Algorithms for finding the minimum norm solution of hierarchical fixed point problems

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Abstract: Let C be a nonempty closed convex subset of a real Hilbert space H, $\{T_k\}_{k=1}^{\infty} : C \to C$ an infinite family of nonexpansive mappings with the nonempty set of common fixed points $\bigcap_{k=1}^{\infty} Fix(T_k)$ and $S : C \to C$ a nonexpansive mapping. In this paper, we introduce an explicit algorithm with strong convergence for finding the minimum norm solution of the following hierarchical fixed point problem

$$\text{Find } x^* \in \bigcap_{k=1}^{\infty} Fix(T_k) \text{ and } \langle (I-S)x^*, x^* - x\rangle \le 0, \ \forall x \in \bigcap_{k=1}^{\infty} Fix(T_k).$$

Key–Words: Hierarchical fixed point, Iterative algorithm, Variational inequality, Minimum norm, Strong convergence

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. Let $f : C \to H$ be a $\alpha-$contraction, where $\alpha \in [0, 1)$; namely,

$$\|f(x) - f(y)\| \le \alpha \|x - y\|, \forall \ x, y \in C.$$

A mapping $T : C \to C$ is said to be nonexpansive, if

$$\|Tx - Ty\| \le \|x - y\|, \forall \ x, y \in C.$$

We use $Fix(T)$ to denote the set of fixed points of T, namely, $Fix(T) = \{x \in C : Tx = x\}$. The metric projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

Let $S, T : C \to C$ be two nonexpansive mappings. Now, we consider the following problem of finding hierarchically a fixed point of a nonexpansive mapping T with respect to another mapping S, namely finding a point x^* with the property

$$\begin{aligned} & x^* \in Fix(T) \ \text{ such that} \\ & \langle (I-S)x^*, x^* - x\rangle \le 0, \ \forall x \in Fix(T). \end{aligned} \tag{1}$$

Problem (1) is very important in the area of optimization and related fields, such as signal processing and image reconstruction (see [1-4]). Recently, for solving (1), Mainge and Moudafi [5] introduced a hybrid iterative method and Lu, Xu and Yin [6] considered a regularization method. Related work in the field can be found in [7-14] and the references therein. It is needed to find a minimum norm solution in many problems. A typical example is the least-squares solution to the constrained linear inverse problem (see [15]). Therefore, it is an interesting problem to find the minimum norm solution of (1). Yao et al. [13] introduced an implicit algorithm and an explicit algorithm as following:

$$x_{s,t} = P_C[s(1-t)Sx_{s,t} + (1-s)Tx_{s,t}],$$

where $s, t \in (0, 1)$, P_C is the metric projection from H to C.

$$x_{n+1} = P_C[\lambda_n(1-\alpha_n)Sx_n + (1-\lambda_n)Tx_n], \ n \ge 0,$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ are two sequences in (0,1) and P_C is the metric projection from H onto C. Under some mild assumptions, Yao et al. [13] proved that $\{x_{s,t}\}$ and $\{x_n\}$ converge strongly to the minimum norm solution x^* of (1).

In order to deal with some problems involving the common fixed points of infinite family of nonexpansive mappings, W-mapping is often used. Let

$\{T_k\}_{k=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings and let $\{\xi_k\}_{k=1}^{\infty}$ be a real number sequence such that $0 \le \xi_k \le 1$ for every $k \in \mathbb{N}$. For any $n \in \mathbb{N}$, we define a mapping W_n of C into itself as follows:

$$
\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \xi_n T_n U_{n,n+1} + (1-\xi_n)I,\\
&U_{n,n-1} = \xi_{n-1} T_{n-1} U_{n,n} + (1-\xi_{n-1})I,\\
&\vdots \cdots\\
&U_{n,k} = \xi_k T_k U_{n,k+1} + (1-\xi_k)I,\\
&U_{n,k-1} = \xi_{k-1} T_{k-1} U_{n,k} + (1-\xi_{k-1})I,\\
&\vdots \cdots\\
&U_{n,2} = \xi_2 T_2 U_{n,3} + (1-\xi_2)I,\\
&W_n = U_{n,1} = \xi_1 T_1 U_{n,2} + (1-\xi_1)I,
\end{aligned}
$$

Such W_n is called the W-mapping generated by $\{T_k\}_{k=1}^{\infty}$ and $\{\xi_k\}_{k=1}^{\infty}$, see[14,16-18].

Now we consider the following hierarchical fixed point problem which includes (1) as a special case.

$$
\begin{array}{l}
\text{Find } x^* \in \bigcap_{k=1}^{\infty} Fix(T_k) \text{ such that}\\
\langle (I-S)x^*, x^*-x\rangle \le 0, \ \forall x \in \bigcap_{k=1}^{\infty} Fix(T_k),
\end{array} \tag{2}
$$

where $\{T_k\}_{k=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings with $\bigcap_{k=1}^{\infty} Fix(T_k) \neq \emptyset$. In [14], Yao, et al. considered an explicit algorithm which generated a iterative sequence $\{x_n\}$ by

$$
\begin{aligned}
x_{n+1} \;=\; & \alpha_n S x_n + (1-\alpha_n) W_n P_C[(1\\
& -\beta_n)x_n],\ n \ge 0,
\end{aligned} \tag{3}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0,1), $W_n : C \to C$ is the W-mapping. Under some mild assumptions, they proved that $\{x_n\}$ generated by (3) converges strongly to the minimum norm solution of hierarchical fixed point problem (2).

Since W-mapping contains many composite operations of $\{T_k\}$, it is complicated and needs large computational work. In this paper, we will introduce a new mapping to take the place of W-mapping for solving hierarchical fixed point problem (2). Let $\{T_k\}_{k=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings. The new mapping is defined as follows:

$$
L_n = \sum_{k=1}^{n} \frac{\omega_k}{S_n} T_k \ \ (n = 1, 2, \ldots), \tag{4}
$$

where $\omega_k > 0$ with $\sum_{k=1}^{\infty} \omega_k = 1$, $S_n = \sum_{k=1}^{n} \omega_k$.

Inspired and motivated by the work in the field, we introduce two explicit algorithms with L_n for finding the minimum norm solution of hierarchical fixed point problem (2). Under certain appropriate conditions, we prove that the two proposed algorithms have strong convergence. Because L_n used in our algorithms doesn't contain many composite operations of $\{T_k\}$ which are included in W-mapping, our introduced algorithms are more brief and need less computational work.

We will use the notations:

- $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
- $\omega_w(x_n) = \{x : \exists\ x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2 Preliminaries

In this section, some lemmas are given which are important to prove our main results.

