frac{(n-1)(n-2)}{2}}. \end{aligned}$$

When $r = 0$,

$$\begin{aligned} &\det B = (-1)^{\frac{n(n-1)}{2}} Q_n^n \\ = &\frac{r(2r - Q_{n+1})^n - (Q_n - 2r)^n}{r^2 + (-1)^n - rQ_n} \times \\ &(-1)^{\frac{(n-1)(n-2)}{2}}. \end{aligned}$$

Next we discuss the singularity of B.

If $r = 0$, then

$$\det B = (-1)^{\frac{n(1-n)}{2}} Q_n^n \neq 0$$

for any $n \in \mathbf{N}_+$, and B is nonsingular.

If $r \neq 0$, A_2 is singular if and only if

$$2 - \frac{1}{r}Q_{n+1} + \left(2 - \frac{1}{r}Q_n\right)|\frac{1}{r}|^{\frac{1}{n}}\omega_k = 0$$

and

$$\left(1 - \alpha_1|\frac{1}{r}|^{\frac{1}{n}}\omega_k\right)\left(1 - \beta_1|\frac{1}{r}|^{\frac{1}{n}}\omega_k\right) \neq 0$$

by Theorem 8. That is

$$2r\rho - \rho Q_{n+1} + (2r - Q_n)\,\omega_k = 0$$

and

$$(\rho - \alpha_1\omega_k)(\rho - \beta_1\omega_k) \neq 0.$$

Furthermore, the matrix Γ is nonsingular with $r \neq 0$. Thus, the proof is completed. □

4 Determinants and singularities of r-circulant and left r-circulant matrix with Jacobsthal numbers

In this section, we first give the explicit determinants of $\mathrm{Circ}_r(J_1, J_2, \ldots, J_n)$ and $\mathrm{LCirc}_r(J_1, J_2, \ldots, J_n)$, and then discuss the singularities of them.

Theorem 10. *Let $A = \mathrm{Circ}_r(J_1, J_2, \ldots, J_n)$. Then*

$$\det A = \frac{(1 - rJ_{n+1})^n - 2^n r^{n+1} J_n^n}{1 + (-2)^n r^2 - rj_n}.$$

Furthermore, A is singular if and only if

$$1 - rJ_{n+1} - 2r\rho\omega_k J_n = 0$$

and

$$(1 - \alpha_2\rho\omega_k)(1 - \beta_2\rho\omega_k) \neq 0,$$

where $\rho = |r|^{\frac{1}{n}}$, $\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$.

Proof. We give the explicit determinant of the matrix A firstly.

By Lemma 3 and ε_k ($k = 1, 2, \ldots, n$) satisfies the equation (1), we can get

$$\begin{aligned} &\lambda_k = J_1 + J_2\varepsilon_k + \cdots + J_n\varepsilon_k^{n-1} \\ = &\frac{\alpha_2 - \beta_2}{\alpha_2 - \beta_2} + \frac{\alpha_2^2 - \beta_2^2}{\alpha_2 - \beta_2}\varepsilon_k + \cdots + \frac{\alpha_2^n - \beta_2^n}{\alpha_2 - \beta_2}\varepsilon_k^{n-1} \\ = &\frac{1}{\alpha_2 - \beta_2}\Big[\alpha_2\left(1 + \alpha_2\varepsilon_k + \cdots + \alpha_2^{n-1}\varepsilon_k^{n-1}\right) \\ &-\beta_2\left(1 + \beta_2\varepsilon_k + \cdots + \beta_2^{n-1}\varepsilon_k^{n-1}\right)\Big] \\ = &\frac{1}{\alpha_2 - \beta_2}\left(\alpha_2\frac{1 - r\alpha_2^n}{1 - \alpha_2\varepsilon_k} - \beta_2\frac{1 - r\beta_2^n}{1 - \beta_2\varepsilon_k}\right) \\ = &\frac{1 - rJ_{n+1} - 2\varepsilon_k rJ_n}{(1 - \alpha_2\varepsilon_k)(1 - \beta_2\varepsilon_k)} \end{aligned}$$

and

$$\begin{aligned} \det A &= \prod_{k=1}^{n} \lambda_k \\ &= \prod_{k=1}^{n} \frac{1 - rJ_{n+1} - 2\varepsilon_k rJ_n}{(1 - \alpha_2\varepsilon_k)(1 - \beta_2\varepsilon_k)}. \end{aligned}$$

According to Lemma 5, we have

$$\begin{aligned} \det A &= \frac{(1 - rJ_{n+1})^n - r\,(2rJ_n)^n}{(1 - r\alpha_2^n)(1 - r\beta_2^n)} \\ &= \frac{(1 - rJ_{n+1})^n - 2^n r^{n+1} J_n^n}{1 + (-2)^n r^2 - rj_n}. \end{aligned}$$

Next, we discuss the singularity of the matrix A.

