Randomly M_t-decomposable Multigraphs and M_2-equipackable Multigraphs

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Abstract: A graph G is called $randomly\ H-decomposable$ if every maximal $H-packing$ in G uses all edges in G. G is called $H-equipackable$ if every maximal $H-packing$ in G is also a maximum $H-packing$ in G. $M_2-decomposable$ graphs, $randomly\ M_2-decomposable$ graphs and $M_2-equipackable$ graphs have been characterized. The definitions could be generalized to multigraphs. And $M_2-decomposable$ multigraphs has been characterized. In this paper, all $randomly\ M_2-decomposable$ multigraphs and $M_2-equipackable$ multigraphs are characterized, and some notes about $randomly\ M_t-decomposable$ multipraphs are given.

Key–Words: Multigraph, packing, decomposable, randomly decomposable, equipackable, matching.

1 Introduction

The common notes and definitions of graphs can be found in [1]. The path and cycle on k vertices are denoted by P_k and C_k, respectively. The star with l edges is denoted by $K_{1,l}$. A matching in a graph is a set of independent edges. By M_t ($t \geq 1$), we denote a matching having t edges. Let H be a subgraph of G. By $G-H$, here we denote the graph left after deleting the edges of H from G and any resulting isolates.

Definition 1 *A collection of disjoint copies of H, say $H_1, H_2, \cdots, H_k$, where each H_i is a subgraph of G, is called an $H-packing$ in G.*

An $H-packing$ in G with k copies $H_1, H_2, \cdots, H_k$ of H is called $maximal$ if $G-\bigcup_{i=1}^{k} E(H_i)$ contains no subgraph isomorphic to H. An $H-packing$ in G with k copies $H_1, H_2, \cdots, H_k$ of H is called $maximum$ if no more than k disjoint copies of H can be packed into G.Let $p(G:H)$ denote the number of copies of H in the maximum H-packing of G.

Definition 2 *A graph G is called $H-decomposable$ if there exists an $H-packing$ of G which uses all edges in G.*

Definition 3 *A graph G is called $randomly\ H-decomposable$ if every maximal $H-packing$ in G uses all edges in G.*

Definition 4 *A graph G is called $H-equipackable$ if every maximal $H-packing$ in G is also a maximum $H-packing$ in G.*

Definition 5 *A graph G is $H-decomposable$, if F is a subgraph of G, and F is $H-decomposable$ but not $randomly\ H-decomposable$, then F is called $H-forbidden$.*

There have been many results on $H-decom-posable$ graphs, $randomly\ H-decomposable$ graphs and $H-equipackable$ graphs. Here give some results.

Theorem 6 *([2]) Let G be a graph of size $2m > 0$ and without isolates. Then G is $M_2-decomposable$ if and only if $\Delta(G) \leq m$ and G is not isomorphic to $K_3 \cup K_2$.*

Theorem 7 *([2]) Let G be a graph of size $2m > 0$ without isolates and G isn't isomorphic to M_2, then G is $randomly\ M_2-decomposable$ if and only if $G \in \mathcal{F}$, where $\mathcal{F} = \{K_4, C_4, 2K_3, K_3 \cup K_{1,3}\} \cup \{2mK_2, 2K_{1,m} | m \geq 2\}$.*

Theorem 8 *([11]) A graph G is randomly $M_3-decomposable$ if and only if it is isomorphic to one of the following: $3nK_2$, $3K_{1,n}$, C_6, $C_4 \bigcup K_{1,2}$, $3K_3$, $2K_3 \bigcup K_{1,3}$, $2K_{1,3} \bigcup K_3$, $K_4 \bigcup K_3$, $K_4 \bigcup K_{1,3}$ or $K_{3,3}$.*

Theorem 9 *([2]) The only connected randomly $K_{1,2}-decomposable$ graphs are the cycle C_4 and the stars $K_{1,2t}$.*

Theorem 10 *([1]) For $r \geq 2$, a connected graph G is randomly $K_{1,r}-$decomposable if and only if it is $K_{r,r}$ or it is bipartite with all degrees in one partite set being multiples of r and all degrees in the other set being less than r.*

Theorem 11 *([1]) Let G be a graph with q edges and maximum degree d. For $q > \frac{8t^2}{3} - 2t$, G is M_t-packable if and only if $t|q$ and $q \geq td$.*

Theorem 12 *([1]) For a given integer $t \geq 2$, a graph with at least $2t^3 - t^2$ edges is randomly M_t-decomposable if and only if it is isomorphic to tH, where H is either nK_2 or $K_{1,n}$ for some $n \geq 1$.*

All $M_2 - equipackable$ graphs have been characterized in [5].

Theorem 13 *([5]) If G is a graph with size $2m$, then G is $M_2 - equipackable$ if and only if G satisfies one of the following:*

(1) $G \cong K_3 \cup K_2$;

(2) $G \in \mathcal{F}$, where $\mathcal{F} = \{K_4, C_4, 2K_3, K_3 \cup K_{1,3}\} \cup \{2mK_2, 2K_{1,m}|m \geq 2\}$;

(3) $\Delta(G) = d > m$, and for any vertex v whose degree is d, the induced subgraph by $E(G - v)$ must be $K_{1,2m-d}$ or K_3.

