On Explicit Determinants of the RFMLR and RLMFL Circulant Matrices Involving Certain Famous Numbers

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Abstract: The row first-minus-last right (RFMLR) circulant matrices and the row last-minus-first left (RLMFL) circulant matrices are two special pattern matrices. By using the inverse factorization of polynomial, we give the explicit determinants of the two pattern matrices involving Fibonacci, Lucas, Pell and Pell-Lucas sequences in terms of finite many terms of these sequences.

Key–Words: Determinant, RFMLR circulant matrix, RLMFL circulant matrix, Fibonacci numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers

1 Introduction

There is a lot of work dealing with the determinant of several special matrices involving some famous numbers. Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [1]. Lind presented the determinants of circulant and skew-circulant involving Fibonacci numbers [2]. Lin gave the determinant of the Fibonacci-Lucas quasi-cyclic matrices in [3]. Shen considered circulant matrices with Fibonacci and Lucas numbers in [4]. By constructing the transformation matrices, he presented the following results:

(i) Let $A_n = \mathrm{Circ}(F_1, F_2, \dots, F_n)$ be circulant matrix, then

$$\det A_n = (1-F_{n+1})^{n-1} + F_n^{n-2}\sum_{k=1}^{n-1} F_k \left(\frac{1-F_{n+1}}{F_n}\right)^{k-1},$$

where F_n is the nth Fibonacci number.

(ii) Let $B_n = \mathrm{Circ}(L_1, L_2, \dots, L_n)$ be circulant matrix, then

$$\det B_n = (1-L_{n+1})^{n-1} + (L_n-2)^{n-2} \times \sum_{k=1}^{n-1}(L_{k+2}-3L_{k+1})\left(\frac{1-L_{n+1}}{L_n-2}\right)^{k-1},$$

where L_n is the nth Lucas number.

The determinant problems of the row first-minus-last right (RFMLR) circulant matrices and row last-minus-first left (RLMFL) circulant matrices involving the Fibonacci, Lucas, Pell and Pell-Lucas sequences are considered in this paper. The explicit determinants are presented by using some terms of these sequences. The techniques used herein are based on the inverse factorization of polynomial. Firstly, we introduce the definitions of the RFMLR and RLMFL circulant matrices, and properties of the related famous numbers. Then, we present the main results and the detailed process.

Definition 1. *[5] A row first-minus-last right* (RFMLR) *circulant matrix with the first row* $(a_1, a_2, \dots, a_n)$*, denoted by* $\mathrm{RFMLRcircfr}(a_1, a_2, \dots, a_n)$*, is meant a square matrix of the form*

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_n & a_1-a_n & \dots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_3 & a_4-a_3 & \dots & a_2 \\ a_2 & a_3-a_2 & \dots & a_1-a_n \end{pmatrix}$$

It can be seen that the matrix has an arbitrary first row and the any other row obtained from the previous one according to the following rule: Get the $(i+1)$-st row by subtracting the last element of the i-th row from the first element of the i-th row, and then shifting the elements of the i-th row (cyclically) one position to the right. Obviously, the RFMLR circulant matrix is determined by its first row.

Note that the RFMLR circulant matrix is a $x^n + x - 1$ circulant matrix [6], which is neither an extension nor special case of the circulant matrix [7, 8].

They are two completely different kinds of special matrices.

We define the matrix $\Theta_{(1,-1)}$ as the basic RFMLR circulant matrix, that is,

$$\Theta_{(1,-1)} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ 1 & -1 & 0 & \dots & 0 \end{pmatrix} = \mathrm{RFMLRcircfr}(0, 1, 0, \dots, 0).$$

Both the minimal polynomial and the characteristic polynomial of $\Theta_{(1,-1)}$ are $g(x) = x^n + x - 1$, which has only simple roots, denoted by ε_i $(i = 1, 2, \dots, n)$. Furthermore, $\Theta_{(1,-1)}$ satisfies $\Theta_{(1,-1)}^j = \mathrm{RFMLRcircfr}(\underbrace{0, \cdots, 0}_{j}, 1, \underbrace{0, \cdots, 0}_{n-j-1})$ and $\Theta_{(1,-1)}^n = I_n - \Theta_{(1,-1)}$, then a matrix A can be written in the form

$$A = f\left(\Theta_{(1,-1)}\right) = \sum_{i=1}^{n} a_i \Theta_{(1,-1)}^{i-1} \tag{1}$$

if and only if A is a RFMLR circulant matrix, where the polynomial $f(x) = \sum_{i=1}^{n} a_i x^{i-1}$ is called the representer of the RFMLR circulant matrix A.

Since $\Theta_{(1,-1)}$ is nonderogatory, then A is a RFMLR circulant matrix if and only if A commutes with $\Theta_{(1,-1)}$, that is, $A\Theta_{(1,-1)} = \Theta_{(1,-1)}A$. Because of the representation (1), RFMLR circulant matrices have very nice structure and the algebraic properties also can be easily attained. Moreover, the product of two RFMLR circulant matrices and the inverse A^{-1} are again RFMLR circulant matrices.

Definition 2. *[5] A row last-minus-first left (*RLMFL*) circulant matrix with the first row $(a_1, a_2, \dots, a_n)$, denoted by* $\mathrm{RLMFLcircfr}(a_1, a_2, \dots, a_n)$*, is meant a square matrix of the form*

$$B = \begin{pmatrix} a_1 & \dots & a_{n-1} & a_n \\ a_2 & \dots & a_n - a_1 & a_1 \\ \vdots & \iddots & \vdots & \vdots \\ a_{n-1} & \dots & a_{n-3} - a_{n-2} & a_{n-2} \\ a_n - a_1 & \dots & a_{n-2} - a_{n-1} & a_{n-1} \end{pmatrix}.$$

It can be seen that the matrix has an arbitrary first row and the any other row obtained from the previous one by the following rule: Get the $(i+1)$-st row by subtracting the first element of the i-th row from the last element of the i-th row, and then shifting the elements of the i-th row (cyclically) one position to the left. Obviously, the RLMFL circulant matrix is determined by its first row.

Let $A = \mathrm{RLMFLcircfr}(a_1, a_2, \dots, a_n)$ and $B = \mathrm{RFMLRcircfr}(a_n, a_{n-1}, \dots, a_1)$. By explicit computation, we find

$$A = B\widehat{I}_n, \tag{2}$$

where $\widehat{I}_n$ is the backward identity matrix of the form

$$\widehat{I}_n = \begin{pmatrix} \mathbf{0} & & & 1 \\ & & 1 & \\ & \iddots & & \\ & 1 & \mathbf{0} & \\ 1 & & & \end{pmatrix}. \tag{3}$$

The Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$ are defined by

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 1, \tag{4}$$

$$L_{n+1} = L_n + L_{n-1}, \quad n \geq 1, \tag{5}$$

with the initial condition $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$.