Lemma 1 *[19] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$; in particular, if $y = 0$, then $x \in Fix(T)$.*

Lemma 2 *[13] Given $x \in H$ and $z \in C$.*
(1) That $z = P_C x$ if and only if there holds the relation:

$$\langle x-z, y-z\rangle \le 0 \quad for\ \ all\ \ y \in C.$$

(2) That $z = P_C x$ if and only if there holds the relation:

$$\|x-z\|^2 \le \|x-y\|^2 - \|y-z\|^2 \quad for\ \ all\ \ y \in C.$$

(3) There holds the relation:

$$\langle P_C x - P_C y, x-y\rangle \ge \|P_C x - P_C y\|^2 \ for\ all\ y \in H.$$

Lemma 3 *[20] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n \delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 4 *[21] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_k\}_{k=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings. Suppose $\bigcap_{k=1}^{\infty} Fix(T_k)$ is nonempty. Let $\{\omega_k\}_{k=1}^{\infty}$ be a sequence in $(0,1)$ with $\sum_{k=1}^{\infty} \omega_k = 1$. Then a mapping L on C defined by $Lx = \sum_{k=1}^{\infty} \omega_k T_k x$ for $x \in C$ is well defined, nonexpansive and $Fix(L) = \bigcap_{k=1}^{\infty} Fix(T_k)$ holds.*

Lemma 5 *[22] Let H be a real Hilbert space, $\{T_k : k \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on H with $\bigcap_{k=1}^{\infty} Fix(T_k) \neq \emptyset$, and $\{\omega_k\}$ be a sequence of positive numbers with $\sum_{k=1}^{\infty} \omega_k = 1$. Let $L = \sum_{k=1}^{\infty} \omega_k T_k$, $L_m = \sum_{k=1}^{m} \frac{\omega_k}{S_m} T_k$, and $S_m = \sum_{k=1}^{m} \omega_k$. Then L_m uniformly converges to L in each bounded subset S of H.*

3 Main result

In this section, we first introduce an explicit scheme for finding the minimum norm solution of hierarchical fixed point problem (2). More precisely, starting with an arbitrary initial guess $x_0 \in C$, we define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = P_C[\beta_n(1-\alpha_n)Sx_n + (1-\beta_n)L_n x_n], \quad n \geq 0, \tag{5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in (0,1), $S : C \to C$ is a nonexpansive mapping, $L_n : C \to C$ is the nonexpansive mapping defined by (4), $P_C : H \to C$ is the metric projection.

Remark 6 *We note that the well-known Mann algorithm $x_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n$ has only weak convergence; please see [23-29] for the related works. This implies that the algorithm*

$$x_{n+1} = \beta_n Sx_n + (1-\beta_n)L_n x_n, \ \ n \geq 0 \tag{6}$$

has only weak convergence. In order to obtain strong convergence, some modifications are needed. We modify the algorithm (6) by adding the factor $1-\alpha_n$(where $\alpha_n \to 0$). However, we note that $(1-\alpha_n)x_n$ may not be in C. Hence, the projection P_C is used in order to guarantee that the sequence $\{x_n\}$ is well-defined.

Next, we will show the strong convergence of the algorithm (5). As a matter of fact, we introduce a general algorithm which includes the algorithm (5) as a special case. For any $x_0 \in C$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = P_C[\beta_n(\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (1-\beta_n)L_n x_n], \quad n \geq 0, \tag{7}$$

where $f : C \to H$ is a $\alpha-$contraction. It is clear that if we take $f = 0$, then (7) reduces to (5). For the strong convergence of the algorithm (5) and (7), we have the following theorem. Throughout, we use Ω to denote the set of solution to (2) and assume that Ω is nonempty.

Theorem 7 *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a α-contraction with $\alpha \in [0,1)$, $S : C \to C$ a nonexpansive mapping, and $\{T_k\}_{k=1}^{\infty} : C \to C$ an infinite family of nonexpansive mapping with $\bigcap_{k=1}^{\infty} Fix(T_k) \neq \emptyset$. Let $L_n = \sum_{k=1}^{n} \frac{\omega_k}{s_n} T_k$, $S_n = \sum_{k=1}^{n} \omega_k$, and $w_k > 0$ with $\sum_{k=1}^{\infty} \omega_k = 1$. Suppose the following conditions are satisfied*

(C1) $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} \frac{\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}}{\alpha_n \beta_n^2} = \lim\limits_{n\to\infty} \frac{\beta_n}{\alpha_n} = \lim\limits_{n\to\infty} \frac{1}{\alpha_n\beta_n}(\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}) = \lim\limits_{n\to\infty} \frac{\omega_n}{\alpha_n\beta_n^2} = 0$;

(C2) $\sum\limits_{n=0}^{\infty} \alpha_n\beta_n = \infty$;

(C3) There exists some constant $\gamma > 0$ such that $\|x - L_n x\| \geq \gamma \mathrm{Dist}(x, \bigcap_{k=1}^{\infty} Fix(T_k))$, where

$$Dist(x, \bigcap_{n=1}^{\infty} Fix(T_k)) = \inf_{y \in \bigcap_{k=1}^{\infty} Fix(T_k)} \|x-y\|.$$

Then the sequence $\{x_n\}$ generated by (7) converges strongly to $x^* \in \bigcap_{k=1}^{\infty} Fix(T_k)$ which is the unique solution of the variational inequality:

$$x^* \in \Omega, \quad \langle (I-f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega. \tag{8}$$

In particular, if we take $f = 0$, then the sequence $\{x_n\}$ generated by (5) converges strongly to $x^* \in \bigcap_{k=1}^{\infty} Fix(T_k)$ which is the minimum norm solution of hierarchical fixed point problem (2).

Proof: We will use six steps to prove the result.

Step 1. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| \to 0$.

For $n \geq 0$, set

$$y_n = \beta_n (\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (1-\beta_n)L_n x_n.$$

Then it holds that

$$\begin{aligned} & y_n - y_{n-1} \\ = \ & \beta_n(\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (1-\beta_n)L_n x_n \\ & -\beta_{n-1}(\alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})Sx_{n-1}) \\ & -(1-\beta_{n-1})L_{n-1}x_{n-1} \\ = \ & \alpha_n\beta_n[f(x_n) - f(x_{n-1})] + \beta_n(1-\alpha_n)(Sx_n \\ & -Sx_{n-1}) + (\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1})[f(x_{n-1}) \\ & -Sx_{n-1}] + (1-\beta_n)(L_n x_n - L_n x_{n-1}) \\ & +(1-\beta_{n-1})(L_n x_{n-1} - L_{n-1}x_{n-1}) + (\beta_n \\ & -\beta_{n-1})(Sx_{n-1} - L_n x_{n-1}). \end{aligned}$$

It follows that

$$\begin{aligned} & \|x_{n+1}-x_n\| \\ = & \|P_C y_n - P_C y_{n-1}\| \le \|y_n - y_{n-1}\| \\ \le & \alpha_n\beta_n\|f(x_n)-f(x_{n-1})\| \\ & +\beta_n(1-\alpha_n)\|Sx_n - Sx_{n-1}\| \\ & +|\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}|\|f(x_{n-1})-Sx_{n-1}\| \\ & +(1-\beta_n)\|L_nx_n - L_nx_{n-1}\| \\ & +(1-\beta_{n-1})\|L_nx_{n-1}-L_{n-1}x_{n-1}\| \\ & +|\beta_n-\beta_{n-1}|\|Sx_{n-1}-L_nx_{n-1}\| \\ \le & \alpha\alpha_n\beta_n\|x_n-x_{n-1}\| \\ & +\beta_n(1-\alpha_n)\|x_n-x_{n-1}\| \\ & +|\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}|\|f(x_{n-1})-Sx_{n-1}\| \\ & +(1-\beta_{n-1})\|L_nx_{n-1}-L_{n-1}x_{n-1}\| \\ & +|\beta_n-\beta_{n-1}|\|Sx_{n-1}-L_nx_{n-1}\| \\ \le & [1-(1-\alpha)\alpha_n\beta_n]\|x_n-x_{n-1}\| \\ & +|\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}|\|f(x_{n-1})-Sx_{n-1}\| \\ & +(1-\beta_{n-1})\|L_nx_{n-1}-L_{n-1}x_{n-1}\| \\ & +|\beta_n-\beta_{n-1}|\|Sx_{n-1}-L_nx_{n-1}\|. \end{aligned}$$

