Strong Convergence of a Hybrid Projection Algorithm for Approximation of a Common Element of Three Sets in Banach Spaces

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Abstract: In this paper, we construct a new iterative scheme by hybrid projection method and prove strong convergence theorems for approximation of a common element of set of common fixed points of an infinite family of asymptotically quasi-ϕ-nonexpansive mappings, set of solutions to a variational inequality problem and set of common solutions to a system of generalized mixed equilibrium problems in a uniformly smooth and 2-uniformly convex real Banach space. Our results extend many important recent results in the literature.

Key–Words: Asymptotically quasi-ϕ-nonexpansive mapping; Generalized mixed equilibrium problem; Uniformly smooth; 2-Uniformly convex; Hybrid projection method; Banach space

1 Introduction

Let C be a closed convex subsets of Banach space E . Let f be a bifunction from $C \times C$ to R , $\varphi : C \to R$ be mapping and $A : C \to E^*$ be a nonlinear mapping. The"so-called" generalized mixed equilibrium problem is to find $z \in C$ such that

$$f(z, y) + \langle Az, y - z\rangle + \varphi(y) - \varphi(z) \geq 0, \forall y \in C. \quad (1)$$

The set of solution of (1) is denoted by $GMEP(f, \varphi)$,i.e.

$$GMEP(f, \varphi) = \{z \in C \mid f(z, y) + \langle Az, y - z\rangle + \varphi(y) - \varphi(z) \geq 0, \forall y \in C\}.$$

Special cases:

(I) If $A = 0$,then the problem (1) is equivalent to find $z \in C$ such that

$$f(z, y) + \varphi(y) - \varphi(z) \geq 0, \forall y \in C. \quad (2)$$

This is called the mixed equilibrium problem. The set of solution of (2) is denoted by $MEP(f, \varphi)$.

(II) If $f = 0$,then the problem (1) is equivalent to find $z \in C$ such that

$$\langle Az, y - z\rangle + \varphi(y) - \varphi(z) \geq 0, \forall y \in C. \quad (3)$$

This is called the mixed variational inequality of Browder type. The set of solution of (3) is denoted by $VI(C, A, \varphi)$. In particular, $VI(C, A, 0)$ is denoted by $VI(C, A)$.

(III) If $\varphi = 0$, then the problem (1) is equivalent to find $z \in C$ such that

$$f(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C. \quad (4)$$

It is called the generalized equilibrium problem. The set of solution of (4) is denoted by $GEP(f)$.

(IV) If $A = 0, \varphi = 0$,then the problem (1) is equivalent to find $z \in C$ such that

$$f(z, y) \geq 0, \forall y \in C. \quad (5)$$

It is called the equilibrium problem. The set of solution of (5) is denoted by $EP(f)$.

An operator $B : C \to E^*$ is called α-inverse-strongly monotone, if there exists a positive real number α such that

$$\langle x - y, Bx - By\rangle \geq \alpha\|Bx - By\|^2, \forall x, y \in C.$$

Obviously, if B is α -inverse-strongly monotone, then B is $\frac{1}{\alpha}$ -continuous. In this paper, we shall assume that

(B1) B is α -inverse-strongly monotone;

(B2) $VI(C, B) \neq \emptyset$;

(B3) $\|By\| \leq \|By - Bu\|$ for all $y \in C$ and $u \in VI(C, B)$.

The generalized mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems and equilibrium problems as special cases (see, for example,[1]). Some methods have been proposed to solve the generalized mixed equilibrium problem(see, for example,[1-5]).Numerous problems in Physics, optimization and economics help to find a solution of problem (5).

Recently, Petrot et al.[6] introduced the following hybrid iterative scheme for approximation of a common fixed point of two relatively quasi-nonexpansive mappings, which is also a solution to generalized mixed equilibrium problem in a uniformly smooth and uniformly convex real Banach space:

$$\begin{cases} x_0 \in C \quad chosen\ arbitrariy, \\ y_n = J^{-1}(\delta_n Jx_n + (1-\delta_n)Jz_n), \\ z_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ f(u_n, y) + \langle Au_n, y-u_n\rangle + \varphi(y) - \varphi(u_n) + \\ \qquad \frac{1}{r_n}\langle y-u_n, Ju_n - Jx\rangle \geq 0, \forall y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases}$$

They proved strong convergence theorem to a common element of set of common fixed points of S and T and set of solutions to the generalized mixed equilibrium problem.

Furthermore, Cholamjiak [7]introduced a hybrid iterative scheme for approximation of a fixed point of relatively quasi-nonexpansive mapping which is also a solution to equilibrium problem and variational inequality problems in a 2-uniformly convex real Banach space, which is also uniformly smooth:

$$\begin{cases} x_0 \in C \quad chosen\ arbitrariy, \\ C_1 = C, x_1 = \Pi_{C_1} x_0, \\ \nu_n = \Pi_C J^{-1}(Jx_n - \delta_n Bx_n), \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JS\nu_n), \\ f(u_n, y) + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases}$$

Then, he proved that $\{x_n\}$ converges strongly to $\Pi_F x_0$,where $F := F(T) \cap F(S) \cap VI(C, B) \cap EP(F) \neq \emptyset$.

In [8], Martinez-Yanes and Xu introduced the following iterative scheme for a single non-expansive mapping T in a Hilbert space H :

$$\begin{cases} x_0 \in C, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|z - y_n\|^2 \leq \alpha_n(\|x_0\|^2 \\ \qquad +2\langle x_n - x_0, z\rangle) + \|z - x_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

where P_C denotes the metric projection of H onto a closed and convex subset C of H . They proved that if $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$, then the sequence $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

In [9], Qin and Su extended the results of Martinez-Yanes and Xu [8] from Hilbert spaces to Banach spaces and proved the following result: Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex Banach space E and let $T : C \to C$ be a relatively non-expansive mapping. Assume that $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{cases} x_0 \in C, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, n \geq 0. \end{cases}$$

If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.

In [10], Plubtieng and Ungchittrakool introduced the following hybrid projection algorithm for a pair of relatively nonexpansive mappings:

$$\begin{cases} x_0 \in C \quad chosen\ arbitrariy, \\ z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n \\ \qquad +\beta_n^{(3)} JSx_n), \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n) + \\ \qquad \alpha_n(\|x_0\|^2 + 2\langle Jx_n - Jx_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \end{cases} \tag{6}$$

where$\{\alpha_n\}$,$\{\beta_n^{(i)}\}$,i=1,2,3, are sequences in $(0,1)$ satisfying $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ and S and T are relatively nonexpansive mappings. They proved under the appropriate conditions on the parameters that the sequence $\{x_n\}$ generated by (6) converges strongly to a common fixed point of S and T .