The Pell and Pell-Lucas sequences $\{P_n\}$ and $\{Q_n\}$ are defined by [10]

$$P_{n+1} = 2P_n + P_{n-1}, \quad n \geq 1, \tag{6}$$

$$Q_{n+1} = 2Q_n + Q_{n-1}, \quad n \geq 1, \tag{7}$$

with the initial condition $P_0 = 0$, $P_1 = 1$ and $Q_0 = 2$, $Q_1 = 2$.

The first few members of these sequences are given as follows:

n	0	1	2	3	4	5	6
F_n	0	1	1	2	3	5	8
L_n	2	1	3	4	7	11	18
P_n	0	1	2	5	12	29	70
Q_n	2	2	6	14	34	82	198

Recurrences (4) and (5) involve the characteristic equation $x^2 - x - 1 = 0$ with roots $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$. Moreover, the Binet form for the Fibonacci sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{8}$$

and the Binet form for Lucas sequence is

$$L_n = \alpha^n + \beta^n. \tag{9}$$

Recurrences (6) and (7) as well imply the characteristic equation $x^2 - 2x - 1 = 0$ with roots $\gamma = 1 + \sqrt{2}$,

$\delta = 1 - \sqrt{2}$. Furthermore, the Binet form for the Pell sequence is

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}, \tag{10}$$

and the Binet form for the Pell-Lucas sequence is

$$Q_n = \gamma^n + \delta^n. \tag{11}$$

2 Main Results

By Proposition 5.1 in [6], we deduce the following lemma.

Lemma 3. *Let $A = \mathrm{RFMLRcircfr}(a_1, a_2, \dots, a_n)$. Then the eigenvalues of A are given by*

$$\lambda_i = f(\varepsilon_i) = \sum_{j=1}^{n} a_j \varepsilon_i^{j-1}, \qquad i = 1, 2, \dots, n,$$

and the determinant of A is given by

$$\det A = \prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n} \sum_{j=1}^{n} a_j \varepsilon_i^{j-1}.$$

Lemma 4. *Suppose ε_i $(i = 1, 2, \dots, n)$ are the roots of the characteristic polynomial of $\Theta_{(1,-1)}$. Then*

$$\prod_{i=1}^{n} (a\varepsilon_i^2 + b\varepsilon_i + c) = c^n + a^{n-1}(a + b + c) + c\left(x_1^{n-1} + x_2^{n-1}\right) - \left(x_1^n + x_2^n\right), \tag{12}$$

where $a, b, c \in \mathbb{R}$ and

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2}; \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2}.$$

Proof: Since ε_i $(i = 1, 2, \dots, n)$ are the roots of the characteristic polynomial of $\Theta_{(1,-1)}$, $g(x) = x^n + x - 1$ can be factored as

$$x^n + x - 1 = \prod_{i=1}^{n} (x - \varepsilon_i). \tag{13}$$

If $a = 0$, then we have

$$\begin{aligned}\prod_{i=1}^{n} (b\varepsilon_i + c) &= (-b)^n \prod_{i=1}^{n} \left(-\frac{c}{b} - \varepsilon_i\right) \\ &= (-b)^n \left[\left(-\frac{c}{b}\right)^n - \frac{c}{b} - 1\right] \\ &= c^n + c(-b)^{n-1} - (-b)^n.\end{aligned}$$

Obviously, the equality (12) holds when $a = 0$.

If $a \neq 0$, then we deduce

$$\begin{aligned}\prod_{i=1}^{n} \left(a\varepsilon_i^2 + b\varepsilon_i + c\right) &= a^n \prod_{i=1}^{n} \left(\varepsilon_i^2 + \frac{b}{a}\varepsilon_i + \frac{c}{a}\right) \\ &= a^n \prod_{i=1}^{n} \left(\frac{x_1}{a} - \varepsilon_i\right)\left(\frac{x_2}{a} - \varepsilon_i\right)\end{aligned}$$

where

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2}; \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2}.$$

By the factored form (13), we have

$$\begin{aligned}&\prod_{i=1}^{n} \left(a\varepsilon_i^2 + b\varepsilon_i + c\right) \\ &= a^n \left[\left(\frac{x_1}{a}\right)^n + \frac{x_1}{a} - 1\right]\left[\left(\frac{x_2}{a}\right)^n + \frac{x_2}{a} - 1\right] \\ &= c^n + a^{n-1}(a + b + c) \\ &\quad + c\left(x_1^{n-1} + x_2^{n-1}\right) - \left(x_1^n + x_2^n\right).\end{aligned}$$

The proof is completed. □

2.1 Determinants of the RFMLR and RLMFL Circulant Matrix Involving Fibonacci Numbers

Theorem 5. *Let $C = \mathrm{RFMLRcircfr}(F_1, F_2, \dots, F_n)$ and $n \geq 3$. If n is even, then*

$$\det C = \frac{(1 - F_{n+1})^n + F_n^{n-1}}{-L_{n-2} + 2} + \frac{(1 - F_{n+1})\left(y_1^{n-1} + y_2^{n-1}\right) - y_1^n - y_2^n}{-L_{n-2} + 2},$$

and if n is odd, then

$$\det C = \frac{(1 - F_{n+1})^n + F_n^{n-1}}{-L_{n-2}} + \frac{(1 - F_{n+1})\left(y_1^{n-1} + y_2^{n-1}\right) - y_1^n - y_2^n}{-L_{n-2}},$$

where

$$y_1 = \frac{-F_{n-1} + \sqrt{F_{n-1}^2 - 4F_n\left(1 - F_{n+1}\right)}}{2}, \tag{14}$$

$$y_2 = \frac{-F_{n-1} - \sqrt{F_{n-1}^2 - 4F_n\left(1 - F_{n+1}\right)}}{2}. \tag{15}$$

Proof: Obviously, C has the form

$$C = \begin{pmatrix} F_1 & F_2 & \dots & F_n \\ F_n & F_1 - F_n & \dots & F_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ F_3 & F_4 - F_3 & \dots & F_2 \\ F_2 & F_3 - F_2 & \dots & F_1 - F_n \end{pmatrix}.$$