If $r = 0$, then all the eigenvalues of the matrix A are 1, and A is nonsingular.

If $r \neq 0$, then the roots of polynomial $g(x) = x^n - r$ are $\rho\omega_k$ $(k = 1, 2, \dots, n)$, where

$$\rho = |r|^{\frac{1}{n}}, \omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}.$$

So we have

$$\begin{aligned} & f(\rho\omega_k) = J_1 + J_2\rho\omega_k + \cdots + J_n(\rho\omega_k)^{n-1} \\ = & \frac{\alpha_2 - \beta_2}{\alpha_2 - \beta_2} + \frac{\alpha_2^2 - \beta_2^2}{\alpha_2 - \beta_2}\rho\omega_k + \cdots + \\ & \frac{\alpha_2^n - \beta_2^n}{\alpha_2 - \beta_2}(\rho\omega_k)^{n-1} \\ = & \frac{1}{\alpha_2 - \beta_2}\left[\alpha_2\frac{1 - r\alpha_2^n}{1 - \alpha_2\rho\omega_k} - \beta_2\frac{1 - r\beta_2^n}{1 - \beta_2\rho\omega_k}\right] \\ = & \frac{1 - rJ_{n+1} - 2r\rho\omega_k J_n}{(1 - \alpha_2\rho\omega_k)(1 - \beta_2\rho\omega_k)}. \end{aligned}$$

By Lemma 4, the matrix A is nonsingular if and only if $f(\rho\omega_k) \neq 0$. That is when

$$(1 - \alpha_2\rho\omega_k)(1 - \beta_2\rho\omega_k) \neq 0,$$

A is nonsingular if and only if

$$1 - rJ_{n+1} - 2r\rho\omega_k J_n \neq 0$$

for any $r \in \mathbf{C}$.

When

$$(1 - \alpha_2\rho\omega_k)(1 - \beta_2\rho\omega_k) = 0,$$

we have $\rho\omega_k = \frac{1}{\alpha_2}$ or $\rho\omega_k = \frac{1}{\beta_2}$.

If

$$\rho\omega_k = \frac{1}{\alpha_2},$$

then the eigenvalue of A is

$$\lambda_k = \frac{\frac{1}{\alpha_2 - \beta_2}\left[\left(\frac{\beta_2}{\alpha_2}\right)^n - 1\right]}{1 - \frac{\beta_2}{\alpha_2}} = \frac{J_n}{\alpha_2^{n-1}(\beta_2 - \alpha_2)} \neq 0,$$

for $\alpha_2 = 2, \beta_2 = -1, n \in \mathbf{N}_+, k = 1, 2, \dots, n$.

If

$$\rho\omega_k = \frac{1}{\beta_2},$$

then the eigenvalue of the matrix A is

$$\lambda_k = \frac{\frac{1}{\alpha_2 - \beta_2}\left[1 - \left(\frac{\beta_2}{\alpha_2}\right)^n\right]}{1 - \frac{\alpha_2}{\beta_2}} = \frac{J_n}{\beta_2^{n-1}(\alpha_2 - \beta_2)} \neq 0,$$

for $\alpha_2 = 2, \beta_2 = -1, n \in \mathbf{N}_+, k = 1, 2, \dots, n$.

So A is nonsingular for $(1 - \alpha_2\rho\omega_k)(1 - \beta_2\rho\omega_k) = 0$. The proof is then completed. □

Theorem 11. *Let $B = \mathrm{LCirc}_r(J_1, J_2, \dots, J_n)$. Then*

$$\det B = \frac{r(r - J_{n+1})^n - 2^n J_n^n}{r^2 + (-2)^n - rj_n}(-1)^{\frac{(n-1)(n-2)}{2}}.$$

Furthermore, B is singular if and only if

$$r\rho - \rho J_{n+1} - 2\omega_k J_n = 0$$

and

$$(\rho - \alpha_2\omega_k)(\rho - \beta_2\omega_k) \neq 0,$$

where $\rho = |r|^{\frac{1}{n}}$, $\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$.