2009, Qin et al. [11] introduced the following hybrid projection algorithm for two families of relatively quasi-nonexpansive mappings, which are more general than relatively nonexpansive mappings in a Banach space:

$$\begin{cases} x_0 \in C \quad chosen\ arbitrariy, \\ z_{n,j} = J^{-1}(\beta_{n,j}^{(1)} Jx_n + \beta_{n,j}^{(2)} JT_i x_n \\ \qquad + \beta_{n,j}^{(3)} JS_i x_n), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 + (1-\alpha_{n,i}) Jz_{n,i}), \\ C_{n,i} = \{z \in C : \phi(z, y_{n,i}) \le \phi(z, x_n) \\ \qquad + \alpha_{n,i}(\|x_0\|^2 + 2\langle Jx_n - Jx_0, z\rangle)\}, \\ C_n = \bigcap_{i=1}^{\infty} C_{n,i}, Q_0 = C, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, Jx_0 \\ \qquad - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{7}$$

They proved under appropriate conditions on the parameters that the sequence $\{x_n\}$ generated by (7) converges strongly to a common fixed point of the two families $\{S_i\}$ and $\{T_i\}$.

Recently, Wangkeeree and Wangkeeree[12] introduced the following hybrid projection algorithm for approximation of common fixed point of two families of relatively quasi-non- expansive mappings, which is also a solution to variational inequality problem in a Banach space:

$$\begin{cases} x_0 \in C \quad chosen\ arbitrariy, \\ C_{1,i} = C, C_1 = \bigcap_{i=1}^{\infty} C_{1,i}, \\ x_i = \Pi_{C_1} x_0, \\ w_{n,i} = \Pi_{C_1} J^{-1}(Jx_n - \lambda_{n,i} Bx_n), \\ z_{n,i} = J^{-1}(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n \\ \qquad + \beta_{n,i}^{(3)} JS_i w_{n,i}), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 + (1-\alpha_{n,i}) Jz_{n,i}), \\ C_{n,i} = \{z \in C : \phi(z, y_{n,i}) \le \phi(z, x_n) + \\ \qquad \alpha_{n,i}(\|x_0\|^2 + 2\langle Jx_n - Jx_0, z\rangle)\}, \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_n+1} x_0. \end{cases} \tag{8}$$

They proved under appropriate conditions on the parameters that the sequence $\{x_n\}$ generated by (8) converges strongly to a common element of the set of common fixed points of the two families $\{S_i\}$ and $\{T_i\}$ and set of solutions to a variational inequality problem.

In 2009, Takahashi and Zembayashi [13] proved strong and weak convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space by using the shrinking projection method.

Motivated by the above mentioned results and the on-going research, we introduce a new hybrid projection algorithm based on the shrinking projection method and prove strong convergence theorem for approximation of a common element of the set of common fixed point of an infinite family of asymptotically quasi-ϕ-nonexpansive mappings, set of solutions to a variational inequality problem and the set of solutions to system of generalized mixed equilibrium problems in a 2-uniformly convex real Banach space which is also uniformly smooth. Our results extend the results of Martinez-Yanes and Xu[8], Plubtieng and Ungchittrakool [10], Takahashi and Zembayashi [13] and many other recent and important results in the literature.

2 Preliminaries

Throughout this paper, we denote by N and R the sets of nonnegative integers and real numbers, respectively. Let E be a Banach space and let E^* be the topological dual of .For all $x \in E$ and $x^* \in E^*$, we denote the value of x^* at x by $\langle x, x^*\rangle$ .The duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}.$$

It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E .Now, let E be a smooth Banach space, we use $\phi : E \times E \to R$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \forall x, y \in E.$$

It is obvious from the definition of ϕ that

$$(A_1)(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2.$$

Following Alber [14], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C x = \arg\min_{y \in C} \phi(y, x), \forall x \in C.$$

If E is a Hilbert space H , then $\phi(y, x) = \|y - x\|^2, x, y \in H$ and Π_C is the metric projection P_C of E onto C .

Let C be a nonempty closed convex subset of E and T be a mapping from C into itself. We denoted $F(T)$ by the set of fixed points of T . A point $p \in C$ is said to be an asymptotic fixed point of T[15] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\overrightarrow{F(T)}$. A mapping T from C into itself is said to be relatively nonexpansive [15,16] if $\overrightarrow{F(T)} = F(T) \neq \emptyset$, and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in$

$F(T)$. T is said to be quasi-ϕ-nonexpansive [16-18]if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The mapping T is said to be asymptotically-ϕ-nonexpansive if there exists a sequence $\{k_n\} \subset [1, +\infty]$ with $\lim_{n\to+\infty} k_n = 1$ such that $\phi(T^n x, T^n y) \leq k_n \phi(x, y)$ for all $x, y \in C$. T is said to be asymptotically quasi-ϕ-nonexpansive [17,18] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, +\infty]$ with $\lim_{n\to+\infty} k_n = 1$ such that $\phi(p, T^n x) \leq k_n \phi(p, x)$ for all $x \in C, p \in F(T)$ and $n \geq 1$.

The class of (asymptotically) quasi-ϕ-nonexpansive mappings is more general than that of relatively nonexpansive mappings which requires the restriction: $\overrightarrow{F(T)} = F(T)$. A quasi-ϕ-nonexpansive mapping with a nonempty fixed point set $F(T)$ is an asymptotically quasi-ϕ-nonexpansive mapping, but the converse may not be true.In the framework of Hilbert spaces,(asymptotically) quasi-ϕ-nonexpansive mappings is reduced to (asymptotically) quasi-nonexpansive mappings.