If $n = 1$, then $\det C = 1$.
If $n = 2$, then

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\det C = -1$.
If $n \geq 3$, then we have

$$\begin{aligned}\det C &= \prod_{i=1}^{n} \left(F_1 + F_2\varepsilon_i + \cdots + F_n\varepsilon_i^{n-1}\right) \\ &= \prod_{i=1}^{n} \left(\frac{\alpha-\beta}{\alpha-\beta} + \frac{\alpha^2-\beta^2}{\alpha-\beta}\varepsilon_i \right. \\ &\quad \left. + \cdots + \frac{\alpha^n-\beta^n}{\alpha-\beta}\varepsilon_i^{n-1}\right) \\ &= \prod_{i=1}^{n} \frac{F_n\varepsilon_i^2 + F_{n-1}\varepsilon_i + 1 - F_{n+1}}{\alpha\beta\varepsilon_i^2 - (\alpha+\beta)\varepsilon_i + 1}.\end{aligned}$$

owing to Lemma 3 and the Binet form (8). By Lemma 4 and the Binet form (9), we obtain

$$\begin{aligned}&\prod_{i=1}^{n} \left(F_n\varepsilon_i^2 + F_{n-1}\varepsilon_i + 1 - F_{n+1}\right) = (1 - F_{n+1})^n \\ &+ F_n^{n-1} + (1 - F_{n+1})\left(y_1^{n-1} + y_2^{n-1}\right) - y_1^n - y_2^n\end{aligned}$$

where y_1, y_2 are defined by (14) and (15), and

$$\prod_{i=1}^{n} \left[\alpha\beta\varepsilon_i^2 - (\alpha+\beta)\varepsilon_i + 1\right] = -L_{n-2} + (-1)^n + 1.$$

Therefore, if n is even, then

$$\begin{aligned}\det C &= \frac{(1 - F_{n+1})^n + F_n^{n-1}}{-L_{n-2} + 2} \\ &\quad + \frac{(1 - F_{n+1})\left(y_1^{n-1} + y_2^{n-1}\right) - y_1^n - y_2^n}{-L_{n-2} + 2},\end{aligned}$$

and if n is odd, then

$$\begin{aligned}\det C &= \frac{(1 - F_{n+1})^n + F_n^{n-1}}{-L_{n-2}} \\ &\quad + \frac{(1 - F_{n+1})\left(y_1^{n-1} + y_2^{n-1}\right) - y_1^n - y_2^n}{-L_{n-2}}.\end{aligned}$$

The proof is complete. □

Theorem 6. *Let* $D = \mathrm{RFMLRcircfr}(F_n, \dots, F_1)$. *If* n *is even, then*

$$\det D = \frac{F_n^n + F_{n+2} - F_n\left(y_3^{n-1} + y_4^{n-1}\right) - y_3^n - y_4^n}{-L_{n+1}},$$

and if n *is odd, then*

$$\det D = \frac{F_n^n + F_{n+2} + F_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{L_{n+1} + 2}.$$

where

$$y_3 = \frac{F_{n+1} - 1 + \sqrt{(F_{n+1} - 1)^2 - 4F_n}}{2}, \quad (16)$$

$$y_4 = \frac{F_{n+1} - 1 - \sqrt{(F_{n+1} - 1)^2 - 4F_n}}{2}. \quad (17)$$

Proof: The matrix D has the form

$$D = \begin{pmatrix} F_n & F_{n-1} & \dots & F_1 \\ F_1 & F_n - F_1 & \dots & F_2 \\ \vdots & \vdots & \ddots & \vdots \\ F_{n-2} & F_{n-3} - F_{n-2} & \dots & F_{n-1} \\ F_{n-1} & F_{n-2} - F_{n-1} & \dots & F_n - F_{n-1} \end{pmatrix}.$$

According to Lemma 3 and the Binet form (8), we have

$$\begin{aligned}\det D &= \prod_{i=1}^{n} \left(F_n + F_{n-1}\varepsilon_i + \cdots + F_1\varepsilon_i^{n-1}\right) \\ &= \prod_{i=1}^{n} \left(\frac{\alpha^n-\beta^n}{\alpha-\beta} + \frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\varepsilon_i \right. \\ &\quad \left. + \cdots + \frac{\alpha-\beta}{\alpha-\beta}\varepsilon_i^{n-1}\right) \\ &= \prod_{i=1}^{n} \frac{-\varepsilon_i^2 + (1 - F_{n+1})\varepsilon_i - F_n}{\varepsilon_i^2 - (\alpha+\beta)\varepsilon_i + \alpha\beta}.\end{aligned}$$

Using Lemma 4 and the Binet form (9), we obtain

$$\begin{aligned}&\prod_{i=1}^{n} \left[-\varepsilon_i^2 + (1 - F_{n+1})\varepsilon_i - F_n\right] = (-F_n)^n \\ &+ (-1)^n F_{n+2} - F_n\left(y_3^{n-1} + y_4^{n-1}\right) - y_3^n - y_4^n,\end{aligned}$$

where y_3, y_4 are defined as (16) and (17), and

$$\prod_{i=1}^{n} \left[\varepsilon_i^2 - (\alpha+\beta)\varepsilon_i + \alpha\beta\right] = -L_{n+1} + (-1)^n - 1.$$

Consequently, if n is even, then

$$\det D = \frac{F_n^n + F_{n+2} - F_n\left(y_3^{n-1} + y_4^{n-1}\right) - y_3^n - y_4^n}{-L_{n+1}},$$

and if n is odd, then

$$\det D = \frac{F_n^n + F_{n+2} + F_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{L_{n+1} + 2}.$$

The conclusion is obtained. □

Theorem 7. *Let* $E = \text{RLMFLcircfr}(F_1, F_2, \ldots, F_n)$ *and* y_3*,* y_4 *be given by* (16) *and* (17)*. If* $n \equiv 0 \pmod 4$*, then*

$$\det E = \frac{F_n^n + F_{n+2} - F_n\left(y_3^{n-1} + y_4^{n-1}\right) - y_3^n - y_4^n}{-L_{n+1}}.$$

If $n \equiv 1 \pmod 4$*, then*

$$\det E = \frac{F_n^n + F_{n+2} + F_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{L_{n+1} + 2}.$$

If $n \equiv 2 \pmod 4$*, then*

$$\det E = \frac{F_n^n + F_{n+2} - F_n\left(y_3^{n-1} + y_4^{n-1}\right) - y_3^n - y_4^n}{L_{n+1}}.$$

If $n \equiv 3 \pmod 4$*, then*

$$\det E = \frac{F_n^n + F_{n+2} + F_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{-L_{n+1} - 2}.$$

Proof: The matrix E has the form

$$E = \begin{pmatrix} F_1 & \ldots & F_{n-1} & F_n \\ F_2 & \ldots & F_n - F_1 & F_1 \\ \vdots & \ddots & \vdots & \vdots \\ F_{n-1} & \ldots & F_{n-3} - F_{n-2} & F_{n-2} \\ F_n - F_1 & \ldots & F_{n-2} - F_{n-1} & F_{n-1} \end{pmatrix}.$$