We observe that

$$\begin{aligned} & \|L_nx_{n-1}-L_{n-1}x_{n-1}\| \\ = & \|\sum_{k=1}^{n}\frac{\omega_k}{S_n}T_kx_{n-1} - \sum_{k=1}^{n-1}\frac{\omega_k}{S_{n-1}}T_kx_{n-1}\| \\ = & \|\frac{\omega_n}{S_n}T_nx_{n-1} + \sum_{k=1}^{n-1}\frac{-\omega_n\omega_k}{S_nS_{n-1}}T_kx_{n-1}\| \\ \le & \|\frac{\omega_n}{S_n}T_nx_{n-1}\| + \|\sum_{k=1}^{n-1}\frac{\omega_n\omega_k}{S_nS_{n-1}}T_kx_{n-1}\| \\ \le & \frac{\omega_n}{S_n}\|T_nx_{n-1}\| + \sum_{k=1}^{n-1}\frac{\omega_n\omega_k}{S_nS_{n-1}}\|T_kx_{n-1}\| \\ \le & \omega_n\frac{\|T_nx_{n-1}\|}{\omega_1} + \sum_{k=1}^{n-1}\frac{\omega_n\omega_k}{S_{n-1}}\frac{\|T_kx_{n-1}\|}{\omega_1} \\ \le & M\omega_n, \end{aligned}$$

where M is a constant such that

$$\begin{aligned} M \ \ge\ & \sup_{1\le k\le n}\{(\|f(x_n)\|+\|Sx_n\|), \frac{2\|T_kx_n\|}{\omega_1}, \\ & (\|Sx_{n-1}-L_nx_{n-1}\|), \|x_n-x_{n-1}\|\}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \|x_{n+1}-x_n\| \\ & \le [1-(1-\alpha)\alpha_n\beta_n]\|x_n-x_{n-1}\| \\ & +[|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}| + (1-\beta_{n-1})\omega_n \\ & \quad +|\beta_n-\beta_{n-1}|]M \\ & \le [1-(1-\alpha)\alpha_n\beta_n]\|x_n-x_{n-1}\| \\ & +\alpha_n\beta_n M\left(\tfrac{|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|}{\alpha_n\beta_n} + \tfrac{\omega_n}{\alpha_n\beta_n} + \tfrac{|\beta_n-\beta_{n-1}|}{\alpha_n\beta_n}\right). \end{aligned} \tag{9}$$

Thus, from (C1), we have $\limsup_{n\to\infty}(\frac{\omega_n}{\alpha_n\beta_n} + \frac{|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|}{\alpha_n\beta_n} + \frac{|\beta_n-\beta_{n-1}|}{\alpha_n\beta_n}) = 0$. Hence, applying Lemma 3 to (9), we conclude immediately that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{10}$$

Step 2. We prove that $\omega_w(x_n) \subset Fix(L) = \bigcap_{k=1}^{\infty} Fix(T_k)$, where $L = \sum_{k=1}^{\infty}\omega_kT_k$.

By (7), we get immediately

$$\begin{aligned} & \|x_{n+1}-L_nx_n\| \\ = & \|P_Cy_n - P_CL_nx_n\| \le \|y_n - L_nx_n\| \\ = & \|\beta_n(\alpha_nf(x_n)+(1-\alpha_n)Sx_n) \\ & +(1-\beta_n)L_nx_n - L_nx_n\| \\ \le & \beta_n\|\alpha_nf(x_n)+(1-\alpha_n)Sx_n - L_nx_n\| \\ \to & 0\ (n\to\infty). \end{aligned} \tag{11}$$

Notice that

$$\begin{aligned} \|x_n - Lx_n\| \ \le\ & \|x_n-x_{n+1}\| + \|x_{n+1}-L_nx_n\| \\ & +\|L_nx_n - Lx_n\|. \end{aligned}$$

Thus, from (10), (11) and Lemma 5, we deduce

$$\lim_{n\to\infty}\|x_n - Lx_n\| = 0. \tag{12}$$

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $\tilde{x} \in H$. Therefore, we have $\tilde{x} \in Fix(L) = \bigcap_{k=1}^{\infty} Fix(T_k)$ by (12) and Lemma 1. Hence, $\omega_w(x_n) \subset Fix(L) = \bigcap_{k=1}^{\infty} Fix(T_k)$.

Step 3. We claim that $\omega_w(x_n) \subset \Omega$.

By (9), we get

$$\begin{aligned} & \frac{\|x_{n+1}-x_n\|}{\beta_n} \\ \le & [1-(1-\alpha)\alpha_n\beta_n]\frac{\|x_n-x_{n-1}\|}{\beta_n} \\ & +M\frac{|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|+\omega_n+|\beta_n-\beta_{n-1}|}{\beta_n} \\ = & [1-(1-\alpha)\alpha_n\beta_n]\frac{\|x_n-x_{jn-1}\|}{\beta_{n-1}} \\ & +[1-(1-\alpha)\alpha_n\beta_n]\Bigg(\frac{\|x_n-x_{n-1}\|}{\beta_n} \\ & -\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}}\Bigg) + M\Bigg(\frac{|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|}{\beta_n} \\ & +\frac{\omega_n+|\beta_n-\beta_{n-1}|}{\beta_n}\Bigg) \end{aligned}$$

$$\begin{aligned} &\leq [1-(1-\alpha)\alpha_n\beta_n]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} \\ &+\alpha_n\beta_n M\Big(\frac{|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|+\omega_n}{\alpha_n\beta_n^2} \\ &+\frac{|\beta_n-\beta n-1|}{\alpha_n\beta_n^2}+\frac{1}{\alpha_n\beta_n}|\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}|\Big). \end{aligned}$$

Thus, by virtue of condition (C1), we have $\lim_{n\to\infty}\left(\frac{|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|+\omega_n}{\alpha_n\beta_n^2}+\frac{|\beta_n-\beta n-1|}{\alpha_n\beta_n^2}+\frac{1}{\alpha_n\beta_n}|\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}|\right)=0$. Hence, applying Lemma 3 to above last inequality, we conclude immediately that

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_n}=0. \quad (13)$$

Rewriting (7) as

$$\begin{aligned} x_{n+1} &= P_C y_n - y_n + \beta_n(\alpha_n f(x_n) \\ &+(1-\alpha_n)Sx_n)+(1-\beta_n)L_n x_n. \end{aligned}$$

We obtain

$$\begin{aligned} x_n - x_{n+1} &= y_n - P_C y_n + \alpha_n\beta_n(I-f)x_n \\ &+\beta_n(1-\alpha_n)(I-S)x_n \\ &+(1-\beta_n)(I-L_n)x_n. \end{aligned}$$