Proof. We give the explicit determinant of the matrix B firstly.

When $r \neq 0$, the matrix B can be written as

$$B = \begin{pmatrix} J_1 & J_2 & \dots & J_n \\ J_2 & \dots & J_n & rJ_1 \\ \vdots & \iddots & \iddots & \vdots \\ J_n & rJ_1 & \dots & rJ_{n-1} \end{pmatrix} = \Gamma^{-1} A_3.$$

where

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{r} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 0 & \frac{1}{r} & \dots & 0 \\ 0 & \frac{1}{r} & 0 & \dots & 0 \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} J_1 & \dots & J_{n-1} & J_n \\ \frac{1}{r}J_n & J_1 & \dots & J_{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{r}J_2 & \dots & \frac{1}{r}J_n & J_1 \end{pmatrix}.$$

Hence, we have

$$\det B = \det \Gamma^{-1} \det A_3,$$

where A_3 is a $\frac{1}{r}$-circulant matrix and its determinant can be gotten from Theorem 10 by replacing r with $\frac{1}{r}$,

$$\det A_3 = \frac{(1 - \frac{1}{r}J_{n+1})^n - 2^n\left(\frac{1}{r}\right)^{n+1} J_n^n}{1 + \frac{(-2)^n}{r^2} - \frac{1}{r}j_n},$$

and

$$\det \Gamma = (-1)^{\frac{(n-1)(n-2)}{2}}\left(\frac{1}{r}\right)^{n-1}.$$

So

$$\begin{aligned}\det B &= \det A_3 \det \Gamma^{-1}\\ &= \frac{(1-\frac{1}{r}J_{n+1})^n - 2^n\left(\frac{1}{r}\right)^{n+1}J_n^n}{1+\frac{(-2)^n}{r^2}-\frac{1}{r}j_n}\times\\ &\quad r^{n-1}(-1)^{\frac{(n-1)(n-2)}{2}}\\ &= \frac{r(r-J_{n+1})^n - 2^nJ_n^n}{r^2+(-2)^n-rj_n}(-1)^{\frac{(n-1)(n-2)}{2}}.\end{aligned}$$

When $r=0$,

$$\begin{aligned}\det B &= (-1)^{\frac{n(n-1)}{2}}J_n^n\\ &= \frac{r(r-J_{n+1})^n - 2^nJ_n^n}{r^2+(-2)^n-rj_n}(-1)^{\frac{(n-1)(n-2)}{2}}.\end{aligned}$$

Next, we discuss the singularity of the matrix B.

If $r=0$, then

$$\det B = (-1)^{\frac{n(1-n)}{2}}J_n^n \neq 0$$

for any $n\in\mathbf{N}_+$, and B is nonsingular.

If $r\neq 0$, A_3 is singular if and only if

$$1-\frac{1}{r}J_{n+1}-\frac{2}{r}|\frac{1}{r}|^{\frac{1}{n}}\omega_kJ_n=0$$

and

$$\left(1-\alpha_2|\frac{1}{r}|^{\frac{1}{n}}\omega_k\right)\left(1-\beta_2|\frac{1}{r}|^{\frac{1}{n}}\omega_k\right)\neq 0$$

by Theorem 10. That is

$$r\rho-\rho J_{n+1}-2\omega_kJ_n=0$$

and

$$(\rho-\alpha_2\omega_k)(\rho-\beta_2\omega_k)\neq 0.$$

Furthermore, the matrix Γ is nonsingular with $r\neq 0$. This completes the proof. □

5 Determinants and singularities of r-circulant and left r-circulant matrix with Jacobsthal-Lucas numbers

In this section, we first give the explicit determinants of $\mathrm{Circ}_r(j_1,j_2,\ldots,j_n)$ and $\mathrm{LCirc}_r(j_1,j_2,\ldots,j_n)$, and then discuss the singularities of them.

Theorem 12. *Let $A=\mathrm{Circ}_r(j_1,j_2,\ldots,j_n)$. Then*

$$\det A = \frac{(1-rj_{n+1})^n-2^nr(rj_n-2)^n}{1+(-2)^nr^2-rj_n}.$$

Furthermore, A is singular if and only if

$$1-rj_{n+1}+2\left(2-rj_n\right)\rho\omega_k=0$$

and

$$(1-\alpha_2\rho\omega_k)(1-\beta_2\rho\omega_k)\neq 0,$$

where $\rho=|r|^{\frac{1}{n}}$, $\omega_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$.