From the relation (2), we have

$$E = D\,\widehat{I}_n,$$

where $\widehat{I}_n$ is the backward identity matrix of the form (3). Thus,

$$\det E = \det D\ \det \widehat{I}_n,$$

and

$$\det \widehat{I}_n = (-1)^{\frac{n(n-1)}{2}}.$$

Define y_3 and y_4 as (16) and (17). According to Theorem 6, we can obtain the following results: If $n \equiv 0 \pmod 4$, then

$$\det E = \frac{F_n^n + F_{n+2} - F_n\left(y_3^{n-1} + y_4^{n-1}\right) - y_3^n - y_4^n}{-L_{n+1}},$$

If $n \equiv 1 \pmod 4$, then

$$\det E = \frac{F_n^n + F_{n+2} + F_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{L_{n+1} + 2}.$$

If $n \equiv 2 \pmod 4$, then

$$\det E = \frac{F_n^n + F_{n+2} - F_n\left(y_3^{n-1} + y_4^{n-1}\right) - y_3^n - y_4^n}{L_{n+1}}.$$

If $n \equiv 3 \pmod 4$, then

$$\det E = \frac{F_n^n + F_{n+2} + F_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{-L_{n+1} - 2}.$$

The proof is complete. □

2.2 Determinants of the RFMLR and RLMFL Circulant Matrix Involving Lucas Numbers

Theorem 8. *Let* $F = \text{RFMLRcircfr}(L_1, \ldots, L_n)$ *and* $n \geq 3$*. If* n *is even, then*

$$\det F = \frac{(1 - L_{n+1})^n + 3L_n^{n-1}}{-L_{n-2} + 2} + \frac{(1 - L_{n+1})\left(y_5^{n-1} + y_6^{n-1}\right) - y_5^n - y_6^n}{-L_{n-2} + 2},$$

and if n *is odd, then*

$$\det F = \frac{(1 - L_{n+1})^n + 3L_n^{n-1}}{-L_{n-2}} + \frac{(1 - L_{n+1})\left(y_5^{n-1} + y_6^{n-1}\right) - y_5^n - y_6^n}{-L_{n-2}},$$

where

$$y_5 = \frac{L_{n-1} + 2 - \sqrt{(L_{n-1} + 2)^2 - 4L_n\left(1 - L_{n+1}\right)}}{-2}, \tag{18}$$

$$y_6 = \frac{L_{n-1} + 2 + \sqrt{(L_{n-1} + 2)^2 - 4L_n\left(1 - L_{n+1}\right)}}{-2}. \tag{19}$$

Proof: The matrix F has the form

$$F = \begin{pmatrix} L_1 & L_2 & \ldots & L_n \\ L_n & L_1 - L_n & \ldots & L_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ L_3 & L_4 - L_3 & \ldots & L_2 \\ L_2 & L_3 - L_2 & \ldots & L_1 - L_n \end{pmatrix}.$$

If $n = 1$, then $\det F = 1$.
If $n = 2$, then

$$F = \begin{pmatrix} 1 & 3 \\ 3 & -2 \end{pmatrix}$$

and $\det F=-11$.

If $n\geq 3$, then we have

$$\begin{aligned}\det F&=\prod_{i=1}^{n}\left(L_1+L_2\varepsilon_i+\cdots+L_n\varepsilon_i^{n-1}\right)\\&=\prod_{i=1}^{n}\big[\alpha+\beta+(\alpha^2+\beta^2)\varepsilon_i+\cdots\\&\quad+(\alpha^n+\beta^n)\varepsilon_i^{n-1}\big]\\&=\prod_{i=1}^{n}\frac{L_n\varepsilon_i^2+(L_{n-1}+2)\varepsilon_i+1-L_{n+1}}{\alpha\beta\varepsilon_i^2-(\alpha+\beta)\varepsilon_i+1}\end{aligned}$$

from Lemma 3 and the Binet form (9).

Using Lemma 4 and (9), we obtain

$$\begin{aligned}&\prod_{i=1}^{n}\left[L_n\varepsilon_i^2+(L_{n-1}+2)\varepsilon_i+1-L_{n+1}\right]\\&=(1-L_{n+1})^n+3L_n^{n-1}+(1-L_{n+1})\\&\quad\times\left(y_5^{n-1}+y_6^{n-1}\right)-y_5^n-y_6^n,\end{aligned}$$

where y_5, y_6 are defined as (18) and (19), and

$$\prod_{i=1}^{n}\left[\alpha\beta\varepsilon_i^2-(\alpha+\beta)\varepsilon_i+1\right]=-L_{n-2}+(-1)^n+1.$$

Hence, if n is even, then

$$\begin{aligned}\det F&=\frac{(1-L_{n+1})^n+3L_n^{n-1}}{-L_{n-2}+2}\\&\quad+\frac{(1-L_{n+1})\left(y_5^{n-1}+y_6^{n-1}\right)-y_5^n-y_6^n}{-L_{n-2}+2},\end{aligned}$$

and if n is odd, then

$$\begin{aligned}\det F&=\frac{(1-L_{n+1})^n+3L_n^{n-1}}{-L_{n-2}}\\&\quad+\frac{(1-L_{n+1})\left(y_5^{n-1}+y_6^{n-1}\right)-y_5^n-y_6^n}{-L_{n-2}}.\end{aligned}$$

The proof is complete. □

Theorem 9. *Let* $G=\mathrm{RFMLRcircfr}(L_n,\ldots,L_1)$*. If* n *is even, then*

$$\begin{aligned}\det G&=\frac{(2-L_n)^n+L_{n+2}}{-L_{n+1}}\\&\quad+\frac{(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n}{-L_{n+1}},\end{aligned}$$

and if n *is odd, then*

$$\begin{aligned}\det G&=\frac{(2-L_n)^n-L_{n+2}}{-L_{n+1}-2}\\&\quad+\frac{(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n}{-L_{n+1}-2},\end{aligned}$$

where

$$y_7=\frac{L_{n+1}+1+\sqrt{L_{n+1}^2-2L_{n-2}+9}}{2},\qquad(20)$$

$$y_8=\frac{L_{n+1}+1-\sqrt{L_{n+1}^2-2L_{n-2}+9}}{2}.\qquad(21)$$

Proof: The matrix G has the form

$$G=\begin{pmatrix}L_n & L_{n-1} & \ldots & L_1\\ L_1 & L_n-L_1 & \ldots & L_2\\ \vdots & \vdots & \ddots & \vdots\\ L_{n-2} & L_{n-3}-L_{n-2} & \ldots & L_{n-1}\\ L_{n-1} & L_{n-2}-L_{n-1} & \ldots & L_n-L_{n-1}\end{pmatrix}.$$