It is well-known that the following conclusions hold:

Lemma 1 *[16] Let E be uniformly convex and smooth Banach space. Let $\{y_n\}$ and $\{z_n\}$ be sequences in E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If $\lim_{n\to+\infty} \phi(y_n, z_n) = 0$,then $\lim_{n\to+\infty} \|y_n - z_n\| = 0$.*

Lemma 2 *[14] Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E , $x \in E$ and $x_0 \in C$ .Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geq 0$, $\forall y \in C$.*

Lemma 3 *[14] Let C be a nonempty closed convex subset of reflexive, strictly convex and smooth Banach space E and $x \in E$, Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \forall y \in C.$$

Lemma 4 *[18] Let E be a nonempty closed convex subset of uniformly convex and smooth Banach space E .Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-non-expansive mapping. Then $F(T)$ is a closed convex subset of C .*

Lemma 5 *[18] Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0) = \{x \in E : \|x\| \leq r\}$. Then for any given infinite subset $\{x_n\} \subset B_r(0)$ and for any given sequence $\{\lambda_n\}$ of positive numbers with $\sum_{n=1}^{+\infty} \lambda_n = 1$,there exists a continuous, strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any $i, j \in N$ with $i < j$.*

$$\|\sum_{n=1}^{+\infty} \lambda_n x_n\|^2 \leq \sum_{n=1}^{+\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|).$$

For solving the equilibrium problem for bifunction $f : C \times C \to R$,let us assume that f satisfies the following conditions:

(C1) $f(x, x) = 0, \forall x \in C$

(C2) f is monotone,i.e. $f(x, y) + f(y, x) \leq 0, \forall x, y \in C$

(C3) $\forall x, y, z \in C, \limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$

(C4) $\forall x \in C, y \mapsto f(x, y)$ is a convex and lower semicontinuous.

If a bifunction $f : C \times C \to R$ satisfies conditions(C1)-(C4), then we have the following two important results.

Lemma 6 *[18] Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach spaces E , let $f : C \times C \to R$ be a bifunction satisfying conditions (C1)-(C4), $\varphi : C \to R$ be a lower semicontinuous and convex functional, $A : C \to E^*$ be a continuous and monotone mapping. For $r > 0$ and $x \in E$,define a mapping $T_r^G : E \to C$ as follows:*

$$T_r^G x = \{z \in E : f(x, y) + \varphi(y) - \varphi(x) + \langle Az, y - z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle, \forall y \in C\}.$$

Where $G(x, y) = f(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle, \forall x, y \in C$. Then, the following holds:

(1) T_r^G is single-valued;

(2) $F(T_r^G) = GMEP(f, \varphi)$;

(3) T_r^G is quasi-ϕ-nonexpansive;

(4) $GMEP(f, \varphi)$ is closed and convex.

Lemma 7 *[14] Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach spaces E , let $f : C \times C \to R$ be a bifunction satisfying conditions(C1)-(C4), and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r^f)$,*

$$\phi(q, T_r^f x) + \phi(T_r^f x, x) \leq \phi(q, x).$$

The function V as studied by Alber [14]: $V(x, x^*) = \|x^2\| - 2\langle x, x^*\rangle + \|x^*\|^2$ for all $x \in E$ and $x^* \in E^*$ .Thus, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 8 *[14] Let E be a reflexive strictly convex Banach space. Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 9 *[19] Let E be a 2-uniformly convex Banach space, then there exists a constant $c > 0$ such that for all $x, y \in E$ and $jx \in Jx, jy \in Jy$,we have*

$$\langle x - y, jx - jy\rangle \geq c\|x - y\|^2.$$

where $\frac{1}{c}$ is the 2-uniformly convexity constant.

3 Main results

Theorem 10 *Let C be a nonempty closed convex subset of 2-uniformly convex and uniformly smooth Banach space E .Suppose $B : C \to E^*$ is an operator satisfying(B1)-(B3).For each $k = 1, 2, \cdots, m$,let $A_k : C \to E^*$ be a continuous and monotone mapping, $\varphi_k : C \to R$ be a lower semi-continuous and convex functional, let $f_k : C \times C \to R$ be a bifunction satisfying(C1)-(C4)and $T_i : C \to C, \forall i \in N$ be an infinite family of closed and asymptotically quasi-ϕ-nonexpansive mapping with sequence $\{k_n^{(i)}\} \subseteq [1, +\infty), \lim_{n\to+\infty} k_n^{(i)} = 1$,where $T_0 = I$ .Assume that $T_i, \forall i \in N$ is asymptotically regular on C , i.e., $\lim_{n\to+\infty} \|T_i^{n+1}x_n - T_i^n x_n\| = 0$ and$F = [\bigcap_{i=0}^{+\infty} F(T_i)] \bigcap [\bigcap_{k=1}^{m} GMEP(f_k, \varphi_k)] \bigcap VI(C, B) \neq \emptyset$. Let x_n be a sequence generated by*

$$\begin{cases} x_0 \in C \quad chosen\ arbitrariy, \\ y_n = J^{-1}(\alpha_n J\Pi_C J^{-1}(Jx_n - \lambda_n Bx_n) + \\ \qquad (1 - \alpha_n)Jz_n), \\ z_n = J^{-1}(\sum_{i=0}^{+\infty} \beta_n^{(i)} JT_i^n x_n), \\ u_n = T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} \cdots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} y_n, \\ C_n = \{z \in C : \phi(z, u_n) \leq (1 - \alpha_n)\phi(z, z_n) \\ \quad +\alpha_n \phi(z, x_n) \leq \phi(z, x_n) \\ \quad +(k_n - 1)M_n\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \end{cases} \tag{9}$$

where $M_n = \sup\{\phi(z, x_n) | z \in F\} < +\infty$ for each $n \geq 0, k_n = \sup_{i\geq 0}\{k_n^{(i)}\}, \{\lambda_n\} \subset [a, b]$,for some a, b with $0 < a < b < c\alpha$, where $\frac{1}{c}$ is 2-uniformly convexity constant of E , for each $k = 1, 2, \cdots, m, \{r_{k,n}\}_{n=1}^{+\infty} \subset (0, +\infty)$ satisfying $\liminf_{n\to+\infty} r_{k,n} > 0$, for all $z, y \in C$, $G_k(z, y) = f_k(z, y) + \varphi_k(y) - \varphi_k(z) + \langle A_k z, y - z\rangle$, $T_{r_{k,n}}^{G_k}(x) = \{z \in C : G_k(z, y) + \frac{1}{r_{k,n}}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}$, $\{\alpha_n\}, \{\beta_n^{(i)}\}, i \in N$ are real sequences in $[0, 1]$ satisfies the conditions: $\forall n \geq 1, 0 \leq \beta_n^{(i)} \leq 1, \sum_{i=0}^{\infty} \beta_n^{(i)} = 1, \liminf_{n\to\infty}(1 - \alpha_n)\beta_n^{(0)}\beta_n^{(i)} > 0, \forall i \in N$ .Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof: We define a bifunction $G_k : C \times C \to R$ by

$$G_k(x, y) = f_k(x, y) + \varphi_k(y) - \varphi_k(x) + \langle A_k x, y - x\rangle,$$

$\forall x, y \in C$ .Then, we prove from Lemma 6 that the bifunction G_k satisfies conditions(C1)-(C4)for each $k = 1, 2, \cdots, m$. Therefore, the generalized mixed equilibrium problem (1) is equivalent to the following equilibrium problem: find $x \in C$ such that

$$G_k(x, y) \geq 0, \forall y \in C.$$

Hence $GMEP(f_k, \varphi_k) = EP(G_k)$, By taking $\theta_n^k = T_{r_{k,n}}^{G_k} T_{r_{k-1,n}}^{G_{k-1}} \cdots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1}, k = 1, 2, \cdots, m$ and $\theta_n^0 = I$ for all $n \geq 1$,we obtain $u_n = \theta_n^m y_n$ .Let $t_n = J^{-1}(Jx_n - \lambda_n Bx_n)$ .We divide the proof of Theorem 1 into five steps:

Step 1 We first show that C_n and Q_n are closed and convex for each $n \geq 0$.