Theorem 14 *([5]) If G is a graph with size $2m + 1$ and $\Delta(G) = d \geq m+2$, then G is $M_2-equipackable$ if and only if for any vertex v whose degree is d, the induced subgraph by $E(G - v)$ must be $K_{1,2m+1-d}$ or K_3.*

Theorem 15 *([5]) If G is a graph with size $2m + 1 > 1$ and $\Delta(G) = d < m + 2$, then G is $M_2 - equipackable$ if and only if G satisfies one of the following:*

(1) Neither K_3 nor $K_{1,3}$ is contained in G as a subgraph.

(2) At least one copy of K_3 or $K_{1,3}$ is contained in G as a subgraph and for any subgraph H of G which is isomorphic to K_3 or $K_{1,3}$, $\Delta(G - H) > m - 1$ or $G - H \cong K_3 \cup K_2$.

A graph is called a $multigraph$ if it contains loops or has two edges joining two common vertices. All multigraphs considered in the following are loopless and without $isolates$. For any pair of adjacent vertices of M, say x and y, let $n(x, y)$ denote the number of multiple edges joining x and y, called the multiplicity of x and y. The $underlying\ simple\ graph$ of M is a simple spanning subgraphs of M, obtained by deleting all loops and all but one multiple edges, such that $n(x, y) = 1$, for any adjacent vertices x and y of M.

Definition 16 *The multistar $S^{w_1,\cdots,w_t}$ is the multigraph, whose underlying graph is $K_{1,t}$, and the multiplicities of its edges are $w_1, \cdots, w_t$.*

Definition 17 *The multitriangle $T^{x_1,\cdots,x_3}$ is the multigraph, whose underlying graph is a triangle C_3, and the multiplicites of its three edges are x_1, x_2, x_3.*

Given a submultigraph (or a subset of vertices) S, let $M[S](= M[x, y, \cdots, z]$ if S comprises vertices $x, y, \cdots, z)$ denote the submultigraph of M induced by vertices of S.

Definition 18 *An edge set in which all edges are mutually adjacent is called a $cluster$ or $edge - clique$.*

Therefore a $cluster$ is a subset of edges of a submultigraph induced by vertices of either a star or a triangle.

Definition 19 *The maximumsize among clusters in M is called the $cluster\ number$ (or $edge - clique\ number$) of M and is denoted by $\omega_1 = \omega_1(M)$.*

Thus $\omega_1 = \omega(L(M))$, the clique number of the line graph $L(M)$ of M. Hence

$$\omega_1(M) = \max\{\Delta(M), \max_{K_3 \subseteq M} e(M[K_3])\}.$$

A cluster of size $e(M)/2$ is called a $critical\ cluster$ in M. By a $critical\ triangle$ and a $critical\ star$ we mean a critical cluster induced by vertices of a triangle and a star, respectively. The center of a critical star is called a $critical\ vertex$ of the multigraph.

The concept of $decomposable$ has been first extended to multigraphs by Skupień.

Theorem 20 *([4]) A multigraph M is M_2-decomposable if and only if the number of edges e(M) is even and every edge-clique includes no more than half of the edges, i.e., $\omega_1(M) \leq e(M)/2$.*

Theorem 21 *([9]) A multigraph M is M_3-decomposable if and only if $3|e(M)$, $\omega_1(M) \leq e(M)/3$ and $\omega_2(M) \leq 2e(M)/3$, where $\omega_2(M) = \max\{\max e(M[V_5]) : V_5 \subseteq V(M), |V_5| = 5\}$.*

In this paper, some of the results will be generalized to multigraphs. All randomly M_2-decomposable multigraphs and M_2-equipackable multigraphs will be characterized in section 2 and section 4, and in section 3, there are some notes about randomly M_t-decomposable multigraphs. When $M \cong K_2$, $M \cong C_2$ or $M \cong P_3$, there exists no copy of M_2 in M, it has no meaning. When $M \cong M_2$, it is obviously randomly M_2-decomposable and M_2-equipackable. So in the following, we consider the multigraphs with size larger than 2.

2 Randomly M_2 -decomposable multigraphs

In order to describe the randomly M_2-decomposable multigraphs, give a family of simple graphs $\mathcal{R}$ defined by

$$\mathcal{R} = \{C_4, K_4, 2K_3, K_3 \cup K_{1,r}, K_{1,r} \cup K_{1,s}, 2nK_2 | r, s = 1, 2, \cdots ; n \geq 2\}.$$

Let $\mathcal{G}$ be a family of multigraphs with $\mathcal{R}$ as the underlying simple graphs.

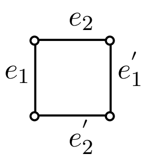
See Figure 1. Here e_i has the same multiplicity with $e_i^{'}$ $(i = 1, 2, 3.)$ in (a) and (b). In (c), (d) and (e),

$$t_c = \sum_{i,j=1, i\neq j}^{3} n(v_i, v_j), t_c^{'} = \sum_{i,j=1, i\neq j}^{3} n(v_i^{'}, v_j^{'});$$

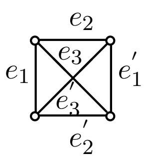
$$t_d = \sum_{i,j=1, i\neq j}^{3} n(v_i, v_j), s_d = \sum_{1\leq i\leq s} n(v^{'}, v_i^{'});$$

$$s_e = \sum_{1\leq i\leq r} n(v, v_i), s_e^{'} = \sum_{1\leq i\leq s} n(v^{'}, v_i^{'}),$$

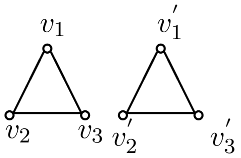
such that, $t_c = t_c^{'}$, $t_d = s_d$, and $s_e = s_e^{'}$. Clearly, (c) is a disjoint union of two critical triangles, (d) is a disjoint union of a critical triangle and a critical star, and (e) shows a disjoint union of two critical stars, (f) is $2nK_2$. Let $\mathcal{G}$ denote the family of multigraphs in Figure 1 satisfying the conditions above.