Set $z_n=\frac{x_n-x_{n+1}}{\beta_n}$ for all $n\geq 0$. That is

$$\begin{aligned} z_n &= \frac{y_n-P_C y_n}{\beta_n}+\alpha_n(I-f)x_n \\ &+(1-\alpha_n)(I-S)x_n \\ &+\frac{(1-\beta_n)}{\beta_n}(I-L_n)x_n. \end{aligned}$$

Pick up $u\in\bigcap_{k=1}^{\infty}Fix(T_k)$, then we have

$$\begin{aligned} &\langle z_n, x_n-u\rangle \\ =\ & \frac{1}{\beta_n}\langle y_n-P_C y_n, P_C y_{n-1}-u\rangle \\ &+\alpha_n\langle (I-f)x_n, x_n-u\rangle \\ &+(1-\alpha_n)\langle (I-S)x_n, x_n-u\rangle \\ &+\frac{1-\beta_n}{\beta_n}\langle (I-L_n)x_n, x_n-u\rangle \\ =\ & \frac{1}{\beta_n}\langle y_n-P_C y_n, P_C y_n-u\rangle \\ &+\frac{1}{\beta_n}\langle y_n-P_C y_n, P_C y_{n-1}-P_C y_n\rangle \\ &+\alpha_n\langle (I-f)x_n, x_n-u\rangle \\ &+(1-\alpha_n)\langle (I-S)u, x_n-u\rangle \\ &+(1-\alpha_n)\langle (I-S)x_n-(I-S)u, x_n-u\rangle \\ &+\frac{1-\beta_n}{\beta_n}\langle (I-L_n)x_n-(I-L_n)u, x_n-u\rangle. \end{aligned}$$

Using the property of the projection (Lemma 2), we have

$$\langle y_n-P_C y_n, P_C y_n-u\rangle\geq 0.$$

Using monotonicity of $I-S$ and $I-L_n$, we derive the that

$$\begin{aligned} \langle (I-S)x_n-(I-S)u, x_n-u\rangle\geq 0, \\ \langle (I-L_n)x_n-(I-L_n)u, x_n-u\rangle\geq 0. \end{aligned}$$

Therefore, we conclude that (noticing $x_n=P_C y_{n-1}$)

$$\begin{aligned} &\langle z_n, x_n-u\rangle \\ \geq\ & \frac{1}{\beta_n}\langle y_n-P_C y_n, P_C y_{n-1}-P_C y_n\rangle \\ &+\alpha_n\langle (I-f)x_n, x_n-u\rangle \\ &+(1-\alpha_n)\langle (I-S)u, x_n-u\rangle \\ =\ & \langle y_n-P_C y_n, z_n\rangle+\alpha_n\langle (I-f)x_n, x_n-u\rangle \\ &+(1-\alpha_n)\langle (I-S)u, x_n-u\rangle. \end{aligned}$$

Since $z_n\to 0$, $\alpha_n\to 0$ and $\{x_n\}$ is bounded by assumption which implies $\{y_n\}$ is bounded, we obtain from the above inequality that

$$\limsup_{n\to\infty}\langle (I-S)u, x_n-u\rangle\leq 0,\ \forall u\in\bigcap_{k=1}^{\infty}Fix(T_k).$$

Therefore, we have

$$\limsup_{j\to\infty}\langle (I-S)u, x_{n_j}-u\rangle\leq 0,\ \forall u\in\bigcap_{k=1}^{\infty}Fix(T_k).$$

Since $x_{n_j}\rightharpoonup\tilde{x}\in\omega_w(x_n)$, we obtain

$$\limsup_{j\to\infty}\langle (I-S)u, x_{n_j}-u\rangle=\langle (I-S)u, \tilde{x}-u\rangle.$$

This implies that every weak cluster point $\tilde{x}\in\bigcap_{k=1}^{\infty}Fix(T_k)$ of the sequence $\{x_n\}$ solves the variational inequality

$$\langle (I-S)u, \tilde{x}-u\rangle\leq 0,\quad \forall u\in\bigcap_{k=1}^{\infty}Fix(T_k).$$

This is equivalent to its dual variational inequality

$$\langle (I-S)\tilde{x}, \tilde{x}-u\rangle\leq 0,\quad \forall u\in\bigcap_{k=1}^{\infty}Fix(T_k). \quad (14)$$

Hence, we get $\omega_w(x_n)\subset\Omega$.

Step 4. We show that $\limsup_{n\to\infty}\langle (I-f)x^*, x_n-x^*\rangle\geq 0$.

Since f is a contraction, the solution set of the variational inequality (8) is a singleton. Let x^* is the unique solution of the variational inequality (8). Now we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying

$$\begin{aligned} & \limsup_{n\to\infty}\langle (I-f)x^*, x_n - x^*\rangle \\ = & \lim_{k\to\infty}\langle (I-f)x^*, x_{n_k} - x^*\rangle. \end{aligned}$$

Without loss of generality, we may further assume that $x_{n_k} \rightharpoonup \bar{x}$, then $\bar{x} \in \Omega$. Therefore, noticing that x^* is the solution of the variational inequality (8), we have

$$\begin{aligned} & \limsup_{n\to\infty}\langle (I-f)x^*, x_n - x^*\rangle \\ = & \langle (I-f)x^*, \bar{x} - x^*\rangle \\ \geq & 0. \end{aligned} \tag{15}$$

Step 5. We show that $\limsup_{n\to\infty}\frac{1}{\alpha_n}\langle Sx^* - x^*, x_{n+1} - x^*\rangle \leq 0$.

We note that

$$\begin{aligned} & \langle Sx^* - x^*, x_{n+1} - x^*\rangle \\ = & \langle Sx^* - x^*, x_{n+1} - P_{\bigcap_{k=1}^\infty Fix(T_k)}x_{n+1}\rangle \\ & +\langle Sx^* - x^*, P_{\bigcap_{k=1}^\infty Fix(T_k)}x_{n+1} - x^*\rangle. \end{aligned}$$

Since $P_{\bigcap_{k=1}^\infty Fix(T_k)}x_{n+1} \in \bigcap_{k=1}^\infty Fix(T_k)$, by (2) we have

$$\langle Sx^* - x^*, P_{\bigcap_{k=1}^\infty Fix(T_k)}x_{n+1} - x^*\rangle \leq 0,$$

and by assumption (C3), we have

$$\begin{aligned} & \langle Sx^* - x^*, x_{n+1} - x^*\rangle \\ \leq & \langle Sx^* - x^*, x_{n+1} - P_{\bigcap_{k=1}^\infty Fix(T_k)}x_{n+1}\rangle \\ \leq & \|Sx^* - x^*\|\,\|x_{n+1} - P_{\bigcap_{k=1}^\infty Fix(T_k)}x_{n+1}\| \\ = & \|Sx^* - x^*\|\, Dist(x_{n+1}, \bigcap_{k=1}^\infty Fix(T_k)) \\ \leq & \frac{1}{\gamma}\|Sx^* - x^*\|\,\|x_{n+1} - L_n x_{n+1}\|. \end{aligned}$$