Proof. We give the explicit determinant of the matrix A firstly. By Lemma 3, we obtain

$$\begin{aligned}\lambda_k &= j_1+j_2\varepsilon_k+\cdots+j_n\varepsilon_k^{n-1}\\ &= (\alpha_2+\beta_2)+\left(\alpha_2^2+\beta_2^2\right)\varepsilon_k+\cdots+\\ &\quad (\alpha_2^n+\beta_2^n)\varepsilon_k^{n-1}\\ &= \alpha_2\left(1+\alpha_2\varepsilon_k+\cdots+\alpha_2^{n-1}\varepsilon_k^{n-1}\right)+\\ &\quad \beta_2\left(1+\beta_2\varepsilon_k+\cdots+\beta_2^{n-1}\varepsilon_k^{n-1}\right)\\ &= \alpha_2\frac{1-r\alpha_2^n}{1-\alpha_2\varepsilon_k}+\beta_2\frac{1-r\beta_2^n}{1-\beta_2\varepsilon_k}\\ &= \frac{(1-rj_{n+1})+2(2-rj_n)\varepsilon_k}{(1-\alpha_2\varepsilon_k)(1-\beta_2\varepsilon_k)},\end{aligned}$$

and

$$\begin{aligned}\det A &= \prod_{k=1}^{n}\lambda_k\\ &= \prod_{k=1}^{n}\frac{(1-rj_{n+1})+2(2-rj_n)\varepsilon_k}{(1-\alpha_2\varepsilon_k)(1-\beta_2\varepsilon_k)}.\end{aligned}$$

According to Lemma 5, we have

$$\begin{aligned}\det A &= \frac{(1-rj_{n+1})^n-2^nr(rj_n-2)^n}{(1-r\alpha_2^n)(1-r\beta_2^n)}\\ &= \frac{(1-rj_{n+1})^n-2^nr(rj_n-2)^n}{1+(-2)^nr^2-rj_n}.\end{aligned}$$

Next, we discuss the singularity of A.

If $r=0$, then all the eigenvalues of A are 1, and A is nonsingular.

If $r\neq 0$, then the roots of polynomial $g(x)=x^n-r$ are $\rho\omega_k$ $(k=1,2,\ldots,n)$, where

$$\rho=|r|^{\frac{1}{n}},\omega_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}.$$

Thus we have

$$f(\rho\omega_k)=j_1+j_2\rho\omega_k+\cdots+j_n(\rho\omega_k)^{n-1}$$

$$\begin{aligned} &= \alpha_2\left(1+\alpha_2\rho\omega_k+\cdots+\alpha_2^{n-1}\left(\rho\omega_k\right)^{n-1}\right)\\ &\quad+\beta_2\left(1+\beta_2\rho\omega_k+\cdots+\beta_2^{n-1}\left(\rho\omega_k\right)^{n-1}\right)\\ &= \alpha_2\frac{1-r\alpha_2^n}{1-\alpha_2\rho\omega_k}+\beta_2\frac{1-r\beta_2^n}{1-\beta_2\rho\omega_k}\\ &= \frac{1-rj_{n+1}+2\left(2-rj_n\right)\rho\omega_k}{(1-\alpha_2\rho\omega_k)(1-\beta_2\rho\omega_k)}. \end{aligned}$$

By Lemma 4, the matrix A is nonsingular if and only if $f(\rho\omega_k)\neq 0$. That is when

$$(1-\alpha_2\rho\omega_k)(1-\beta_2\rho\omega_k)\neq 0,$$

A is nonsingular if and only if

$$1-rj_{n+1}+2\left(2-rj_n\right)\rho\omega_k\neq 0$$

for any $r\in\mathbf{C}$.

When $(1-\alpha_2\rho\omega_k)(1-\beta_2\rho\omega_k)=0$, we have $\rho\omega_k=\frac{1}{\alpha_2}$ or $\rho\omega_k=\frac{1}{\beta_2}$.

If $\rho\omega_k=\frac{1}{\alpha_2}$, then the eigenvalue of the matrix A is

$$\lambda_k=\frac{\beta_2^n}{\alpha_2^{n-1}\left(\beta_2-\alpha_2\right)}\neq 0,$$

for $\alpha_2=2,\beta_2=-1,k=1,2,\ldots,n$.