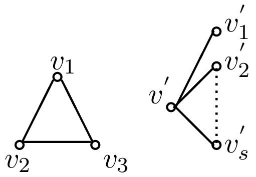
(a)



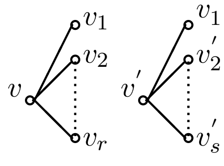
(b)



(c)



(d)



(e)

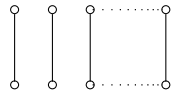

(f)

Figure 1 : the multigraphs set $\mathcal{G}$

Lemma 22 *Let M be a multigraph of size $2m(m \geq 1)$. Suppose that $M \notin \mathcal{G}$(described in Figure 1). Then M can be decomposed into one copy of P_3 or C_2 and $m - 1$ copies of M_2 if and only if $\omega_1(M) \leq m + 1$.*

Proof: At first, it is not hard to note that the lemma holds for $m = 1, 2$. Then we assume that $m \geq 3$.

Suppose that M has been decomposed into one copy of P_3 or C_2 and $m-1$ copies of M_2. If $\omega_1(M) \leq m + 1$ is not true, we consider for $\omega_1(M) \geq m + 2$. There are two cases:

Case 1: M can be decomposed into one copy of P_3 and $m - 1$ copies of M_2. Consider two subcases.

Subcase 1: $\omega_1(M) = \Delta(M)$.

Then M contains a vertex v of degree $\Delta(M) \geq m + 2$. If we delete a copy of P_3, which contains v as an end-vertex, there must exist $m + 1$ edges incident with v which belong to different M_2. If we delete a copy of P_3 with v as the center, there must exist m edges incident with v which belong to different M_2. So at least m edges incident with v belong to different M_2. Then $e(M) \geq n + 2 + n > 2n$. This is a contradiction, so $\omega_1(M) \leq m + 1$.

Subcase 2: $\omega_1(M) = \max_{K_3 \subseteq M} e(M[K_3])$.

Then M contains a submultigraph $M[K_3]$, say T, with size $e(T) \geq m + 2$. Clearly, $\Delta(M) \leq e(T)$. We take any two edges of T, forming a copy of P_3, the remaining m edges of T must belong to different $2K_2$. Similar to case 1, we get a contradiction.

Case 2: M can be decomposed into one copy of C_2 and $m - 1$ copies of M_2, as the case 1, there are two cases.

Subcase 1: $\omega_1(M) = \Delta(M) \geq m + 2$.

Let v be a vertex with the biggest degree, then v must be the end-vertex of C_2. As case 1, delete the copy of C_2, there are m edges incident with v which belong to different M_2. A contradiction.

Subcase 2: $\omega_1(M) = \max_{K_3 \subseteq M} e(M[K_3])$.

Then M contains a submultigraph $M[K_3]$, say T, with size $e(T) \geq m + 2$. So T's any two multiple edges between any two vertices of T, form a copy of C_2. As case 1, get a contradiction.

Conversely, suppose that $M \notin \mathcal{G}$ and $\omega_1(M) \leq m+1$, then we have three cases:

Case 1: $\omega_1(M) \leq m-1$.

Since $M \notin \mathcal{G}$, so $M \neq 2mK_2$, then M must contain P_3 or C_2, whose removal results in a submultigraph M', which has $2(m-1)$ edges, and $\omega_1(M') \leq \omega_1(M) \leq m-1$. By Theorem 4, M' has an M_2-decomposition. So M can be decomposed into one copy of P_3 or C_2 and $m-1$ copies of M_2.

Case 2: $\omega_1(M) = m$.

Then M contains either a vertex with degree m or a $M[K_3]$ with size m. Since $M \notin \mathcal{G}$, M is not a disjoint union of two critical triangles or two critical stars, M is not a disjoint union of a critical triangle and a critical star, either. Then M at most contains a union of two critical stars or two critical triangles with a common vertex, or, contains a union of a critical stars and a critical triangle with a common vertex. Let u denote the common vertex. We can delete such a P_3(i.e., $K_{1,2}$) with u as the center vertex and its two edges belong to the two critical submultigraphs, respectively. Then we get the remaining submultigraph M', which has $2(m-1)$ edges and $\omega_1(M') \leq \omega_1(M) \leq m-1$. By Theorem 4, M can be decomposed into one copy of P_3 and $m-1$ copies of M_2.

Case 3: $\omega_1(M) = m+1$.

Then M contains only one vertex u with degree $m+1$ or one $M[K_3]$, say $M[u,v,w]$ of size $m+1$. For the former, delete two edges uv and uw from M, where u and w have the two largest degrees among those vertices adjacent to u. For the latter, delete two edges uv and uw from T, where u has the largest degree in M among x, y, z. Since $\omega_1(M - uv - uw) \leq m-1$. we proceed as before to obtain the decomposition of M. □

Theorem 23 *Let M be a multigraph of size $2m(m \geq 1)$, then M is randomly M_2-decomposable if and only if $M \in \mathcal{G}$.*

Proof: If $M \in \mathcal{G}$, clearly, M is randomly M_2 - decomposable.