According to Lemma 3 and the Binet form (9), we have

$$\begin{aligned}\det G&=\prod_{i=1}^{n}\left(L_n+L_{n-1}\varepsilon_i+\cdots+L_1\varepsilon_i^{n-1}\right)\\&=\prod_{i=1}^{n}\big[\alpha^n+\beta^n+(\alpha^{n-1}+\beta^{n-1})\varepsilon_i\\&\quad+\cdots+(\alpha+\beta)\varepsilon_i^{n-1}\big]\\&=\prod_{i=1}^{n}\frac{-\varepsilon_i^2-\left(L_{n+1}+1\right)\varepsilon_i+2-L_n}{\varepsilon_i^2-(\alpha+\beta)\varepsilon_i+\alpha\beta}.\end{aligned}$$

Using Lemma 4 and (9), we obtain

$$\begin{aligned}&\prod_{i=1}^{n}\left[-\varepsilon_i^2-\left(L_{n+1}+1\right)\varepsilon_i+2-L_n\right]=(2-L_n)^n\\&+(-1)^nL_{n+2}+(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n,\end{aligned}$$

where y_7, y_8 are defined as (20) and (21), and

$$\prod_{i=1}^{n}\left[\varepsilon_i^2-(\alpha+\beta)\varepsilon_i+\alpha\beta\right]=-L_{n+1}+(-1)^n-1.$$

Consequently, if n is even, then

$$\begin{aligned}\det G&=\frac{(2-L_n)^n+L_{n+2}}{-L_{n+1}}\\&\quad+\frac{(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n}{-L_{n+1}}\end{aligned}$$

and if n is odd, then

$$\det G = \frac{(2-L_n)^n - L_{n+2}}{-L_{n+1}-2} + \frac{(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n}{-L_{n+1}-2}.$$

Theorem 10. *Let* H = RLMFLcircfr$(L_1,\dots,L_n)$ *and* y_7*,* y_8 *be defined as* (20) *and* (21)*. If* $n \equiv 0 \pmod 4$*, then*

$$\det H = \frac{(2-L_n)^n + L_{n+2}}{-L_{n+1}} + \frac{(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n}{-L_{n+1}}.$$

If $n \equiv 1 \pmod 4$*, then*

$$\det H = \frac{(2-L_n)^n - L_{n+2}}{-L_{n+1}-2} + \frac{(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n}{-L_{n+1}-2}.$$

If $n \equiv 2 \pmod 4$*, then*

$$\det H = \frac{(2-L_n)^n + L_{n+2}}{L_{n+1}} + \frac{(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n}{L_{n+1}}.$$

If $n \equiv 3 \pmod 4$*, then*

$$\det H = \frac{(2-L_n)^n - L_{n+2}}{L_{n+1}+2} + \frac{(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n}{L_{n+1}+2}.$$

Proof: The matrix H has the form

$$H = \begin{pmatrix} L_1 & \dots & L_{n-1} & L_n \\ L_2 & \dots & L_n - L_1 & L_1 \\ \vdots & \ddots & \vdots & \vdots \\ L_{n-1} & \dots & L_{n-3}-L_{n-2} & L_{n-2} \\ L_n - L_1 & \dots & L_{n-2}-L_{n-1} & L_{n-1} \end{pmatrix}$$

and can be written as

$$H = G\,\widehat{I}_n,$$

where $\widehat{I}_n$ is the backward identity matrix of the form (3). Thus,

$$\det H = \det G \, \det \widehat{I}_n,$$

and

$$\det \widehat{I}_n = (-1)^{\frac{n(n-1)}{2}}.$$

Define y_7 and y_8 as (20) and (21), respectively. According to Theorem 9, we have the following results: If $n \equiv 0 \pmod 4$, then

$$\det H = \frac{(2-L_n)^n + L_{n+2}}{-L_{n+1}} + \frac{(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n}{-L_{n+1}}.$$

If $n \equiv 1 \pmod 4$, then

$$\det H = \frac{(2-L_n)^n - L_{n+2}}{-L_{n+1}-2} + \frac{(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n}{-L_{n+1}-2}.$$

If $n \equiv 2 \pmod 4$, then

$$\det H = \frac{(2-L_n)^n + L_{n+2}}{L_{n+1}} + \frac{(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n}{L_{n+1}}.$$

If $n \equiv 3 \pmod 4$, then

$$\det H = \frac{(2-L_n)^n - L_{n+2}}{L_{n+1}+2} + \frac{(2-L_n)\left(y_7^{n-1}+y_8^{n-1}\right)-y_7^n-y_8^n}{L_{n+1}+2}.$$

The proof is complete. □

2.3 Determinants of the RFMLR and RLMFL Circulant Matrix Involving Pell Numbers

Theorem 11. *Let* J = RFMLRcircfr$(P_1,\dots,P_n)$*. If* n *is even, then*

$$\det J = \frac{(1-P_{n+1})^n + P_n^{n-1}}{-4P_{n-1}+3} + \frac{(1-P_{n+1})\left(z_1^{n-1}+z_2^{n-1}\right)-z_1^n-z_2^n}{-4P_{n-1}+3}$$

and if n *is odd, then*

$$\det J = \frac{(1-P_{n+1})^n + P_n^{n-1}}{-4P_{n-1}-1} + \frac{(1-P_{n+1})\left(z_1^{n-1}+z_2^{n-1}\right)-z_1^n-z_2^n}{-4P_{n-1}-1},$$

where

$$z_1 = \frac{-Q_n + \sqrt{Q_{n+1}^2 - 16P_n}}{4}, \quad (22)$$

and

$$z_2 = \frac{-Q_n - \sqrt{Q_{n+1}^2 - 16P_n}}{4}. \quad (23)$$

Proof: Obviously, J has the form

$$J = \begin{pmatrix} P_1 & P_2 & \dots & P_n \\ P_n & P_1 - P_n & \dots & P_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ P_3 & P_4 - P_3 & \dots & P_2 \\ P_2 & P_3 - P_2 & \dots & P_1 - P_n \end{pmatrix}.$$

According to Lemma 3 and the Binet form (10), we have

$$\det J = \prod_{i=1}^{n} \left(P_1 + P_2\varepsilon_i + \cdots + P_n\varepsilon_i^{n-1}\right)$$
$$= \prod_{i=1}^{n} \left(\frac{\gamma - \delta}{\gamma - \delta} + \frac{\gamma^2 - \delta^2}{\gamma - \delta}\varepsilon_i + \cdots + \frac{\gamma^n - \delta^n}{\gamma - \delta}\varepsilon_i^{n-1}\right)$$
$$= \prod_{i=1}^{n} \frac{P_n\varepsilon_i^2 + (P_{n+1} - P_n)\varepsilon_i + 1 - P_{n+1}}{\gamma\delta\varepsilon_i^2 - (\gamma + \delta)\varepsilon_i + 1}.$$