In fact,for $z \in C_m$,we see that

$$\begin{aligned} &\phi(z, u_m) \leq \alpha_n \phi(z, x_m) + (1 - \alpha_m)\phi(z, z_m) \\ \leq\ &\phi(z, x_m) + (k_m - 1)M_m \end{aligned}$$

is equivalent to

$$\begin{aligned} &2\langle z, \alpha_m Jx_m + (1 - \alpha_m)Jz_m - Ju_m\rangle \\ \leq\ &\alpha_m\|x_m\|^2 + (1 - \alpha_m)\|z_m\|^2 - \|u_m\|^2 \end{aligned}$$

and

$$\begin{aligned} &2(1 - \alpha_m)\langle z, Jx_m - Jz_m\rangle \\ \leq\ &(1 - \alpha_m)(\|x_m\|^2 - \|z_m\|^2) + (k_n - 1)M_n. \end{aligned}$$

The last two inequalities are the affine with respect to z ,so C_n is closed and convex. From the definition of Q_n ,we may obtain that Q_n is closed and convex for each $n \geq 0$.

Step 2 Next, we show that $F \subset C_n \bigcap Q_n$ for each $n \geq 0$.

First we show that $F \subset C_n$ for each $n \geq 0$.

In fact, by the definition of $\phi(\cdot, \cdot)$ and (9), for each $p \subset F$, we obtain

$$\begin{aligned} &\phi(p, z_n) = \phi(p, J^{-1}(\sum_{i=0}^{+\infty} \beta_n^{(i)} JT_i^n x_n)) \\ =\ &\|p\|^2 - 2\langle p, \sum_{i=0}^{+\infty} \beta_n^{(i)} JT_i^n x_n\rangle \end{aligned}$$

$$
\begin{aligned}
&+ \quad \|J^{-1}(\sum_{i=0}^{+\infty}\beta_n^{(i)}JT_i^n x_n)\|^2 \\
&\leq \quad \|p\|^2 - 2\sum_{i=0}^{+\infty}\beta_n^{(i)}\langle p, JT_i^n x_n\rangle + \sum_{i=0}^{+\infty}\beta_n^{(i)}\|T_i^n x_n\|^2 \\
&= \quad \sum_{i=0}^{+\infty}\beta_n^{(i)}\phi(p, T_i^n x_n) \\
&\leq \quad \sum_{i=0}^{+\infty}\beta_n^{(i)}k_n^{(i)}\phi(p, x_n) \\
&= \quad \sum_{i=0}^{+\infty}\beta_n^{(i)}[1 + (k_n^{(i)} - 1)]\phi(p, x_n) \\
&= \quad \phi(p, x_n) + \sum_{i=0}^{+\infty}\beta_n^{(i)}(k_n^{(i)} - 1)\phi(p, x_n) \\
&\leq \quad \phi(p, x_n) + (k_n - 1)M_n. \qquad (10)
\end{aligned}
$$

Observe that $p \subset F$ implies $p \subset C$, by Lemma 3, Lemma 8 and (9), for all $p \subset C$, we have

$$
\begin{aligned}
&\phi(x_n, \Pi_C t_n) \leq \phi(x_n, t_n) - \phi(\Pi_C t_n, t_n) \\
\leq \quad & \phi(p, t_n) = V(p, Jx_n - \lambda_n Bx_n) \\
\leq \quad & V(p, (Jx_n - \lambda_n Bx_n) + \lambda_n Bx_n) \\
& -2\langle J^{-1}(Jx_n - \lambda_n Bx_n) - p, \lambda_n Bx_n\rangle \\
= \quad & V(p, Jx_n) - 2\lambda_n\langle t_n - p, Bx_n\rangle \\
= \quad & \phi(p, x_n) - 2\lambda_n\langle x_n - p, Bx_n\rangle \\
+ \quad & 2\langle t_n - x_n, -\lambda_n Bx_n\rangle. \qquad (11)
\end{aligned}
$$

From condition (B1) and $p \in VI(C, B)$, we obtain

$$
\begin{aligned}
& -2\lambda_n\langle x_n - p, Bx_n\rangle \\
= \quad & -2\lambda_n\langle x_n - p, Bx_n - Bp\rangle - 2\lambda_n\langle x_n - p, Bp\rangle \\
\leq \quad & -2\lambda_n\alpha\|Bx_n - Bp\|^2. \qquad (12)
\end{aligned}
$$

By Lemma 9 and condition (B1), we also obtain

$$
\begin{aligned}
& 2\langle t_n - x_n, -\lambda_n Bx_n\rangle \leq 2\|t_n - x_n\| \cdot \lambda_n\|Bx_n\| \\
\leq \quad & \frac{2}{c}\|Jt_n - Jx_n\| \cdot \lambda_n\|Bx_n\| \\
= \quad & \frac{2}{c}\lambda_n^2 \cdot \|Bx_n\|^2 \leq \frac{2}{c}\lambda_n^2 \cdot \|Bx_n - Bp\|^2. \qquad (13)
\end{aligned}
$$

Combining (11)-(13) and $0 < b < c\alpha$,we obtain

$$
\begin{aligned}
& \phi(p, \Pi_C t_n) \leq \phi(p, t_n) \\
\leq \quad & \phi(p, x_n) + 2\lambda_n(\frac{b}{c} - \alpha) \cdot \|Bx_n - Bp\|^2 \\
\leq \quad & \phi(p, x_n). \qquad (14)
\end{aligned}
$$