Conversely, let M is randomly M_2-decomposable. We claim that $M \in \mathcal{G}$. On the contrary, suppose that $M \notin \mathcal{G}$. By Theorem20, since M is M_2-decomposable, $\omega_1(M) \leq m$. It follows that M can be decomposed into one copy of P_3 or C_2 and $m-1$ copies of M_2, by Lemma 22. These $m-1$ copies of M_2 do not belong to any M_2-decomposition of M, contradicting to the fact that M is randomly M_2-decomposable. Therefore, $M \in \mathcal{G}$. □

3 Some notes about randomly M_t-decomposable multigraphs

The following lemma is useful to our work.

Lemma 24 *([1]) If G is a randomly $H-$decomposable graph (or multigraph) and F is an $H-$decomposable subgraph (or submultigraph),*

(1) If G is randomly $H-$decomposable, so is F;

(2) If F is $H-$forbidden, then G isn't randomly $H-$decomposable.

Lemma 25 *A graph M is randomly M_t-decomposable, then the following conditions are necessary:*

(1) $t|e(M)$,

(2) $\omega_1(M) \leq \frac{e(M)}{t}$.

The conditions above are not sufficient, for example, $C_2 \bigcup P_4 \bigcup K_2$ satisfies both the two conditions for $t = 3$, however it isn't randomly M_3-decomposable.

Theorem 26 *([6]) A path P_n is M_t-equipackable if and only if $n = kt$ $(k \in N, k \geq 2)$.*

Theorem 27 *([6]) A circle C_n is M_t-equipackable if and only if $2t \leq n \leq 3t-2$, or $n = kt+t-1$ $(k \in N, k \geq 2)$.*

It is easy to prove the lemma below by the Definition 3 and Definition 4.

Lemma 28 *If graph G if $H-$decomposable, then G is randomly $H-$decomposable if and only if G is $H-$equipackable.*

By Theorem 26, Theorem 27 and Lemma 28, get the following corollary.

Corollary 29 *Only C_{2t} is a randomly M_t-decomposable path or circle.*

Theorem 30 *Let a multigraph M be a connected graph which doesn't contain submultigraphs isomorphic to K_3, $K_{1,3}$ or $S^{1,2}$, then M is M_t-decomposable if and only if $M \cong C_{2t}$.*

Proof: If M is a connected graph satisfying the conditions, then M is a simple path or circle. So by Corollary 29, $M \cong C_{2t}$. □

Theorem 31 *If a multigraph M is both randomly M_p-decomposable and randomly M_q-decomposable, and $p(M : M_p) = m$, $p(M : M_q) = l$, where $p, q, m, l \in N$, $p \neq q$, and $p * m = q * l = k$. Then $M \cong M_k$.*

Proof: Without losing of generality, let $p > q$, so $m < l$.

Assume that M isn't isomorphic with M_k, then M must contain P_3 or C_2 as submultigraphs. Let the two edges of the P_3 or C_2 are e and f, respectively. Then e and f must belong to different M_p's copies and M_q's copies in M. Let H be a submultigraph of M composed of the copy of M_p containing e , say H_1, and the copy of M_p containing f, say H_2. As M is randomly M_p-decomposable, by Lemma 24, H is also randomly M_p-decomposable. Obviously, in H_1 there is a copy of M_q containing e, say F_1, and in H_2 there is a copy of M_q containing f, say F_2. Consider two cases:

Case 1: Consider subgraph P_3. There are two subcases.

Subcase 1: No edge except e in F_1 has neighbor edge in F_2.

Then replace any edge in F_1, say g, with f, where $g \neq e$. Now F_2 is still isomorphic with M_q, while F_1 contains P_3. And $M - F_1 - F_2$ is the union of $l - 2$ copies of M_q. So there is a maximal M_q-packing in M, which doesn't use up all edges of M, that means M is not randomly M_q-decomposable. A contradiction.

Subcase 2: There exits at least one edge, which is adjacent to F_2, except e in F_1. See Figure 2, where $g, e \in F_1$ and $f, h \in F_2$.

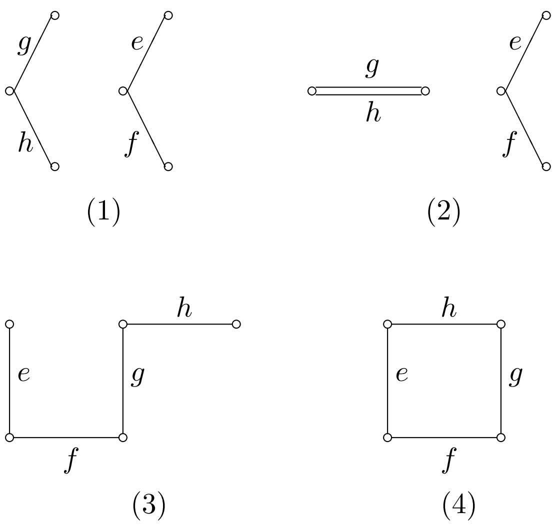


Figure 2

(1) and (2): g is adjacent to h, but not adjacent to f.

If $F_1 \bigcup F_2$ is an union of q copies of clusters with size of 2, mean $K_{1,2}$ or C_2. Then M must be the union of q clusters of size l. (Because otherwise, there exists a cluster with size a, where $a < l$. We just consider the worst case: $a = l - 1$. And by Lemma 25, $\omega_1(M) \leq l$. So M is the disjoint union of $q - 1$ copies of $K_{1,l}$, one copy of $K_{1,l-1}$ and one copy of K_2. Then we can take the K_2 and the other $q - 1$ edges form the copy of $K_{1,l-1}$ and from $q - 2$ copies of $K_{1,l}$. These q edges form a copy of M_q. Obviously the remaining graph has only $q - 2$ copies of M_q. So M has $q - 1$ copies of $K_{1,l}$, a contradict.) So go back to $M \cong qK_{1,l}$. In this condition, $\omega_1(M) = \frac{e(M)}{q} = l > m$. However, M is also randomly M_p-decomposable, and $\omega_1(M) = \frac{e(M)}{q} = m$, a contradiction.