If $\rho\omega_k=\frac{1}{\beta_2}$, then the eigenvalue of the matrix A is

$$\lambda_k=\frac{\alpha_2^n}{\beta_2^{n-1}\left(\alpha_2-\beta_2\right)}\neq 0,$$

for $\alpha_2=2,\beta_2=-1,k=1,2,\ldots,n$.

So the matrix A is nonsingular for $(1-\alpha_2\rho\omega_k)(1-\beta_2\rho\omega_k)=0$. Hence, the proof is completed. □

Theorem 13. *Let* $B=\mathrm{LCirc}_r(j_1,j_2,\ldots,j_n)$. *Then*

$$\det B=\frac{r(r-j_{n+1})^n-2^n(j_n-2r)^n}{r^2+(-2)^n-rj_n}(-1)^{\frac{(n-1)(n-2)}{2}}.$$

Furthermore, B *is singular if and only if*

$$r\rho-\rho j_{n+1}+2\left(2r-j_n\right)\omega_k=0$$

and

$$\left(\rho-\alpha_2\omega_k\right)\left(\rho-\beta_2\omega_k\right)\neq 0,$$

where $\rho=|r|^{\frac{1}{n}}$, $\omega_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$.

Proof. We give the explicit determinant of the matrix B firstly.

When $r\neq 0$, the matrix B can be written as

$$B=\begin{pmatrix} j_1 & j_2 & \cdots & j_n\\ j_2 & \cdots & j_n & rj_1\\ \vdots & \iddots & \iddots & \vdots\\ j_n & rj_1 & \cdots & rj_{n-1}\end{pmatrix}=\Gamma^{-1}A_4.$$

where

$$\Gamma=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & \frac{1}{r}\\ \vdots & \iddots & \iddots & \iddots & \vdots\\ 0 & 0 & \frac{1}{r} & \cdots & 0\\ 0 & \frac{1}{r} & 0 & \cdots & 0\end{pmatrix},$$

and

$$A_4=\begin{pmatrix} j_1 & \cdots & j_{n-1} & j_n\\ \frac{1}{r}j_n & j_1 & \cdots & j_{n-1}\\ \vdots & \ddots & \ddots & \vdots\\ \frac{1}{r}j_2 & \cdots & \frac{1}{r}j_n & j_1\end{pmatrix}.$$

Thus, we have

$$\det B=\det\Gamma^{-1}\det A_4,$$

where A_4 is a $\frac{1}{r}$-circulant matrix and its determinant can be obtained from Theorem 12 by replacing r with $\frac{1}{r}$,

$$\det A_4=\frac{\left(1-\frac{1}{r}j_{n+1}\right)^n-2^n\frac{1}{r}\left(\frac{1}{r}j_n-2\right)^n}{1+\frac{(-2)^n}{r^2}-\frac{1}{r}j_n},$$

where

$$\det\Gamma=(-1)^{\frac{(n-1)(n-2)}{2}}\left(\frac{1}{r}\right)^{n-1}.$$

So

$$\begin{aligned}\det B&=\det\Gamma^{-1}\det A_4\\ &=\frac{r(r-j_{n+1})^n-2^n(j_n-2r)^n}{r^2+(-2)^n-rj_n}(-1)^{\frac{(n-1)(n-2)}{2}}.\end{aligned}$$

When $r=0$,

$$\begin{aligned}\det B&=(-1)^{\frac{n(n-1)}{2}}j_n^n\\ &=\frac{r(r-j_{n+1})^n-2^n(j_n-2r)^n}{r^2+(-2)^n-rj_n}(-1)^{\frac{(n-1)(n-2)}{2}}.\end{aligned}$$

Next we discuss the singularity of B.

If $r=0$, then

$$\det B=(-1)^{\frac{n(1-n)}{2}}j_n^n\neq 0$$

for any $n\in\mathbf{N}_+$, and B is nonsingular.

If $r\neq 0$, A_4 is singular if and only if

$$1-\frac{1}{r}j_{n+1}+2\left(2-\frac{1}{r}j_n\right)|\frac{1}{r}|^{\frac{1}{n}}\omega_k=0$$

and

$$\left(1-\alpha_2|\frac{1}{r}|^{\frac{1}{n}}\omega_k\right)\left(1-\beta_2|\frac{1}{r}|^{\frac{1}{n}}\omega_k\right)\neq 0$$

by Theorem 12. That is

$$r\rho-\rho j_{n+1}+2\left(2r-j_n\right)\omega_k=0$$

and

$$\left(\rho-\alpha_2\omega_k\right)\left(\rho-\beta_2\omega_k\right)\neq 0.$$

Furthermore, the matrix Γ is nonsingular with $r\neq 0$. Thus, the proof is completed. □

6 Conclusion

In this section, we give two identities of Pell and Pell-Lucas numbers and two identities of Jacobsthal and Jacobsthal-Lucas numbers.