Thus, by (9), (10), (14), Lemma7, Lemma6 and the fact that $T_{r_{k,n}}^{G_k}(k = 1, 2, \cdots, m)$ is quasi-ϕ-nonexpansive mapping, for each $p \subset F$, we obtain

$$
\begin{aligned}
& \phi(p, u_n) = \phi(p, \theta_n^m y_n) \\
\leq \quad & \phi(p, y_n) \\
= \quad & \phi(p, J^{-1}(\alpha_n J\Pi_C t_n + (1 - \alpha_n)Jz_n)) \\
= \quad & \|p\|^2 - 2\langle p, \alpha_n J\Pi_C t_n + (1 - \alpha_n)Jz_n\rangle \\
+ \quad & \|J^{-1}(\alpha_n J\Pi_C t_n + (1 - \alpha_n)Jz_n)\|^2 \\
= \quad & \|p\|^2 - 2\alpha_n\langle p, J\Pi_C t_n\rangle - 2(1 - \alpha_n)\langle p, Jz_n\rangle \\
+ \quad & \|\alpha_n J\Pi_C t_n + (1 - \alpha_n)Jz_n\|^2 \\
\leq \quad & \|p\|^2 - 2\alpha_n\langle p, J\Pi_C t_n\rangle - 2(1 - \alpha_n)\langle p, Jz_n\rangle \\
+ \quad & \alpha_n\|\Pi_C t_n\|^2 + (1 - \alpha_n)\|z_n\|^2 \\
= \quad & \alpha_n(\|p\|^2 - 2\langle p, J\Pi_C t_n\rangle + \|\Pi_C t_n\|^2) \\
+ \quad & (1 - \alpha_n)(\|p\|^2 - 2\langle p, Jz_n\rangle + \|z_n\|^2) \\
= \quad & \alpha_n\phi(p, \Pi_C t_n) + (1 - \alpha_n)\phi(p, z_n) \\
\leq \quad & \alpha_n\phi(p, x_n) + (1 - \alpha_n)\phi(p, z_n) \\
\leq \quad & \alpha_n\phi(p, x_n) + (1 - \alpha_n)[\phi(p, x_n) \\
+ \quad & (k_n - 1)M_n] \\
\leq \quad & \phi(p, x_n) + (k_n - 1)M_n. \qquad (15)
\end{aligned}
$$

So, $p \subset C_n$ .This implies that $F \subset C_n, \forall n \geq 0$.

Second we show that $F \subset Q_n$ for each $n \geq 0$. In fact, for $n = 0, F \subset C = Q_0$ is obvious. Suppose that $F \subset Q_n$ for some positive integer n , it follows from $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ and Lemma2 that

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1}\rangle \geq 0, \forall z \in C_n \cap Q_n.$$

From $F \subset Q_n$, we obtain $F \subset C_n \cap Q_n$ .In particular, for all $z \subset F$, the last inequality should be held. Combining the definition of Q_{n+1} , we obtain that $F \subset Q_{n+1}$ .So we have that $F \subset C_n \cap Q_n, \forall n \geq 0$.

Step 3 Now,we show that $\{x_n\}$ is Cauchy sequence.

In fact, by the construction of Q_n and Lemma 2, we have that $x_n = \Pi_{Q_n} x_0$, it then follows from Lemma 3 that

$$
\begin{aligned}
& \phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \\
\leq \quad & \phi(p, x_0) - \phi(p, x_n) \\
\leq \quad & \phi(p, x_0).
\end{aligned}
$$

for each $p \in F \subset Q_n, \forall n \geq 0$ .Hence, the sequence $\phi(x_n, x_0)$ is bounded.

Combining $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in Q_n$ and Lemma 3, we obtain

$$0 \leq \phi(x_n, x_{n+1}) \leq \phi(x_n, x_0) - \phi(x_{n-1}, x_0).$$

for all $n \geq 0$ .Thus, the sequence $\phi(x_n, x_0)$ is nondecreasing. It follows from the boundedness of $\phi(x_n, x_0)$ that the limit of $\phi(x_n, x_0)$ exists.

For any positive integer m , it then follows from Lemma 3 that

$$
\begin{aligned}
& \phi(x_{n+m}, x_n) = \phi(x_{n+m}, \Pi_{Q_n} x_0) \\
\leq \quad & \phi(x_{n+m}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) \\
= \quad & \phi(x_{n+m}, x_0) - \phi(x_n, x_0). \qquad (16)
\end{aligned}
$$

it follows from (16) that $\phi(x_{n+m}, x_0) \to 0$ as $m, n \to \infty$. we have from (A1) and the boundedness of $\phi(x_n, x_0)$ that $\{x_n\}$ is bounded, combining Lemma 1, we obtain

$$x_{n+m} - x_n \to 0, m, n \to \infty.$$

Hence, the sequence $\{x_n\}$ is Cauchy in C . Since E is a Banach space and C is closed convex, then there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$. Now, since $\phi(x_{n+m}, x_0) \to 0$ as $m, n \to \infty$,we have in particular that $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$ and this further implies that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Since $x_{n+1} = \Pi_{C_n\cap Q_n} x_0 \in C_n$, we have

$$\begin{aligned} 0 &\le \phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n) \\ &+ (k_n - 1)M_n \to 0, n \to \infty. \end{aligned}$$

From Lemma 1, we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.$$

Therefore

$$\begin{aligned} &\|x_n - u_n\| \le \|x_n - x_{n+1}\| \\ &+ \|x_{n+1} - u_n\| \to 0. \end{aligned} \quad (17)$$

It follows from $\lim_{n\to+\infty} \|x_n - p\| = 0$ and (17) that

$$u_n \to p, n \to \infty. \quad (18)$$

Step 4 Now we prove that

$$p \in [\bigcap_{i=0}^{+\infty} F(T_i)] \bigcap [\bigcap_{k=1}^{m} GMEP(f_k, \varphi_k)] \bigcap VI(C, B).$$

(a) First we prove that $p \in \bigcap_{i=0}^{+\infty} F(T_i)$.