If $F_1 \bigcup F_2$ isn't an union of q copies of clusters with size of 2, then $F_1 \bigcup F_2$ may contain subgraphs as P_4 or K_2, but not clusters with size of 3, for definitions of F_1 and F_2. So if there is a copy of P_4 or K_2, e or f can be replaced, by subcase 1, get a contradiction.

So (1) and (2) are not correct.

(3): g is adjacent to h and f, and h isn't adjacent with e. So replace h with g. As same as subcase 1, $F_1 \cong M_q$, but F_2 has P_3 as a subgraph, so $F_1 \bigcup F_2$ is not randomly M_q-decomposable. A contradiction.

(4): e, f, g and h form a copy of C_4. Because of Lemma 25, the copy of C_4 is independent of other edges of H. Except e, f, g and h, if in F_1 (or F_2) there exists an edge, say e', which isn't adjacent to any of edges in F_2 (or F_1), then replaced h (or g) with e', as above get a contradiction. Otherwise, all edges in F_1 have adjacent edges in F_2 and all edges in F_2 have adjacent edges in F_1, and note that by Lemma 25, $\Delta(F_1 \bigcup F_2) \leq 2$. Then $F_1 \bigcup F_2$ is the union of paths and cycles. Obviously, if odd cycles or odd paths are contained in $F_1 \bigcup F_2$, then $F_1 \bigcup F_2$ is not randomly M_q-decomposable. So $F_1 \bigcup F_2$ is the union of even paths and even cycles. If $F_1 \bigcup F_2$ contains copies of P_5, we can take the two end-edges of a copy of P_5 as a M_2 belonging to a copy of M_q, then the remaining edges can't form a copy of M_q. A contradiction. So $F_1 \bigcup F_2$ contains no paths with size larger than 2. It's easy to see that if $F_1 \bigcup F_2$ is the union of even cycles and P_3 or C_2, it is randomly M_q-decomposable. As a result, H is also the union of even cycles, P_3 or C_2 by the same analytical procedure. Now, because $p > q$, so at least there is a component in H not belonging to $F_1 \bigcup F_2$. This component would be a even cycle, P_3 or C_2, whatever, we can replace one edge of the components with e or f, then new $F_1 \bigcup F_2$ is not randomly M_q-decomposable, contradict to Lemma 24.

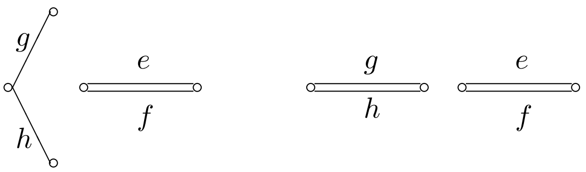
So that M doesn't contain P_3 as a subgraph has been proved. Now consider the case 2.

Case 2: Consider subgraph C_2. As case 1, there are also two subcases.

Subcase 1: No edge except e in F_1 has neighbor edge in F_2.

As same as subcase 1 in case 1, get a contradiction.

Subcase 2: There exits at least one edge, which is adjacent to F_2, except e in F_1. By Lemma 25, see Figure 3, where $g, e \in F_1$ and $f, h \in F_2$.



(1) (2)

Figure 3

In (1) and (2) it is easy to get a contradict by the same thought process of subcase 2 in case 1.

Above all, it has been proved that M doesn't contain P_3 or C_2, so $M \cong M_k$. □

The lemmas given below is obvious.

Lemma 32 *ntK_2 is randomly M_t-decomposable, where $n, t \in N$ and $n \geq 1$.*

Lemma 33 *The disjoint union of t clusters, whose sizes are equal to each other, is randomly M_t-decomposable, where $t \in N$.*

Lemma 34 *Let multigraph F_1 is randomly M_p-decomposable with n copies of M_p, and multigraph F_2 is randomly M_q-decomposable with n copies of M_q, where $n \in N$. Then the disjoint union of F_1 and F_2 is randomly $M_{p+q}-$decomposable with n copies of M_{p+q}.*

Proof: Let $F_1 + F_2$ denote the disjoint union of F_1 and F_2, and let $t = p + q$.

Assume that $F_1 + F_2$ isn't randomly M_t-decomposable. Then, a maximal M_t-packing of $F_1 + F_2$ doesn't use all of edges of $F_1 + F_2$, that is $p(F_1 + F_2 : M_t) = m < n$. So there are at least t edges that cannot form a copy of M_t. As a result $F_1 + F_2$ must contain copies of P_3 or C_2 as subgraphs. Consider P_3 and C_2 in two cases respectively.

Case 1: At least one copy of P_3 is contained in $F_1 + F_2$.

So the copy of P_3 belongs to either F_1 or F_2. Without loss of generality, let the copy of $P_3 \subseteq F_1$. Let in $F_1 - P_3$ there be k copies of M_p, then $n - 2 \leq k \leq n - 1$.

Subcase 1: $k = n - 1$.

Then $e(F_1) - ke(M_p) = ne(M_p) - (n - 1)e(M_p) = e(M_p) = p$ ($e(G)$ is the number of the edges of G, where G is a graph.), it means that the copy of P_3 is in the p edges. This is contradict to that F_1 is randomly M_p-decomposable.