Furthermore, by straightforward computation, we can obtain the following identities:

$$P_{n+1} - P_n = P_n + P_{n-1} = \frac{1}{2}Q_n, \quad (24)$$
$$Q_{n+1} - Q_n = Q_n + Q_{n-1} = 4P_n. \quad (25)$$

From Lemma 4, (11) and (24) follows that

$$\prod_{i=1}^{n} \left[P_n\varepsilon_i^2 + (P_{n+1} - P_n)\varepsilon_i + 1 - P_{n+1}\right]$$
$$= (1 - P_{n+1})^n + P_n^{n-1}$$
$$+ (1 - P_{n+1})\left(z_1^{n-1} + z_2^{n-1}\right) - z_1^n - z_2^n,$$

where z_1, z_2 are defined as (22) and (23).

According to Lemma 4, the Binet form (11) and the identity (25), we have

$$\prod_{i=1}^{n} \left[\gamma\delta\varepsilon_i^2 - (\gamma + \delta)\varepsilon_i + 1\right]$$
$$= Q_{n-1} - Q_n + 2(-1)^n + 1$$
$$= -4P_{n-1} + 2(-1)^n + 1.$$

Therefore, if n is even, then

$$\det J = \frac{(1 - P_{n+1})^n + P_n^{n-1}}{-4P_{n-1} + 3} + \frac{(1 - P_{n+1})\left(z_1^{n-1} + z_2^{n-1}\right) - z_1^n - z_2^n}{-4P_{n-1} + 3}$$

and if n is odd, then

$$\det J = \frac{(1 - P_{n+1})^n + P_n^{n-1}}{-4P_{n-1} - 1} + \frac{(1 - P_{n+1})\left(z_1^{n-1} + z_2^{n-1}\right) - z_1^n - z_2^n}{-4P_{n-1} - 1}.$$

The proof is complete. □

Theorem 12. *Let* $K = \mathrm{RFMLRcircfr}(P_n, \dots, P_1)$. *If* n *is even, then*

$$\det K = \frac{2P_n^n + Q_{n+1}}{-8P_n - 2} + \frac{P_n\left(z_3^{n-1} + z_4^{n-1}\right) + z_3^n + z_4^n}{4P_n + 1},$$

and if n *is odd, then*

$$\det K = \frac{2P_n^n + Q_{n+1}}{8P_n + 6} + \frac{P_n\left(z_3^{n-1} + z_4^{n-1}\right) + z_3^n + z_4^n}{4P_n + 3},$$

where

$$z_3 = \frac{P_{n+1} - 1 + \sqrt{(P_{n+1} - 1)^2 - 4P_n}}{2}, \quad (26)$$
$$z_4 = \frac{P_{n+1} - 1 - \sqrt{(P_{n+1} - 1)^2 - 4P_n}}{2}. \quad (27)$$

Proof: The matrix K has the form

$$K = \begin{pmatrix} P_n & P_{n-1} & \dots & P_1 \\ P_1 & P_n - P_1 & \dots & P_2 \\ \vdots & \vdots & \ddots & \vdots \\ P_{n-2} & P_{n-3} - P_{n-2} & \dots & P_{n-1} \\ P_{n-1} & P_{n-2} - P_{n-1} & \dots & P_n - P_1 \end{pmatrix}.$$

According to Lemma 3 and the Binet form (10), we have

$$\begin{aligned}\det K &= \prod_{i=1}^{n}\left(P_n + P_{n-1}\varepsilon_i + \cdots + P_1\varepsilon_i^{n-1}\right)\\ &= \prod_{i=1}^{n}\left(\frac{\gamma^n-\delta^n}{\gamma-\delta} + \frac{\gamma^{n-1}-\delta^{n-1}}{\gamma-\delta}\varepsilon_i \right.\\ &\quad \left. + \cdots + \frac{\gamma-\delta}{\gamma-\delta}\varepsilon_i^{n-1}\right)\\ &= \prod_{i=1}^{n}\frac{-\varepsilon_i^2 + (1-P_{n+1})\varepsilon_i - P_n}{\varepsilon_i^2 - (\gamma+\delta)\varepsilon_i + \gamma\delta}.\end{aligned}$$

Considering Lemma 4, (11), (24) and (25), we obtain

$$\begin{aligned}&\prod_{i=1}^{n}\left[-\varepsilon_i^2 + (1-P_{n+1})\varepsilon_i - P_n\right]\\ &= (-P_n)^n + (-1)^n(P_{n+1}+P_n)\\ &\quad - P_n\left(z_3^{n-1} + z_4^{n-1}\right) - z_3^n - z_4^n\\ &= (-P_n)^n + (-1)^n\frac{Q_{n+1}}{2}\\ &\quad - P_n\left(z_3^{n-1} + z_4^{n-1}\right) - z_3^n - z_4^n,\end{aligned}$$

where z_3, z_4 are defined as (26) and (27), and

$$\begin{aligned}&\prod_{i=1}^{n}\left[\varepsilon_i^2 - (\gamma+\delta)\varepsilon_i + \gamma\delta\right]\\ &= -Q_n - Q_{n-1} + (-1)^n - 2\\ &= -4P_n + (-1)^n - 2.\end{aligned}$$

Consequently, if n is even, then

$$\det K = \frac{2P_n^n + Q_{n+1}}{-8P_n - 2} + \frac{P_n\left(z_3^{n-1} + z_4^{n-1}\right) + z_3^n + z_4^n}{4P_n + 1},$$

and if n is odd, then

$$\det K = \frac{2P_n^n + Q_{n+1}}{8P_n + 6} + \frac{P_n\left(z_3^{n-1} + z_4^{n-1}\right) + z_3^n + z_4^n}{4P_n + 3}.$$

The proof is complete. □

Theorem 13. *Let* L = RLMFLcircfr$(P_1, \ldots, P_n)$ *and* z_3*,* z_4 *be defined as* (26) *and* (27)*. If* $n \equiv 0 \pmod 4$*, then*

$$\det L = \frac{2P_n^n + Q_{n+1}}{-8P_n - 2} + \frac{P_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{4P_n + 1},$$

If $n \equiv 1 \pmod 4$*, then*

$$\det L = \frac{2P_n^n + Q_{n+1}}{8P_n + 6} + \frac{P_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{4P_n + 3}.$$