Since E is uniformly smooth space, we have that J is uniformly norm-norm continuous on any bounded sets and (17),we obtain

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \quad (19)$$

It follows from the boundedness of the sequences $\{x_n\}$and$\{k_n\}$, $\phi(p, T_i^n x_n) \le k_n \phi(p, x_n)$ for each$p \in F$and$i \in N$that the sequences $\{JT_i^n x_n\}$ are bounded. Thus there exists $r > 0$ such that $\{JT_i^n x_n\} \subset B_r(0)$. For each $p \in F$, we have from Lemma 5, Lemma 6, Lemma 7 and (14) that

$$\begin{aligned}
&\phi(p, u_n) = \phi(p, \theta_n^m y_n) \\
\le\ & \phi(p, y_n) \\
=\ & \phi(p, J^{-1}(\alpha_n J\Pi_C t_n + (1-\alpha_n)Jz_n)) \\
\le\ & \alpha_n \phi(p, \Pi_C t_n) + (1-\alpha_n)\phi(p, z_n) \\
=\ & \alpha_n \phi(p, \Pi_C t_n) + (1-\alpha_n) \cdot (\|p\|^2 \\
-\ & 2\langle p, \sum_{i=0}^{+\infty} \beta_n^{(i)} JT_i^n x_n \rangle + \| \sum_{i=0}^{+\infty} \beta_n^{(i)} JT_i^n x_n\|^2) \\
\le\ & \alpha_n \phi(p, \Pi_C t_n) + (1-\alpha_n) \cdot \\
& (\|p\|^2 - 2\sum_{i=0}^{+\infty} \beta_n^{(i)} \langle p, JT_i^n x_n\rangle \\
+\ & \sum_{i=0}^{+\infty} \beta_n^{(i)} \|JT_i^n x_n\|^2) \\
-\ & \beta_n^{(0)} \beta_n^{(i)} g(\|JT_0^n x_n - JT_i^n x_n\|)) \\
=\ & \alpha_n \phi(p, \Pi_C t_n) + (1-\alpha_n) \cdot (\sum_{i=0}^{+\infty} \beta_n^{(i)} \phi(p, T_i^n x_n) \\
-\ & \beta_n^{(0)} \beta_n^{(i)} g(\|JT_0^n x_n - JT_i^n x_n\|)) \\
\le\ & \alpha_n \phi(p, x_n) + (1-\alpha_n) \\
\cdot\ & (\sum_{i=0}^{+\infty} \beta_n^{(i)} [1 + (k_n^{(i)} - 1)] \phi(p, x_n) \\
-\ & \beta_n^{(0)} \beta_n^{(i)} g(\|JT_0^n x_n - JT_i^n x_n\|)) \\
\le\ & \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, x_n) + (k_n - 1)M_n \\
-\ & (1-\alpha_n)\beta_n^{(0)} \beta_n^{(i)} g(\|JT_0^n x_n - JT_i^n x_n\|) \\
=\ & \phi(p, x_n) + (k_n - 1)M_n \\
-\ & (1-\alpha_n)\beta_n^{(0)} \beta_n^{(i)} g(\|JT_0^n x_n - JT_i^n x_n\|).
\end{aligned}$$

This implies that

$$\begin{aligned} & 0 \le (1-\alpha_n)\beta_n^{(0)} \beta_n^{(i)} g(\|JT_0^n x_n - JT_i^n x_n\|) \\ \le\ & \phi(p, x_n) - \phi(p, u_n) + (k_n - 1)M_n \end{aligned} \quad (20)$$

On the other hand, we have

$$\begin{aligned} & \phi(p, x_n) - \phi(p, u_n) \\ =\ & \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n\rangle \\ \le\ & \|x_n - u_n\| \cdot (\|x_n\| + \|u_n\|) \\ +\ & 2\|p\| \cdot \|Jx_n - Ju_n\|. \end{aligned}$$

In view of (17) and (19), we obtain

$$\phi(p, x_n) - \phi(p, u_n) \to 0, n \to \infty. \quad (21)$$

Combining(20)-(21), $\lim_{n\to+\infty} (k_n - 1)M_n = 0, T_0 = I$ and the assumption $\lim_{n\to\infty} (1-\alpha_n)\beta_n^{(0)}\beta_n^{(i)} > 0$,we have

$$g(\|Jx_n - JT_i^n x_n\|) \to 0, n \to \infty.$$

It follows from the property of g that

$$\lim_{n\to+\infty} \|Jx_n - JT_i^n x_n\| = 0 \quad (22)$$

Since $x_n \to p$ as $n \to \infty$ and J is uniformly norm-norm continuous on any bounded sets, we obtain that.

$$\|Jx_n - Jp\| \to 0, n \to \infty. \quad (23)$$

Note that

$$\|JT_i^n x_n - Jp\| \leq \|Jx_n - JT_i^n x_n\| + \|Jx_n - Jp\|.$$

From (22) and (23), we see that

$$\lim_{n\to+\infty} \|JT_i^n x_n - Jp\| = 0. \quad (24)$$

Note that J^{-1} is also uniformly norm-norm continuous on any bounded sets. It follows from (24) that

$$\lim_{n\to+\infty} \|T_i^n x_n - p\| = 0. \quad (25)$$

Note that $\|T_i^{n+1}x_n - p\| \leq \|T_i^{n+1}x_n - T_i^n x_n\| + \|T_i^n x_n - p\|$, the asymptotic regularity of T and (25), we have $\lim_{n\to+\infty} \|T_i^{n+1}x_n - p\| = 0$ .That is, $T_i(T_i^n x_n) \to p$ as $n \to \infty$,it follows from the closeness of T_i that $T_i p = p, \forall i \in N$, i.e. $p \in \bigcap_{i=0}^{+\infty} F(T_i)$.

(b) Now we prove that

$$p \in \bigcap_{k=1}^{m} GMEP(f_k, \varphi_k) = \bigcap_{k=1}^{m} EP(G_k).$$

In fact, in view of $u_n = \theta_n^m y_n$,(15) and Lemma 7, for each $q \in F(\theta_n^k)$,we have

$$\begin{aligned} & 0 \leq \phi(u_n, y_n) = \phi(\theta_n^m y_n, y_n) \\ \leq \; & \phi(p, y_n) - \phi(p, \theta_n^m y_n) \\ \leq \; & \phi(p, x_n) - \phi(p, u_n) + (k_n - 1)M_n. \end{aligned}$$

It follows from (21) and $\lim_{n\to+\infty}(k_n - 1)M_n = 0$ that $\phi(u_n, y_n) \to 0$ as $n \to \infty$ .Using Lemma 1, we see that $\|u_n - y_n\| \to 0$ as $n \to \infty$ .Furthermore, $\|x_n - y_n\| \leq \|x_n - u_n\| + \|u_n - y_n\| \to 0$ as $n \to \infty$ .Since $x_n \to p, n \to \infty$ and $\|x_n - y_n\| \to 0, n \to \infty$,then $y_n \to p, n \to \infty$. By the fact that $\theta_n^k, k = 1, 2, \cdots, m$ is relatively nonexpansive and using Lemma7 again, we have that

$$\begin{aligned} & 0 \leq \phi(\theta_n^k y_n, y_n) \\ \leq \; & \phi(p, y_n) - \phi(p, \theta_n^k y_n) \\ \leq \; & \phi(p, x_n) - \phi(p, \theta_n^k y_n) + (k_n - 1)M_n. \quad (26) \end{aligned}$$