Subcase 2: $k = n - 2$.

Then $e(F_1) - ke(M_p) = ne(M_p) - (n - 2)e(M_p) = 2e(M_p) = 2p$. Different from subcase 1, the two edges of P_3 belongs to the 2 copies of M_p separately. So the remaining $2p$ edges are union of p clusters of size 2 or even cycles. (Otherwise, we can choose a M_p, but remain p edges which cannot form a M_p. So it is not randomly M_p-decomposable.) If so, there is no copy of M_p containing P_3. This is contradict to the assumption.

Case 2: At least one copy of C_2 is contained in $F_1 + F_2$. By case 1, easy to get a contradict.

All above, we prove the lemma. □

Theorem 35 *Let M is a non-connected multigraph with k components, when $t \geq 2$, M is randomly M_t-decomposable and $p(M : M_t) = n$ ($n \geq 1, n \in N$), if and only if $M \in \Psi$, where $\Psi = \{\sum_{j=2}^{k} F_j | F_j \in RD(M_{\sigma(j)}, n), \sigma(j) \in N, \sum \sigma(j) = t\}$, where $F_j \in RD(M_{\sigma(j)}, n)$ means graph $F_j (2 \leq j \leq k)$ is randomly $M_{\sigma(j)}-$decomposable and $p(F_j : M_{\sigma(j)}) = n$. Especially, $ntK_2 \in \Psi$ and t clusters with size of n belong to Ψ*

Proof: By Lemma 32, Lemma 33 and Lemma 34, sufficiency of the theorem is obvious, here only prove it's necessity.

Let $M \in RD(M_t, n)$, and M is non-connected. Consider number of components of M, denoted by $w(M)$, in three cases.

Case 1: $w(M) > t$.

Now M must contain more than two copies of M_t. Assume that $M \neq ntK_2$, then at least one component of M contains P_3 or C_2. There is always a subgraph H composed of two copies of M_t, such that H contains the copy of P_3 or C_2, and each of the two edges of P_3 or C_2 belongs to the two copies of M_t separately. If H is not union of clusters with size of 2 (means P_3 or C_2) and even cycles, then H is M_t-forbidden. If H is the union of clusters with size of 2 and even cycles, then arbitrarily replace one edge of other components of M except these t components with one edge of H. Easy to see, H is M_t-forbidden. So, $M \cong ntK_2$.

Case 2: $w(M) = t$.

Because that randomly M_t-decomposable, so a subgraph say H of M, composed by any two copies of

M_t, is still randomly M_t-decomposable by Lemma 24. Let W_i denote the ith$(1 \leq i \leq t)$ component of M and let $e(W_1)$ is largest. Assume that exits one of the components in M is not a cluster with size of n. And by Lemma 25, $\omega_1(M) \leq n$, then there must be a copy of P_4 as a subgraph.

Claim that in $W_2, \cdots, W_t$, at least one of them contains P_3 or C_2.

(The existing of P_3 or C_2 is true. Otherwise, let $M = M' + (t-1)K_2)$, where M' is W_1 and other components are K_2. So, in W_1 there must be a copy of M_t, and because W_1 is connected, any two edges of the copy of M_t must be in a same path. Then this copy of M_t and any common neighbor edge of it's two edges, with other $(t-1)K_2$ form a subgraph H' of M, such a subgraph is M_t-forbidden. So $M \neq M' + (t-1)K_2$).)

Let $W_2 \supseteq P_3$ or $W_2 \supseteq C_2$, and let $P_4 \subseteq W_1$, and $P_4 = v_1ev_2fv_3gv_4$, where e,f and g are edges of P_4. Take e and g from W_1, and take one edge each from $W_2, \cdots, W_{t-1}$, the t edges form a copy of M_t. And take f, with $t-1$ edges from $W_2, \cdots, W_t$, form a copy of M_t, where make sure at least one edge from W_2. And the two copies of M_t form a subgraph H of M and the two edges taken from W_2 is adjacent, that is P_3 or C_2. From H, take edges each from $W_3, \cdots, W_t$ with e and f form a copy of M_t. Obviously, the t edges left can't form a copy of M_t. So H is M_t-forbidden. A contradiction. So all components of M are clusters with same size of $\frac{e(M)}{t}$.

Case 3: $2 \leq w(M) < t$.

Let the components of M is $W_1, \cdots, W_k(2 \leq k < t)$. $W_1 \bigcup \cdots \bigcup W_k$ is randomly M_t-decomposable, then W_j must be randomly $M_{\sigma(j)}-$decomposable, where $1 \leq j \leq k$, $\sigma(j) \in N$ and $\sum_{j=1}^{k} \sigma(j) = t$. Otherwise, assume that W_1 is not randomly $M_{\sigma(1)}-$decomposable. Then in W_1, there is a maximal $M_{\sigma(1)}-$packing which doesn't use up all edges in W_1, with other maximal $M_{\sigma(j)}-$packing of $W_j(2 \leq j \leq k)$, form a M_t-packing is till maximal and doesn't use all edges of M, A contradiction. □

The non-connected randomly M_t-decomposable multigraphs have been characterized. Frankly speaking, it is hard to character connected randomly M_t-decomposable multigraphs. If once they are characterized, then Theorem 35 would be rewrite more clearly.

4 M_2 -equipackable multigraphs

Firstly, we prove a lemma.