We note that

$$\begin{aligned} & \|x_{n+1} - L_n x_{n+1}\| \\ \leq & \|x_{n+1} - L_n x_n\| + \|L_n x_n - L_n x_{n+1}\| \\ \leq & \|P_C y_n - P_C L_n x_n\| + \|L_n x_n - L_n x_{n+1}\| \\ \leq & \beta_n\|\alpha_n f(x_n) + (1-\alpha_n)Sx_n - L_n x_n\| \\ & +\|x_n - x_{n+1}\|. \end{aligned}$$

Therefore, we get

$$\begin{aligned} & \frac{1}{\alpha_n}\|x_{n+1} - L_n x_{n+1}\| \\ \leq & \frac{\beta_n}{\alpha_n}\|\alpha_n f(x_n) + (1-\alpha_n)Sx_n - L_n x_n\| \\ & +\frac{\beta_n}{\alpha_n}\frac{\|x_n - x_{n+1}\|}{\beta_n} \to 0. \end{aligned}$$

It follows that

$$\limsup_{n\to\infty}\frac{1}{\alpha_n}\langle Sx^* - x^*, x_{n+1} - x^*\rangle \leq 0. \tag{16}$$

Step 6. Finally, we prove that $x_n \to x^*$ as $n \to \infty$.

From (7), we deduce that (noticing $x_{n+1} = P_C y_n$)

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ = & \langle y_n - x^*, x_{n+1} - x^*\rangle \\ & +\langle P_C y_n - y_n, P_C y_n - x^*\rangle \\ \leq & \langle y_n - x^*, x_{n+1} - x^*\rangle \\ \leq & \langle \beta_n(\alpha_n f(x_n) + (1-\alpha_n)Sx_n) \\ & +(1-\beta_n)L_n x_n - x^*, x_{n+1} - x^*\rangle \\ = & \alpha_n\beta_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\beta_n\langle Sx_n - x^*, x_{n+1} - x^*\rangle \\ & +(1-\beta_n)\langle L_n x_n - x^*, x_{n+1} - x^*\rangle \\ = & \alpha_n\beta_n\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle \\ & +\alpha_n\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\beta_n\langle Sx_n - Sx^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\beta_n\langle Sx^* - x^*, x_{n+1} - x^*\rangle \\ & +(1-\beta_n)\langle L_n x_n - x^*, x_{n+1} - x^*\rangle \\ \leq & \alpha\alpha_n\beta_n\|x_n) - x^*\|\|x_{n+1} - x^*\| \\ & +\alpha_n\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\beta_n\|x_n - x^*\|\|x_{n+1} - x^*\| \\ & +(1-\alpha_n)\beta_n\langle Sx^* - x^*, x_{n+1} - x^*\rangle \\ & +(1-\beta_n)\|x_n - x^*\|\|x_{n+1} - x^*\| \\ = & [1 - (1-\alpha)\alpha_n\beta_n]\|x_n - x^*\|\|x_{n+1} - x^*\| \\ & +\alpha_n\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\beta_n\langle Sx^* - x^*, x_{n+1} - x^*\rangle \\ \leq & [1 - (1-\alpha)\alpha_n\beta_n]\frac{\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2}{2} \\ & +\alpha_n\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\beta_n\langle Sx^* - x^*, x_{n+1} - x^*\rangle. \end{aligned}$$

It turns out that

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ \leq & \frac{1 - (1-\alpha)\alpha_n\beta_n}{1 + (1-\alpha)\alpha_n\beta_n}\|x_n - x^*\|^2 \\ & +\frac{2\beta_n}{1 + (1-\alpha)\alpha_n\beta_n}[\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\langle Sx^* - x^*, x_{n+1} - x^*\rangle] \\ \leq & [1 - (1-\alpha)\alpha_n\beta_n]\|x_n - x^*\|^2 \\ & +\frac{2\beta_n}{1 + (1-\alpha)\alpha_n\beta_n}[\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\langle Sx^* - x^*, x_{n+1} - x^*\rangle] \end{aligned}$$

$$\leq [1-(1-\alpha)\alpha_n\beta_n]\|x_n-x^*\|^2 + \frac{2\alpha_n\beta_n}{1+(1-\alpha)\alpha_n\beta_n}[\langle f(x^*)-x^*, x_{n+1}-x^*\rangle + (1-\alpha_n)\frac{1}{\alpha_n}\langle Sx^*-x^*, x_{n+1}-x^*\rangle].$$

Thus, from (15) and (16), we have $\limsup_{n\to\infty}[(1-\alpha_n)\frac{1}{\alpha_n}\langle Sx^*-x^*, x_{n+1}-x^*\rangle + \langle f(x^*)-x^*, x_{n+1}-x^*\rangle] = 0$. Therefor, we can apply Lemma 3 to above last inequality to conclude that $x_n \to x^*\ as\ n\to\infty$.

In particular, if we take $f = 0$, variational inequality (8) is reduced to the inequality

$$x^* \in \Omega, \quad \langle x^*, x-x^*\rangle \geq 0, \quad x\in\Omega.$$

This is equivalent to $\|x^*\| \leq \|x\|$ for all $x\in\Omega$. It implies that x^* is the minimum norm element of Ω; i.e., the minimum norm solution of hierarchical fixed point problem (2). This completes the proof. □

Remark 8 *We can choose the following parameters satisfying conditions (C1) and (C2), for instance,*

$$\alpha_n = \frac{1}{(n+1)^{\frac{1}{5}}}, \quad \beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}, \quad \omega_n = \frac{1}{2^{n+1}}.$$

Remark 9 *(1) The assumption (C3) was used in [30] by Senter and Dotson so as to obtain a strong convergence result for Mann iterates. Later Maiti and Ghosh [31], Xu and Tan [32] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the condition introduced in [20] and point out that this assumption is weaker than the requirement that the mapping is demi-compact.*

(2) We would like to note that thanking to a result generated by Lemaire in [33], (C3) is in convex minimization setting equivalent to

$$\forall x\in H, \quad \varphi(x) - \min\varphi \geq \gamma Dist(x, argmin\varphi)^{\frac{1}{2}}$$

which is exactly one of the assumptions used in [3] to obtain convergence results of a proximal method for hierachical minimization problems. In [3], the convergence results are valid in the finite dimensional case.

Next, we introduce another explicit scheme in which W-mapping in (3) is replaced by L_n defined in (4) for finding the minimum norm solution of hierarchical fixed point problem (2).