If $n \equiv 2 \pmod 4$*, then*

$$\det L = \frac{2P_n^n + Q_{n+1}}{8P_n + 2} - \frac{P_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{4P_n + 1}.$$

If $n \equiv 3 \pmod 4$*, then*

$$\det L = \frac{2P_n^n + Q_{n+1}}{-8P_n - 6} - \frac{P_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{4P_n + 3}.$$

Proof: The matrix L has the form

$$L = \begin{pmatrix} P_1 & \ldots & P_{n-1} & P_n \\ P_2 & \ldots & P_n - P_1 & P_1 \\ \vdots & \ddots & \vdots & \vdots \\ P_{n-1} & \ldots & P_{n-3} - P_{n-2} & P_{n-2} \\ P_n - P_1 & \ldots & P_{n-2} - P_{n-1} & P_{n-1} \end{pmatrix}$$

and can be written as

$$L = K\,\widehat{I}_n$$

where $\widehat{I}_n$ is the backward identity matrix of the form (3). Therefore,

$$\det L = \det K\ \det \widehat{I}_n,$$

and

$$\det \widehat{I}_n = (-1)^{\frac{n(n-1)}{2}}.$$

Define y_3, y_4 as (16) and (17), respectively. According to Theorem 12, we obtain the following conclusions: If $n \equiv 0 \pmod 4$, then

$$\det L = \frac{2P_n^n + Q_{n+1}}{-8P_n - 2} + \frac{P_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{4P_n + 1}.$$

If $n \equiv 1 \pmod 4$, then

$$\det L = \frac{2P_n^n + Q_{n+1}}{8P_n + 6} + \frac{P_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{4P_n + 3}.$$

If $n \equiv 2 \pmod 4$, then

$$\det L = \frac{2P_n^n + Q_{n+1}}{8P_n + 2} - \frac{P_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{4P_n + 1}.$$

If $n \equiv 3 \pmod 4$, then

$$\det L = \frac{2P_n^n + Q_{n+1}}{-8P_n - 6} - \frac{P_n\left(y_3^{n-1} + y_4^{n-1}\right) + y_3^n + y_4^n}{4P_n + 3}.$$

The proof is complete. □

2.4 Determinants of the RFMLR and RLMFL Circulant Matrix Involving Pell-Lucas Numbers

Theorem 14. *Let* $M = \mathrm{RFMLRcircfr}(Q_1, \ldots, Q_n)$. *If* n *is even, then*

$$\det M = \frac{(Q_{n+1} - 2)^n + 4Q_n^{n-1}}{-4P_{n-1} + 3} + \frac{(2 - Q_{n+1})\left(z_5^{n-1} + z_6^{n-1}\right) - z_5^n - z_6^n}{-4P_{n-1} + 3},$$

and if n *is odd, then*

$$\det M = \frac{(Q_{n+1} - 2)^n - 4Q_n^{n-1}}{4P_{n-1} + 1} - \frac{(2 - Q_{n+1})\left(z_5^{n-1} + z_6^{n-1}\right) - z_5^n - z_6^n}{4P_{n-1} + 1},$$

where

$$z_5 = -(2P_n + 1) + \sqrt{(2P_n + 1)^2 - Q_n\left(2 - Q_{n+1}\right)}, \quad (28)$$

$$z_6 = -(2P_n + 1) - \sqrt{(2P_n + 1)^2 - Q_n\left(2 - Q_{n+1}\right)}. \quad (29)$$

Proof: The matrix M has the form

$$M = \begin{pmatrix} Q_1 & Q_2 & \cdots & Q_n \\ Q_n & Q_1 - Q_n & \cdots & Q_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ Q_3 & Q_4 - Q_3 & \cdots & Q_2 \\ Q_2 & Q_3 - Q_2 & \cdots & Q_1 - Q_n \end{pmatrix}.$$

According to Lemma 3 and the Binet form (11), we have

$$\det M = \prod_{i=1}^{n}\left(Q_1 + Q_2\varepsilon_i + \cdots + Q_n\varepsilon_i^{n-1}\right)$$
$$= \prod_{i=1}^{n}\left[\gamma + \delta + (\gamma^2 + \delta^2)\varepsilon_i + \cdots + (\gamma^n + \delta^n)\varepsilon_i^{n-1}\right]$$
$$= \prod_{i=1}^{n}\frac{Q_n\varepsilon_i^2 + (Q_{n+1} - Q_n + 2)\varepsilon_i + 2 - Q_{n+1}}{\gamma\delta\varepsilon_i^2 - (\gamma + \delta)\varepsilon_i + 1}.$$

From Lemma 4, (10) and (25), we obtain

$$\prod_{i=1}^{n}\left[Q_n\varepsilon_i^2 + (Q_{n+1} - Q_n + 2)\varepsilon_i + 2 - Q_{n+1}\right] = (2 - Q_{n+1})^n + 4Q_n^{n-1} + (2 - Q_{n+1})\left(z_5^{n-1} + z_6^{n-1}\right) - z_5^n - z_6^n,$$

where z_5 and z_6 are defined as (28) and (29), and

$$\prod_{i=1}^{n}\left[\gamma\delta\varepsilon_i^2 - (\gamma + \delta)\varepsilon_i + 1\right] = Q_{n-1} - Q_n + 2(-1)^n + 1 = -4P_{n-1} + 2(-1)^n + 1.$$

Hence, if n is even, then

$$\det M = \frac{(Q_{n+1} - 2)^n + 4Q_n^{n-1}}{-4P_{n-1} + 3} + \frac{(2 - Q_{n+1})\left(z_5^{n-1} + z_6^{n-1}\right) - z_5^n - z_6^n}{-4P_{n-1} + 3},$$

and if n is odd, then

$$\det M = \frac{(Q_{n+1} - 2)^n - 4Q_n^{n-1}}{4P_{n-1} + 1} - \frac{(2 - Q_{n+1})\left(z_5^{n-1} + z_6^{n-1}\right) - z_5^n - z_6^n}{4P_{n-1} + 1}.$$