Let $C=\mathrm{Circ}(P_1,P_2,\ldots,P_n)$ and $D=\mathrm{Circ}(Q_1,Q_2,\ldots,Q_n)$ be circulant matrices. Jiang [22] got

$$\det C=(1-P_{n+1})^{n-1}+P_n^{n-2}\sum_{k=1}^{n-1}P_k\left(\frac{1-P_{n+1}}{P_n}\right)^{k-1},$$

and

$$\det D=2\times\left[(2-Q_{n+1})^{n-1}+(Q_n-2)^{n-2}\Delta_1\right],$$

where

$$\Delta_1=\sum_{k=1}^{n-1}(Q_{k+2}-3Q_{k+1})\Big(\frac{Q_n-2}{2-Q_{n+1}}\Big)^{-k+1}.$$

We have $\det C=\frac{\left(1-P_{n+1}\right)^n-P_n^n}{1+(-1)^n-Q_n}$ in Theorem 6 and $\det D=\frac{\left(2-Q_{n+1}\right)^n-\left(Q_n-2\right)^n}{1+(-1)^n-Q_n}$ in Theorem 8 when $r=1$.

Similarly, let $E=\mathrm{Circ}(J_1,J_2,\ldots,J_n)$ and $F=\mathrm{Circ}(j_1,j_2,\ldots,j_n)$ be circulant matrices. Gong [23] got

$$\det E=(1-J_{n+1})^{n-1}+(2J_n)^{n-2}\Delta_2,$$

where

$$\Delta_2=\sum_{k=1}^{n-1}(2J_k)\left(\frac{1-J_{n+1}}{2J_n}\right)^{k-1},$$

and

$$\det F=(1-j_{n+1})^{n-1}+(2j_n-4)^{n-2}\Delta_3,$$

where

$$\Delta_3=\sum_{k=1}^{n-1}(j_{k+2}-5j_{k+1})\left(\frac{1-j_{n+1}}{2j_n-4}\right)^{k-1}.$$

We have $\det E=\frac{\left(1-J_{n+1}\right)^n-2^nJ_n^n}{1+(-2)^n-j_n}$ in Theorem 10 and $\det F=\frac{\left(1-j_{n+1}\right)^n-2^n\left(j_n-2\right)^n}{1+(-2)^n-j_n}$ in Theorem 12 when $r=1$.

So we have the following identities of P_n and Q_n, and of J_n and j_n:

$$(1-P_{n+1})^{n-1}+P_n^{n-2}\sum_{k=1}^{n-1}P_k\left(\frac{1-P_{n+1}}{P_n}\right)^{k-1}=\frac{\left(1-P_{n+1}\right)^n-P_n^n}{1+(-1)^n-Q_n}, \tag{3}$$

$$2\times\left[(2-Q_{n+1})^{n-1}+(Q_n-2)^{n-2}\Delta_1\right]=\frac{(2-Q_{n+1})^n-(Q_n-2)^n}{1+(-1)^n-Q_n}, \tag{4}$$

$$(1-J_{n+1})^{n-1}+(2J_n)^{n-2}\Delta_2=\frac{(1-J_{n+1})^n-2^nJ_n^n}{1+(-2)^n-j_n}, \tag{5}$$

and

$$(1-j_{n+1})^{n-1}+(2j_n-4)^{n-2}\Delta_3=\frac{(1-j_{n+1})^n-2^n(j_n-2)^n}{1+(-2)^n-j_n} \tag{6}$$

where

$$\Delta_1=\sum_{k=1}^{n-1}(Q_{k+2}-3Q_{k+1})\Big(\frac{Q_n-2}{2-Q_{n+1}}\Big)^{-k+1},$$

$$\Delta_2=\sum_{k=1}^{n-1}(2J_k)\left(\frac{1-J_{n+1}}{2J_n}\right)^{k-1},$$

and

$$\Delta_3=\sum_{k=1}^{n-1}(j_{k+2}-5j_{k+1})\left(\frac{1-j_{n+1}}{2j_n-4}\right)^{k-1}.$$

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