Theorem 10 *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $f : C\to H$ be a α-contraction with $\alpha\in[0,1)$, $S : C\to C$ a nonexpansive mapping, and $\{T_k\}_{k=1}^\infty : C\to C$ an infinite family of nonexpansive mapping with $\bigcap_{k=1}^\infty Fix(T_k)\neq\emptyset$. Let $L_n = \sum_{k=1}^n \frac{\omega_k}{s_n}T_k$, $S_n = \sum_{k=1}^n \omega_k$, and $w_k > 0$ with $\sum_{k=1}^\infty \omega_k = 1$. Give $x_0\in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n Sx_n + (1-\alpha_n)L_nP_C[\beta_n f(x_n) + (1-\beta_n)x_n], n\geq 0. \tag{17}$$

If $f = 0$, then (17) is reduced to the iterative scheme:

$$x_{n+1} = \alpha_n Sx_n + (1-\alpha_n)L_nP_C[(1-\beta_n)x_n], n\geq 0. \tag{18}$$

Suppose the following conditions are satisfied

(A1) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\frac{\beta_n}{\alpha_n} = \lim_{n\to\infty}\frac{\alpha_n^2}{\beta_n} = \lim_{n\to\infty}\frac{\beta_n-\beta_{n-1}}{\alpha_n\beta_n} = \lim_{n\to\infty}\frac{1}{\beta_n}(\frac{1}{\alpha_n}-\frac{1}{\alpha_{n-1}}) = \lim_{n\to\infty}\frac{\omega_n}{\alpha_n\beta_n} = 0$;

(A2) $\sum_{n=0}^\infty \beta_n = \infty$;

(A3) There exists some constant $\gamma > 0$ such that $\|x - L_nx\| \geq \gamma Dist(x, \bigcap_{k=1}^\infty Fix(T_k))$, where $Dist(x, \bigcap_{n=1}^\infty Fix(T_k)) = \inf_{y\in\bigcap_{k=1}^\infty Fix(T_k)}\|x-y\|$.

Then the sequence $\{x_n\}$ generated by (17) converges strongly to $x^*\in\bigcap_{k=1}^\infty Fix(T_k)$ which is the unique solution of the variational inequality:

$$x^*\in\Omega, \quad \langle (I-f)x^*, x-x^*\rangle \geq 0, \quad \forall x\in\Omega. \tag{19}$$

In particular, if we take $f = 0$, then the sequence $\{x_n\}$ generated by (18) converges strongly to $x^*\in\bigcap_{k=1}^\infty Fix(T_k)$ which is the minimum norm solution of hierarchical fixed point problem (2).

Proof: We will use six steps to complete the proof.

Step 1. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|\to 0$.

Setting $y_n = \beta_n f(x_n) + (1-\beta_n)x_n$ for all $n\geq 0$, that is

$$y_n - y_{n-1} = \beta_n(f(x_n)-f(x_{n-1})) + (\beta_n-\beta_{n-1})f(x_{n-1}) + (1-\beta_n)(x_n-x_{n-1}) + (\beta_{n-1}-\beta_n)x_{n-1}.$$

It follows that

$$\|y_n-y_{n-1}\| \leq [1-(1-\alpha)\beta_n]\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|(\|f(x_{n-1}\| + \|x_{n-1}\|).$$

From (17), we have

$$\begin{aligned} & x_{n+1}-x_n \\ = \;& \alpha_n Sx_n+(1-\alpha_n)L_nP_Cy_n-\alpha_{n-1}Sx_{n-1} \\ & -(1-\alpha_{n-1})L_{n-1}P_Cy_{n-1} \\ = \;& \alpha_n(Sx_n-Sx_{n-1})+(1-\alpha_n)(L_nP_Cy_n \\ & -L_nP_Cy_{n-1})+(1-\alpha_n)(L_nP_Cy_{n-1} \\ & -L_{n-1}P_Cy_{n-1})+(\alpha_n-\alpha_{n-1})Sx_{n-1} \\ & +(\alpha_{n-1}-\alpha_n)L_{n-1}P_Cy_{n-1}. \end{aligned}$$

Then, we obtain

$$\begin{aligned} & \|x_{n+1}-x_n\| \\ \le \;& \alpha_n\|x_n-x_{n-1}\|+(1-\alpha_n)\|y_n-y_{n-1}\| \\ & +(1-\alpha_n)\|L_nP_Cy_{n-1}-L_{n-1}P_Cy_{n-1}\| \\ & +|\alpha_{n-1}-\alpha_n|(\|Sx_{n-1}\|+\|L_{n-1}P_Cy_{n-1}\|). \end{aligned}$$

From theorem 7's proof, we can get

$$\|L_nP_Cy_{n-1}-L_{n-1}P_Cy_{n-1}\|\le M\omega_n,$$

where M is some constant such that

$$\begin{aligned} M \;\ge\; & \sup_{1\le k\le n}\{(\|f(x_n)\|+\|Sx_n\|),\frac{2\|T_kx_{n-1}\|}{\omega_1}, \\ & (\|Sx_{n-1}\|+\|L_nx_{n-1}\|),\|x_n-x_{n-1}\|\}. \end{aligned}$$

Hence, we have

$$\begin{aligned} & \|x_{n+1}-x_n\| \\ \le \;& [1-(1-\alpha)\beta_n(1-\alpha_n)]\|x_n-x_{n-1}\| \\ & +|\beta_n-\beta n-1|(\|f(x_{n-1})+\|x_{n-1}\|) \\ & +|\alpha_n-\alpha_{n-1}|(\|Sx_{n-1}\|+\|L_{n-1}P_Cy_{n-1}\|) \\ & +M\omega_n. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{\|x_{n+1}-x_n\|}{\alpha_n} \\ \le \;& [1-(1-\alpha)\beta_n(1-\alpha_n)]\frac{\|x_n-x_{n-1}\|}{\alpha_n} \\ & +\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}(\|f(x_{n-1})+\|x_{n-1}\|) \\ & +\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}(\|Sx_{n-1}\|+\|L_{n-1}P_Cy_{n-1}\|) \\ & +\frac{M\omega_n}{\alpha_n} \\ \le \;& [1-(1-\alpha)\beta_n(1-\alpha_n)]\frac{\|x_n-x_{n-1}\|}{\alpha_{n-1}} \\ & +(1-\alpha)\beta_n(1-\alpha_n)\frac{M}{(1-\alpha)(1-\alpha_n)} \\ & \times\Bigg(\frac{1}{\beta_n}|\frac{1}{\alpha_n}-\frac{1}{\alpha_{n-1}}|+\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n\beta_n} \\ & +\frac{\beta_n-\beta_{n-1}}{\alpha_n\beta_n}+\frac{\omega_n}{\alpha_n\beta_n}\Bigg). \end{aligned}$$

Thus, from (A1), we have $\limsup_{n\to\infty}(\frac{1}{\beta_n}|\frac{1}{\alpha_n}-\frac{1}{\alpha_{n-1}}|+\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n\beta_n}+\frac{\beta_n-\beta_{n-1}}{\alpha_n\beta_n}+\frac{\omega_n}{\alpha_n\beta_n})=0$. Hence, applying Lemma 3 to above last inequality, we conclude immediately that

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\alpha_n}=0.$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{20}$$

Step 2. We prove that $\omega_w(x_n)\subset Fix(L)=\bigcap_{k=1}^{\infty}Fix(T_k)$.