The proof is complete. □

Theorem 15. *Let* $N = \mathrm{RFMLRcircfr}(Q_n, \ldots, Q_1)$. *If* n *is even, then*

$$\det N = \frac{(2 - Q_n)^n + 2^{n+1}P_{n+1}}{-4P_n - 1} - \frac{(2 - Q_n)\left(z_7^{n-1} + z_8^{n-1}\right) - z_7^n - z_8^n}{4P_n + 1},$$

and if n is odd, then

$$\det N = \frac{(2-Q_n)^n - 2^{n+1}P_{n+1}}{-4P_n-3} - \frac{(2-Q_n)\left(z_7^{n-1}+z_8^{n-1}\right)-z_7^n-z_8^n}{4P_n+3},$$

where

$$z_7 = \frac{Q_{n+1}+\sqrt{Q_{n+1}^2-8Q_n+16}}{2}, \quad (30)$$

$$z_8 = \frac{Q_{n+1}-\sqrt{Q_{n+1}^2-8Q_n+16}}{2}. \quad (31)$$

Proof: The matrix N has the form

$$N = \begin{pmatrix} Q_n & Q_{n-1} & \cdots & Q_1 \\ Q_1 & Q_n - Q_1 & \cdots & Q_2 \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n-2} & Q_{n-3}-Q_{n-2} & \cdots & Q_{n-1} \\ Q_{n-1} & Q_{n-2}-Q_{n-1} & \cdots & Q_n - Q_1 \end{pmatrix}.$$

According to Lemma 3 and (11), we have

$$\begin{aligned} \det N &= \prod_{i=1}^{n}\left(Q_n + Q_{n-1}\varepsilon_i + \cdots + Q_1\varepsilon_i^{n-1}\right) \\ &= \prod_{i=1}^{n}\left[\gamma^n+\delta^n+(\gamma^{n-1}+\delta^{n-1})\varepsilon_i \right.\\ &\quad \left. + \cdots + (\gamma+\delta)\varepsilon_i^{n-1}\right] \\ &= \prod_{i=1}^{n}\frac{-2\varepsilon_i^2 - Q_{n+1}\varepsilon_i + 2 - Q_n}{\varepsilon_i^2-(\gamma+\delta)\varepsilon_i+\gamma\delta}. \end{aligned}$$

By Lemma 4, the Binet form (11) and the identity (25), we obtain

$$\prod_{i=1}^{n}\left(-2\varepsilon_i^2 - Q_{n+1}\varepsilon_i + 2 - Q_n\right) = (2-Q_n)^n$$
$$-(-2)^{n+1}P_{n+1} + (2-Q_n)\left(z_7^{n-1}+z_8^{n-1}\right) - z_7^n - z_8^n,$$

where z_7 and z_8 are defined as (30) and (31), and

$$\prod_{i=1}^{n}\left[\varepsilon_i^2-(\gamma+\delta)\varepsilon_i+\gamma\delta\right] = -4P_n + (-1)^n - 2.$$

Thus, if n is even, then

$$\det N = \frac{(2-Q_n)^n + 2^{n+1}P_{n+1}}{-4P_n-1} - \frac{(2-Q_n)\left(z_7^{n-1}+z_8^{n-1}\right)-z_7^n-z_8^n}{4P_n+1},$$

and if n is odd, then

$$\det N = \frac{(2-Q_n)^n - 2^{n+1}P_{n+1}}{-4P_n-3} - \frac{(2-Q_n)\left(z_7^{n-1}+z_8^{n-1}\right)-z_7^n-z_8^n}{4P_n+3}.$$

The proof is complete. □

Theorem 16. *Let* $R = \text{RLMFLcircfr}(Q_1, \ldots, Q_n)$ *and* z_7, z_8 *are defined by* (30) *and* (31), *respectively. If* $n \equiv 0 \pmod 4$, *then*

$$\det R = \frac{(2-Q_n)^n + 2^{n+1}P_{n+1}}{-4P_n-1} - \frac{(2-Q_n)\left(z_7^{n-1}+z_8^{n-1}\right)-z_7^n-z_8^n}{4P_n+1}.$$

If $n \equiv 1 \pmod 4$, *then*

$$\det R = \frac{(2-Q_n)^n - 2^{n+1}P_{n+1}}{-4P_n-3} - \frac{(2-Q_n)\left(z_7^{n-1}+z_8^{n-1}\right)-z_7^n-z_8^n}{4P_n+3}.$$

If $n \equiv 2 \pmod 4$, *then*

$$\det R = \frac{(2-Q_n)^n + 2^{n+1}P_{n+1}}{4P_n+1} + \frac{(2-Q_n)\left(z_7^{n-1}+z_8^{n-1}\right)-z_7^n-z_8^n}{4P_n+1}.$$

If $n \equiv 3 \pmod 4$, *then*

$$\det R = \frac{(2-Q_n)^n - 2^{n+1}P_{n+1}}{4P_n+3} + \frac{(2-Q_n)\left(z_7^{n-1}+z_8^{n-1}\right)-z_7^n-z_8^n}{4P_n+3}.$$

Proof: The matrix R has the form

$$R = \begin{pmatrix} Q_1 & \cdots & Q_{n-1} & Q_n \\ Q_2 & \cdots & Q_n - Q_1 & Q_1 \\ \vdots & \ddots & \vdots & \vdots \\ Q_{n-1} & \cdots & Q_{n-3}-Q_{n-2} & Q_{n-2} \\ Q_n - Q_1 & \cdots & Q_{n-2}-Q_{n-1} & Q_{n-1} \end{pmatrix}$$

and can be written as

$$R = N\,\widehat{I}_n$$

where $\widehat{I}_n$ is the backward identity matrix of the form (3). Accordingly,

$$\det R = \det N \, \det \widehat{I}_n,$$

and

$$\det \widehat{I}_n = (-1)^{\frac{n(n-1)}{2}}.$$

Define z_7, z_8 as (30) and (31), respectively. From Theorem 15, we obtain the following results: If $n \equiv 0 \pmod 4$, then

$$\det R = \frac{(2-Q_n)^n + 2^{n+1}P_{n+1}}{-4P_n - 1} - \frac{(2-Q_n)\left(z_7^{n-1} + z_8^{n-1}\right) - z_7^n - z_8^n}{4P_n + 1}.$$

If $n \equiv 1 \pmod 4$, then

$$\det R = \frac{(2-Q_n)^n - 2^{n+1}P_{n+1}}{-4P_n - 3} - \frac{(2-Q_n)\left(z_7^{n-1} + z_8^{n-1}\right) - z_7^n - z_8^n}{4P_n + 3}.$$

If $n \equiv 2 \pmod 4$, then

$$\det R = \frac{(2-Q_n)^n + 2^{n+1}P_{n+1}}{4P_n + 1} + \frac{(2-Q_n)\left(z_7^{n-1} + z_8^{n-1}\right) - z_7^n - z_8^n}{4P_n + 1}.$$

If $n \equiv 3 \pmod 4$, then

$$\det R = \frac{(2-Q_n)^n - 2^{n+1}P_{n+1}}{4P_n + 3} + \frac{(2-Q_n)\left(z_7^{n-1} + z_8^{n-1}\right) - z_7^n - z_8^n}{4P_n + 3}.$$

The proof is complete. □

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