Lemma 36 *Let M be a multigraph with size n, and cluster number $\omega_1(M) = \omega_1$. If $\omega_1 > \lceil \frac{n}{2} \rceil$, then the number of M_2 in the maximum $M_2 - packing$ of M is $n - \omega_1$, i.e., $p(M : M_2) = n - \omega_1$.*

Proof: Since $\omega_1(M) = \max\{\Delta(M), \max_{K_3 \subseteq M} e(M[K_3])\}$, we have two cases:

Case 1: $\omega_1(M) = \Delta(M) = \omega_1 > \lceil \frac{n}{2} \rceil$.

Assume that v is a vertex with degree ω_1 and has k neighbor vertices. Let the set of v's neighbor vertices is

$$\{v_i | \sum_{i=1}^{k} n(v, v_i) = \omega_1, i = 1, 2, \cdots, k.\}.$$

Let E_1 be the edges set which are adjacent to v, and $E_2 = E(M) - E_1 = \{e_1, e_2, \ldots, e_{n-\omega_1}\}$. $|E_1| = \omega_1, |E_2| = n - \omega_1 < \lceil \frac{n}{2} \rceil$. It's obvious that each edge of E_2 is adjacent to at most two edges of E_1. Let $n(v_i, v_j) = l, i \neq j$. We have $n(v, v_i) + n(v, v_j) + l \leq \omega_1$. Hence, $l \leq \omega_1 - n(v, v_i) - n(v, v_j)$. That is, for any $e_i \in E_2$, there exists an edge $e_i' \in E_1$ such that $\{e_i, e_i'\}$ forms a copy of M_2. Remove $\{e_i, e_i'\}$. Let $E_1^{(1)} = E_1 - \{e_i'\}$ and $E_2^{(1)} = E_2 - \{e_i\}$. In the same way, each edge in $E_2^{(1)}$ has at most two neighbors in $E_1^{(1)}$, we can get another copy of M_2. We remove it and repeat this process $(n - \omega_1)$ times, thus removing all the edges of E_2 each of which along with one edge of E_1 forms a copy of M_2, while the remaining edges in E_1 contains no M_2. So the removed $n - \omega_1$ copies of M_2 form a maximal M_2-packing of M. Because all edges in E_1 are adjacent and there are no edges in E_2, this M_2-packing is maximum.

Case 2: $\omega_1(M) = \max_{K_3 \subseteq M} e(M[K_3]) = \omega_1$.

Let T be a $M[K_3]$ of size ω_1, $E_1 = E(T)$, $|E_1| = \omega_1$, $V(T) = \{x, y, z\}$. $E_2 = E(M) - E_1 = n - \omega_1$. Obviously, any edge in E_2 is adjacent to at most one of $\{x, y, z\}$. Without loss of generality, we take any vertex of T, say x, let e_x be the edge set adjacent to x in E_2 and T_x be edges between y and z, which are not adjacent to x in T, that is, $|T_x| = n(y, z)$. we have $e_x \leq \Delta(M) - n(x, y) - n(x, z) \leq \omega_1(M) - n(x, y) - n(x, z) = \omega_1 - (\omega_1 - T_x) = T_x$. Hence, for any $e \in E_2$, there exists an edge $e_i' \in E_1$ such that $\{e_i, e_i'\}$ forms a copy of M_2. The proof is completed similar to Case 1. □

Theorem 37 *Let M be a multigraph of size n, $\omega_1 > \lceil \frac{n}{2} \rceil$, then M is M_2-equipackable if and only if M satisfies one of the following:*

(1) $\omega_1(M) = \Delta(M) = \omega_1 > \lceil \frac{n}{2} \rceil$, and for any vertex v whose degree is ω_1, the submultigraph $M[M - v]$ must be a member of the multigraphs family $M[K_{1,r}]$ or $M[K_3]$, where $e(M[K_{1,r}]) = e(M[K_3]) = n - \omega_1$.

(2) $\omega_1(M) = \max_{K_3 \subseteq M} e(M[K_3]) = \omega_1 > \lceil \frac{n}{2} \rceil$, *and for any submultigraph* $T = M[K_3]$ *of size* ω_1, *the submultigraph* $M[M-T]$ *must be a member of the multigraphs family* $M[K_{1,r}]$ *or* $M[K_3]$*, where* $e(M[K_{1,r}]) = e(M[K_3]) = n - \omega_1$.

Proof: We can easily verify that the multigraphs satisfying (1) or (2) are all M_2-equipackable.

Conversely, let M be an M_2-equipackable multigraph.

Case 1: $\omega_1(M) = \Delta(M) = \omega_1 > \lceil \frac{n}{2} \rceil$.

By Lemma 36, the number of M_2 in the maximum $M_2 - packing$ in M is $n - \omega_1$. Let v be a vertex with maximum degree. If there are two edges (say $\{e, f\}$) in $M - v$ which are not adjacent, then after removing $\{e, f\}$ (we denote $M - \{e, f\}$ by M_1), $\Delta(M_1) = \omega_1 > \lceil \frac{n}{2} \rceil > \lceil \frac{n-2}{2} \rceil$. The graph M_1 also satisfies Lemma 36. So we can get a maximum M_2-packing in M_1 with $n - 2 - \omega_1$ copies of M_2 which along with $\{e, f\}$ form a maximal M_2-packing of M. Obviously this resulting maximal packing with only $n-1-\omega_1$ copies of M_2 is not maximum, which contradicts that M is M_2-equipackable. So all edges in $M - v$ are mutually adjacent. That is, the submultigraph $M[M - v]$ must be a member of the multigraphs family $M[K_{1,r}]$ or $M[K_3]$, where $e(M[K_{1,r}]) = e(M[K_3]) = n - \omega_1$.