From (17) and (20), we have

$$\lim_{n\to\infty}\|x_n-L_nP_Cy_n\|=0. \tag{21}$$

By (17), we get

$$\begin{aligned} \|P_Cy_n-x_n\| &\le \|y_n-x_n\| \\ &= \|\beta_n(f(x_n)-x_n)\| \\ &\to 0. \end{aligned} \tag{22}$$

Notice that

$$\begin{aligned} & \|x_n-Lx_n\| \\ \le \;& \|x_n-L_nP_Cy_n\|+\|L_nP_Cy_n-L_nx_n\| \\ & +L_nx_n-Lx_n\| \\ \le \;& \|x_n-L_nP_Cy_n\|+\|P_Cy_n-x_n\| \\ & +L_nx_n-Lx_n\|. \end{aligned} \tag{23}$$

By (21)-(23) and Lemma 5, we get

$$\lim_{n\to\infty}\|x_n-Lx_n\|=0. \tag{24}$$

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $\tilde{x}\in H$. Therefore, we have $\tilde{x}\in Fix(L)=\bigcap_{k=1}^{\infty}Fix(T_k)$ by (24) and Lemma 1. Hence, $\omega_w(x_n)\subset Fix(L)=\bigcap_{k=1}^{\infty}Fix(T_k)$.

Step 3. We claim that $\omega_w(x_n)\subset\Omega$.

Rewriting (17) as

$$\begin{aligned} x_n-x_{n+1} \;=\; & \alpha_n(x_n-Sx_n)+(1-\alpha_n)(P_Cy_n \\ & -L_nP_Cy_n)+(1-\alpha_n)(y_n-P_Cy_n) \\ & +(1-\alpha_n)(x_n-y_n), \end{aligned}$$

that is

$$\begin{aligned} \frac{x_n-x_{n+1}}{\alpha_n} \;=\; & (I-S)x_n+\frac{1-\alpha_n}{\alpha_n}(P_Cy_n \\ & -L_nP_Cy_n)+\frac{1-\alpha_n}{\alpha_n}(I-P_C)y_n \\ & +\frac{\beta_n(1-\alpha_n)}{\alpha_n}(I-f)x_n. \end{aligned}$$

Set $z_n = \frac{x_n - x_{n-1}}{\alpha_n}$ and pick up $u \in \bigcap_{k=1}^{\infty}$. Then, we have

$$\begin{aligned} & \langle z_n, x_n - u\rangle \\ = & \langle (I-S)x_n, x_n - u\rangle \\ & + \frac{1-\alpha_n}{\alpha_n}\langle P_C y_n - L_n P_C y_n, x_n - u\rangle \\ & + \frac{1-\alpha_n}{\alpha_n}\langle (I-P_C)y_n, x_n - u\rangle \\ & + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-f)x_n, x_n - u\rangle \\ = & \langle (I-S)x_n - (I-S)u, x_n - u\rangle \\ & + \langle (I-S)u, x_n - u\rangle + \frac{1-\alpha_n}{\alpha_n}\langle (I-L_n)P_C y_n \\ & -(I-L_n)u, P_C y_n - u\rangle \\ & + \frac{1-\alpha_n}{\alpha_n}\langle (I-L_n)P_C y_n, x_n - P_C y_n\rangle \\ & + \frac{1-\alpha_n}{\alpha_n}\langle (I-P_C)y_n, x_n - P_C y_n\rangle \\ & + \frac{1-\alpha_n}{\alpha_n}\langle (I-P_C)y_n, P_C y_n - u\rangle \\ & + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-f)x_n, x_n - u\rangle. \end{aligned}$$

Using the property of the projection(Lemma 2.2), we have

$$\langle (I-P_C)y_n, P_C y_n - u\rangle \geq 0.$$

Using monotonicity of $I - W_n$ and $I - S$, we derive that

$$\langle (I-S)x_n - (I-S)u, x_n - u\rangle \geq 0 \quad and$$

$$\langle (I-L_n)P_C y_n - (I-L_n)u, P_C y_n - u\rangle \geq 0.$$

At the same time, we observe that

$$\begin{aligned} & y_n - L_n P_C y_n \\ = & \beta_n f(x_n) + (1-\beta_n)x_n - L_n P_C y_n \\ = & \beta_n[f(x_n) - x_{n+1}] + (1-\beta_n)(x_n - x_{n+1}) \\ & + x_{n+1} - L_n P_C y_n \\ = & \beta_n[f(x_n) - x_{n+1}] + (1-\beta_n)(x_n - x_{n+1}) \\ & + \alpha_n(Sx_n - L_n P_C y_n). \end{aligned}$$

Therefore,

$$\begin{aligned} & \langle z_n, x_n - u\rangle \\ \geq & \langle (I-S)u, x_n - u\rangle \\ & + \frac{1-\alpha_n}{\alpha_n}\langle (I-L_n)P_C y_n, x_n - P_C y_n\rangle \\ & + \frac{1-\alpha_n}{\alpha_n}\langle (I-P_C)y_n, x_n - P_C y_n\rangle \\ & + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-f)x_n, x_n - u\rangle \\ = & \langle (I-S)u, x_n - u\rangle \\ & + \frac{1-\alpha_n}{\alpha_n}\langle y_n - L_n P_C y_n, x_n - P_C y_n\rangle \\ & + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-f)x_n, x_n - u\rangle \\ = & \langle (I-S)u, x_n - u\rangle \\ & + \frac{(1-\alpha_n)\beta_n}{\alpha_n}\langle f(x_n) - x_{n+1}, x_n - P_C y_n\rangle \\ & + (1-\alpha_n)(1-\beta_n)\langle \frac{x_n - x_{n+1}}{\alpha_n}, x_n - P_C y_n\rangle \\ & + (1-\alpha_n)\langle Sx_n - L_n P_C y_n, x_n - P_C y_n\rangle \\ & + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-f)x_n, x_n - u\rangle. \end{aligned}$$

But, since $z_n \to 0$, $\frac{\beta_n}{\alpha_n} \to 0$, $\frac{x_n - x_{n+1}}{\alpha_n} \to 0$ and $(x_n - P_C y_n) \to 0$, we obtain from the above inequality that

$$\limsup_{n\to\infty}\langle (I-S)u, x_n - u\rangle \leq 0, \quad u \in \bigcap_{k=1}^{\infty} Fix(T_k).$$

Therefore,

$$\limsup_{j\to\infty}\langle (I-S)u, x_{n_j} - u\rangle \leq 0, \quad u \in \bigcap_{k=1}^{\infty} Fix(T_k).$$

Since $x_{n_j} \rightharpoonup \tilde{x}$, we have

$$\limsup_{j\to\infty}\langle (I-S)u, x_{n_j} - u\rangle = \langle (I-S)u, \tilde{x} - u\rangle.$$

This implies that every weak cluster point $\tilde{x} \in \bigcap_{k=1}^{\infty} Fix(T_k)$ of the sequence $\{x_n\}$ solves the variational inequality

$$\langle (I-S)u, \tilde{x} - u\rangle \leq 0, \quad \forall u \in \bigcap_{k=1}^{\infty} Fix(T_k).$$

This is equivalent to its dual variational inequality

$$\langle (I-S)\tilde{x}, \tilde{x} - u\rangle \leq 0, \quad \forall u \in \bigcap_{k=1}^{\infty} Fix(T_k).$$

Hence, we get $\omega_w(x_n) \subset \Omega$.

Step 4. We show that $\limsup_{n\to\infty}\langle (I-f)x^*, x_n - x^*\rangle \geq 0$.