Observe that

$$\begin{aligned} & \phi(p, u_n) = \phi(p, \theta_n^k y_n) \\ = \; & \phi(p, T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} \cdots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} y_n) \\ = \; & \phi(p, T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} \cdots \theta_n^k y_n) \\ \leq \; & \phi(p, \theta_n^k y_n). \quad (27) \end{aligned}$$

Using (27) in (26), we obtain that

$$\begin{aligned} & 0 \leq \phi(\theta_n^k y_n, y_n) \leq \phi(p, x_n) - \phi(p, u_n) \\ & +(k_n - 1)M_n \to 0, n \to \infty. \end{aligned}$$

Then Lemma 1 implies that $\lim_{n\to\infty} \|\theta_n^k y_n - y_n\| = 0, k = 1, 2, \cdots, m$ .Now

$$\begin{aligned} & \|\theta_n^k y_n - p\| \leq \|\theta_n^k y_n - y_n\| + \|y_n - p\| \\ & \to 0, n \to \infty, k = 1, 2, \cdots, m. \end{aligned}$$

Similarly, $\lim_{n\to+\infty} \|\theta_n^{k-1} y_n - p\| = 0$, $k = 1, 2, \cdots, m$ .This further implies that

$$\lim_{n\to+\infty} \|\theta_n^{k-1} y_n - \theta_n^k y_n\| = 0. \quad (28)$$

Also, since J is uniformly norm-norm continuous on any bounded sets and using (28), we obtain that $\lim_{n\to+\infty} \|J\theta_n^{k-1} y_n - J\theta_n^k y_n\| = 0$. .From the assumption $\{r_{k,n}\}_{n=1}^{+\infty} \subset (0, +\infty)$ satisfying $\liminf_{n\to+\infty} r_{k,n} > 0$ for each $k = 1, 2, \cdots, m$,we see that

$$\lim_{n\to+\infty} \frac{\|J\theta_n^{k-1} y_n - J\theta_n^k y_n\|}{r_{k,n}} = 0. \quad (29)$$

By Lemma 6, we have that for each $k = 1, 2, \cdots, m$,

$$\begin{aligned} & \frac{1}{r_{k,n}} \langle y - \theta_n^k y_n, J\theta_n^k y_n - J\theta_n^{k-1} y_n \rangle \\ & +G_k(\theta_n^k y_n, y) \geq 0, \forall y \in C. \end{aligned}$$

Furthermore, replacing n by n_j in the last inequality and using condition (C2), we obtain

$$\begin{aligned} & \|y - \theta_{n_j}^k y_{n_j}\| \cdot \frac{\|J\theta_{n_j}^k y_{n_j} - J\theta_{n_j}^{k-1} y_{n_j}\|}{r_{k,n_j}} \\ \geq \; & \frac{1}{r_{k,n_j}} \langle y - \theta_{n_j}^k y_{n_j}, J\theta_{n_j}^k y_{n_j} - J\theta_{n_j}^{k-1} y_{n_j} \rangle \\ \geq \; & -G_k(\theta_{n_j}^k y_{n_j}, y) \geq G_k(y, \theta_{n_j}^k y_{n_j}), \forall y \in C. \end{aligned}$$

By taking the limit as $j \to +\infty$ in the above inequality, for each $k = 1, 2, \cdots, m$ we have from the condition(C4),(29)and $\theta_{n_j}^k y_{n_j} \to p$ that $G_k(y, p) \leq 0, \forall y \in C$.

For $0 < t \leq 1$ and $y \in C$, define $y_t = ty + (1 - t)p$. It follows from $y, p \in C$ that $y_t \in C$ which yields that $G_k(y_t, p) \leq 0$. It follows from the conditions (C1) and (C4) that

$$\begin{aligned} & 0 = G_k(y_t, y_t) \\ \leq \; & tG_k(y_t, y) + (1 - t)G_k(y_t, p) \\ \leq \; & tG_k(y_t, y). \end{aligned}$$

That is

$$G_k(y_t, y) \geq 0.$$

Let $t \to 0^+$, from the condition(C3), then we obtain that $G_k(p, y) \geq 0, \forall y \in C$ .This implies that $p \in \bigcap_{k=1}^{m} EP(G_k), k = 1, 2, \cdots, m$, i.e.

$$p \in \bigcap_{k=1}^{m} GMEP(f_k, \varphi_k) = \bigcap_{k=1}^{m} EP(G_k).$$

(c) Next we prove that $\lim_{n\to\infty} \|x_n - \Pi_C t_n\| = 0$.

In fact, it follows from Lemma 3, Lemma 8, (13), (17), (18) and $\frac{1}{\alpha}-$ Lipschitzian of B that

$$\begin{aligned}
& \phi(x_n, \Pi_C t_n) \leq \phi(x_n, t_n) - \phi(\Pi_C t_n, t_n) \\
\leq \ & \phi(x_n, t_n) = V(x_n, Jx_n - \lambda_n Bx_n) \\
\leq \ & V(x_n, Jx_n - \lambda_n Bx_n + \lambda_n Bx_n) \\
& -2\langle J^{-1}(Jx_n - \lambda_n Bx_n) - x_n, \lambda_n Bx_n\rangle \\
= \ & \phi(x_n, x_n) - 2\langle t_n - x_n, \lambda_n Bx_n\rangle \\
= \ & -2\langle t_n - x_n, \lambda_n Bx_n\rangle \leq \frac{2}{c}\lambda_n^2\|Bx_n - Bp\|^2 \\
\leq \ & \frac{2}{c\alpha^2}\lambda_n^2\|x_n - p\|^2 \to 0 (n \to \infty).
\end{aligned}$$

So, from Lemma 1, we have $\lim_{n\to\infty} \phi(x_n, \Pi_C t_n) = 0$ which implies that

$$\lim_{n\to\infty} \|x_n - \Pi_C t_n\| = 0. \tag{30}$$

Thus, by the uniform continuity on any bounded sets of J , we obtain that

$$\lim_{n\to\infty} \|Jx_n - J\Pi_C t_n\| = 0. \tag{31}$$

(d) Now we prove that $p \in VI(C, B)$.