Since f is a contraction, the solution set of the variational inequality (19) is a singleton. Let x^* is the unique solution of the variational inequality (19). Now we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying

$$\begin{aligned} & \limsup_{n\to\infty}\langle (I-f)x^*, x_n - x^*\rangle \\ = & \lim_{k\to\infty}\langle (I-f)x^*, x_{n_k} - x^*\rangle. \end{aligned}$$

Without loss of generality, we may further assume that $x_{n_k} \rightharpoonup \bar{x}$, then $\bar{x} \in \Omega$. Therefore, noticing that x^* is the solution of the variational inequality (19), we have

$$\begin{aligned} & \limsup_{n\to\infty}\langle (I-f)x^*, x_n - x^*\rangle \\ = & \langle (I-f)x^*, \bar{x} - x^*\rangle \\ \geq & 0. \end{aligned} \tag{25}$$

Step 5. We show that $\limsup_{n\to\infty} \frac{\alpha_n}{\beta_n}\langle Sx^* - x^*, x_{n+1} - x^*\rangle \leq 0$.

We note that

$$\begin{aligned} & \langle Sx^* - x^*, x_{n+1} - x^*\rangle \\ = & \langle Sx^* - x^*, x_{n+1} - P_{\bigcap_{k=1}^\infty Fix(T_k)} x_{n+1}\rangle \\ & + \langle Sx^* - x^*, P_{\bigcap_{k=1}^\infty Fix(T_k)} x_{n+1} - x^*\rangle. \end{aligned}$$

Since $P_{\bigcap_{k=1}^\infty Fix(T_k)} x_{n+1} \in \bigcap_{k=1}^\infty Fix(T_k)$, by (2) we have

$$\langle Sx^* - x^*, P_{\bigcap_{k=1}^\infty Fix(T_k)} x_{n+1} - x^*\rangle \leq 0,$$

and by assumption $(A3)$, we have

$$\begin{aligned} & \langle Sx^* - x^*, x_{n+1} - x^*\rangle \\ \leq & \langle Sx^* - x^*, x_{n+1} - P_{\bigcap_{k=1}^\infty Fix(T_k)} x_{n+1}\rangle \\ \leq & \|Sx^* - x^*\|\, \|x_{n+1} - P_{\bigcap_{k=1}^\infty Fix(T_k)} x_{n+1}\| \\ = & \|Sx^* - x^*\|\, Dist(x_{n+1}, \bigcap_{k=1}^{\infty} Fix(T_k)) \\ \leq & \frac{1}{\gamma}\|Sx^* - x^*\|\, \|x_{n+1} - L_n x_{n+1}\|. \end{aligned}$$

We note that

$$\begin{aligned} & \|x_{n+1} - L_n x_{n+1}\| \\ \leq & \|x_{n+1} - L_n P_C y_n\| + \|L_n P_C y_n - L_n x_n\| \\ & + \|L_n x_n - L_n x_{n+1}\| \\ \leq & \alpha_n\|Sx_n - L_n P_C y_n\| + \|y_n - x_n\| \\ & + \|x_{n+1} - x_n| \\ \leq & \alpha_n\|Sx_n - L_n P_C y_n\| + \beta_n\|f(x_n) - x_n\| \\ & + \|x_{n+1} - x_n\|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \frac{\alpha_n}{\beta_n}\|x_{n+1} - L_n x_{n+1}\| \\ \leq & \frac{\alpha_n^2}{\beta_n}\|Sx_n - L_n P_C y_n\| + \alpha_n\|f(x_n) - x_n\| \\ & + \frac{\alpha_n^2}{\beta_n}\frac{\|x_n - x_{n+1}\|}{\alpha_n} \\ \to & 0. \end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \frac{\alpha_n}{\beta_n}\langle Sx^* - x^*, x_{n+1} - x^*\rangle \leq 0. \tag{26}$$

Step 6. We prove that $x_n \to x^*\ as\ n \to \infty$. From (17), we have

$$\begin{aligned} x_{n+1} - x^* = & \alpha_n(Sx_n - Sx^*) + (1-\alpha_n)(L_n P_C y_n \\ & - x^*) + \alpha_n(Sx^* - x^*). \end{aligned}$$

Thus, we have

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ \leq & \|\alpha_n(Sx_n - Sx^*) + (1-\alpha_n)(L_n P_C y_n - x^*)\|^2 \\ & + 2\alpha_n\langle Sx_n - x^*, x_{n+1} - x^*\rangle \\ \leq & \alpha_n\|Sx_n - Sx^*\|^2 + (1-\alpha_n)\|L_n P_C y_n - x^*\|^2 \\ & + 2\alpha_n\langle Sx_n - x^*, x_{n+1} - x^*\rangle \\ \leq & \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|y_n - x^*\|^2 \\ & + 2\alpha_n\langle Sx_n - x^*, x_{n+1} - x^*\rangle. \end{aligned} \tag{27}$$

At the same time, we observe that

$$\begin{aligned} & \|y_n - x^*\|^2 \\ = & \|(1-\beta_n)(x_n - x^*) + \beta_n(f(x_n) - f(x^*)) \\ & + \beta_n(f(x^*) - x^*)\|^2 \\ \leq & \|(1-\beta_n)(x_n - x^*) + \beta_n(f(x_n) - f(x^*))\|^2 \\ & + 2\beta_n\langle f(x^*) - x^*, y_n - x^*\rangle \\ \leq & (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|f(x_n) - f(x^*)\|^2 \\ & + 2\beta_n\langle f(x^*) - x^*, y_n - x^*\rangle \\ \leq & (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\alpha^2\|x_n - x^*\|^2 \\ & + 2\beta_n\langle f(x^*) - x^*, y_n - x^*\rangle \\ = & [1 - (1-\alpha^2)\beta_n]\|x_n - x^*\|^2 \\ & + 2\beta_n\langle f(x^*) - x^*, y_n - x^*\rangle. \end{aligned} \tag{28}$$

Substituting (28) into (27), we have

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ \leq & \alpha_n\|x_n - x^*\|^2 \\ & + (1-\alpha_n)[1 - (1-\alpha^2)\beta_n]\|x_n - x^*\|^2 \\ & + 2\beta_n(1-\alpha_n)\langle f(x^*) - x^*, y_n - x^*\rangle \\ & + 2\alpha_n\langle Sx^* - x^*, x_{n+1} - x^*\rangle \\ = & [1 - (1-\alpha^2)\beta_n(1-\alpha_n)]\|x_n - x^*\|^2 \\ & + 2\beta_n(1-\alpha_n)\langle f(x^*) - x^*, y_n - x^*\rangle \\ & + 2\alpha_n\langle Sx^* - x^*, x_{n+1} - x^*\rangle \\ = & [1 - (1-\alpha^2)\beta_n(1-\alpha_n)]\|x_n - x^*\|^2 \\ & + (1-\alpha^2)\beta_n(1-\alpha_n) \\ & \times\Big(\tfrac{2}{1-\alpha^2}\langle f(x^*) - x^*, y_n - x^*\rangle \\ & + \tfrac{2}{(1-\alpha^2)(1-\alpha_n)}\tfrac{\alpha_n}{\beta_n}\langle Sx^* - x^*, x_{n+1} - x^*\rangle\Big). \end{aligned} \tag{29}$$

Therefore, we can apply Lemma 3 to (29) to conclude that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

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