Define $D : E \to 2^{E^*}$ as follows:

$$Dv = \begin{cases} Bv + N_C(v), v \in C, \\ \emptyset, v \notin C. \end{cases}$$

where $N_C(v) = \{w \in E : \langle v - u, w\rangle \geq 0, \forall u \in C\}$ is the normal cone to C at $v \in C$. Then the multi-valued mapping D is maximal monotone and $D^{-1}0 = VI(C, B)$. Let $G(D)$ denote the graph of D and let $(v, w) \in G(D)$,then we have $w \in Dv = Bv + N_C(v)$ and hence $w - Bv \in N_C(v)$ .Therefore, by $\Pi_C t_n \in C$,we have

$$\langle v - \Pi_C t_n, w - Bv\rangle \geq 0. \tag{32}$$

On the other hand, it follows from Lemma 2 that

$$\langle v - \Pi_C t_n, J\Pi_C t_n - Jt_n\rangle \geq 0.$$

That is

$$\langle v - \Pi_C t_n, \frac{Jx_n - J\Pi_C t_n}{\lambda_n} - Bx_n\rangle \leq 0. \tag{33}$$

It follows from (32) and (33) that

$$\begin{aligned}
& \langle v - \Pi_C t_n, w\rangle \geq \langle v - \Pi_C t_n, Bv\rangle \\
\geq \ & \langle v - \Pi_C t_n, Bv\rangle \\
+ \ & \langle v - \Pi_C t_n, \frac{Jx_n - J\Pi_C t_n}{\lambda_n} - Bx_n\rangle \\
= \ & \langle v - \Pi_C t_n, Bv - B\Pi_C t_n\rangle \\
+ \ & \langle v - \Pi_C t_n, B\Pi_C t_n - Bx_n\rangle \\
+ \ & \langle v - \Pi_C t_n, \frac{Jx_n - J\Pi_C t_n}{\lambda_n}\rangle \\
\geq \ & -\|v - \Pi_C t_n\| \cdot \frac{\|\Pi_C t_n - x_n\|}{\alpha} \\
- \ & \|v - \Pi_C t_n\| \cdot \frac{\|J\Pi_C t_n - Jx_n\|}{a} \\
\geq \ & -M(\frac{\|\Pi_C t_n - x_n\|}{\alpha} + \frac{\|J\Pi_C t_n - Jx_n\|}{a}).
\end{aligned}$$

Where $M = \sup\{\|v - \Pi_C t_n\|, n \in N\}$, letting $n = n_k$ and $k \to +\infty$, using (17),(18),(30) and (31), we obtain that $\langle v-p, w\rangle \geq 0$. Since D is maximal monotone, we have $p \in D^{-1}0$ and hence $p \in VI(C, B)$. Thus we have $p \in F$.

Step 5 Finally, we prove that $p = \Pi_F x_0$.

From Lemma 2 and the definition of Q_n , we see that $x_n = \Pi_{Q_n} x_0$ and $\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \forall z \in Q_n$ .Since $F \subset Q_n$ for each $n \geq 0$, we have

$$\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0, \forall w \in F.$$

Let $n \to +\infty$ in the last inequality, we see that $\langle p - w, Jx_0 - Jp\rangle \geq 0, \forall w \in F$. In view of Lemma 2, we can obtain that $p = \Pi_F x_0$. This completes the proof of Theorem 10.

In the spirit of Theorem 10, we can prove the following strong convergence theorem.

Theorem 11 *Let C be a nonempty closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E . Suppose $B : C \to E^*$ is an operator satisfying(B1)-(B3). For each $k = 1, 2, \cdots, m$,let $A_k : C \to E^*$ be a continuous and monotone mapping, $\varphi_k : C \to R$ be a lower semi-continuous and convex functional, let $f_k : C \times C \to R$ be a bifunction satisfying(C1)-(C4)and $T_i : C \to C, \forall i \in N$ be an infinite family of closed and quasi-ϕ-nonexpansive mapping, where*

$$F = [\bigcap_{i=0}^{+\infty} F(T_i)] \bigcap [\bigcap_{k=1}^{m} GMEP(f_k, \varphi_k)]$$

$$\bigcap VI(C,B) \neq \emptyset,$$

$T_0 = I$ *.Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 \in C \quad chosen\ arbitrariy, \\ y_n = J^{-1}(\alpha_n J\Pi_C J^{-1}(Jx_n - \lambda_n Bx_n) + \\ \qquad (1-\alpha_n)Jz_n), \\ y_n = J^{-1}(\sum_{i=0}^{+\infty} \beta_n^{(i)} JT_i x_n), \\ u_n = T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} \cdots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} y_n, \\ C_n = \{z \in C : \phi(z,u_n) \leq (1-\alpha_n)\phi(z,z_n) + \\ \qquad \alpha_n \phi(z,x_n) \leq \phi(z,x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_n - Jx_0 \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \bigcap Q_n}(x_0) \end{cases} \tag{34}$$

where $\lambda_n \subset [a,b]$ *for some* a, b *with* $0 < a < b < c\alpha$ *, where* $\frac{1}{c}$ *is 2-uniformly convexity constant of* E *, for each* $k = 1, 2, \cdots, m$, $\{r_{k,n}\}_{n=1}^{+\infty} \subset (0, +\infty)$ *satisfying* $\liminf_{n\to+\infty} r_{k,n} > 0$ *,*

$$T_{r_{k,n}}^{G_k}(x) = \{z \in C : \frac{1}{r_{k,n}} \langle y - z, Jz - Jx \rangle +$$

$$G_k(z,y) \geq 0, \forall y \in C\},$$

$\{\alpha_n\}, \{\beta_n^{(i)}\}, i \in N$ *are real sequences in [0,1] satisfies the conditions:* $\forall n \geq 1, 0 \leq \beta_n^{(i)} \leq 1, \sum_{i=0}^{\infty} \beta_n^{(i)} = 1, \liminf_{n\to\infty}(1-\alpha_n)\beta_n^{(0)}\beta_n^{(i)} > 0, \forall i \in N$ *.Where* $G_k(z,y) = f_k(z,y) + \varphi_k(y) - \varphi_k(z) + \langle A_k z, y - z \rangle, \forall z, y \in C$ *.Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$ *.*

Remark 12 *Theorem 11 improves Theorem 3.1 of Takahashi and Zembayashi [13] in the following aspects:*

(a) From a relatively nonexpansive mapping to an infinite family of quasi-ϕ-nonexpansive mapping.

(b) Considering the variational inequality problem from zero to one.

(c) From an equilibrium problem to a system of generalized mixed equilibrium problem.

Remark 13 *It is worth pointing out that Theorem3.1 and Theorem3.2 of Yang, Zhao and Kim[18] need to be held in the framework of the uniformly smooth and uniformly convex real Banach space. Since, the proofs of Theorem 3.1 and Theorem 3.2 in [18] make use of Lemma 5, but Lemma 5 holds under the uniformly convex space.*

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