Case 2: $\omega_1(M) = \max_{K_3 \subseteq M} e(M[K_3]) = \omega_1 > \lceil \frac{n}{2} \rceil$.

Let T be a submultigraph $M[K_3]$ of size ω_1. Similar to Case 1, suppose there are two edges (say $\{e, f\}$) in $M - T$ which are not adjacent, by Lemma 36, we can get a maximal, but not a maximum M_2-packing, which contradicts to the fact that M is M_2-equipackable. Hence, all edges in $M - T$ are mutually adjacent. □

Now, we shall consider the multigraphs M of size n, with $\omega_1 \leq \lceil \frac{n}{2} \rceil$.

Theorem 38 *Let* M *be a multigraph of size* $2m$*, and* $\omega_1(M) \leq m$*. Then* M *is* M_2-*equipackable if and only if* $M \in \mathcal{G}$.

Proof: Obviously, any multigraph $M \in \mathcal{G}$ is M_2-equipackable.

Conversely, let M be an M_2-equipackable multigraph with size $2m$, then by Theorem 20, M is M_2-decomposable. So $p(M : M_2) = m$. If M is not randomly M_2-decomposable, then there exists a maximal M_2-packing which does not use all edges in M and consequently which is not maximum. It contradicts to the fact that M is M_2-equipackable. So M must be randomly M_2-decomposable. By Theorem 23, $M \in \mathcal{G}$. □

Lemma 39 *Let* M *be a multigraph with size* $2m + 1$*, and* F *be a maximal* $M_2 - packing$*. If* F *satisfies one of the following:*

(1) F *omits all the edges of a subgraph* K_3*;*

(2) F *contains a copy (say* $\{e, f\}$*) of* M_2 *such that neither* e *nor* f *is incident with the center of the star* $K_{1,3}$*;*

(3) F *contains a copy (say* $\{e, f\}$*) of* M_2 *such that neither* e *nor* f *is incident with the center of the submultigraph* $S^{1,2}$*. Then the multigraph* M *is not* M_2-*equipackable.*

Proof: We just prove (3). Now we assume that F satisfies (3).

Without loss of generality, we denote the edges of the submultigraph by $S^{1,2}$ by vv_1, vv_2 and vv_2', where $n(v, v_1) = 1$ and $n(v, v_2) = 2$. Neither e nor f is incident with v. For the edge e, there are three subcases:

Subcase 1: The edge e is not incident with any vertex of $\{v_1, v_2\}$.

Since there are at most two vertices of $\{v_1, v_2\}$ which are incident with f, say v_1, we can replace $\{e, f\}$ with $\{f, vv_2\}$(or $\{f, vv_2'\}$) and $\{e, vv_1\}$ to get a maximum M_2-packing whose size is larger than that of the given maximal M_2-packing. So M is not M_2-equipackable.

Subcase 2: The edge e is incident with v_1.

Then f is not incident with v_1 since e and f are independent. We can replace $\{e, f\}$ with $\{vv_1, f\}$ and $\{e, vv_2\}$ (or $\{e, vv_2'\}$) to get another larger maximal M_2-packing which is maximum. So M is not M_2-equipackable.

Subcase 3: The edge e is incident with v_2.

Then v_2 is not incident with f since e and f are independent. We can replace $\{e, f\}$ with $\{f, vv_2\}$(or $\{f, vv_2'\}$) and $\{e, vv_1\}$ to get another larger maximal M_2-packing. So M is not M_2-equipackable. □

Lemma 40 *Let* M *be a multigraph with size* $2m + 1$ *and cluster number* $\omega_1(M) = \omega_1$, $2 < \omega_1 \leq m+1$*. If* M *is* M_2-*equipackable, then for any submultigraph* H *of* M *which is isomorphic to* K_3, $K_{1,3}$*, or* $S^{1,2}$*, then* $G - H$ *is not* M_2-*decomposable.*

Proof: Here we just give the proof about $S^{1,2}$. Assume that the Lemma is not true, that is, $M - H$ is M_2-decomposable, where H is isomorphic to $S^{1,2}$. So $M - H$ can be the union of $(m - 1)$ copies of M_2. There must exists a copy (say $\{e, f\}$) of M_2 in $M - H$ such that neither e nor f is incident with the center of H. Otherwise, each copy of M_2 in $M - H$ has an edge incident with the center (say v) of H, then $\omega_1 \geq \Delta(M) \geq d_M(v) \geq m - 1 + 3 = m + 2$, which contradicts to the fact that $\omega_1 \leq m + 1$. By Lemma

39, M is not M_2-equipackable, a contradiction. Similarly, the other two cases for K_3 and $K_{1,3}$ can also lead contradictions, respectively. □

Theorem 41 *Let M be a multigraph with size $2m + 1 > 1$ and $\omega_1(M) = \omega_1 \leq m + 1$, then M is $M_2 - equipackable$ if and only if M satisfies one of the following:*

(1) None of K_3, $K_{1,3}$ or $S^{1,2}$ is contained in M as a submultigraph;

(2) M is isomorphic to $M[K_{1,r_1}] \cup M[K_{1,r_2}] \bigcup e$, where e isn't incident with both of the two centers,

or,M is isomorphic to $M[K_{1,r_1}] \cup M[K_3] \bigcup e$, where e is neither incident with the star's center nor incident with two vertices of the triangle at the same time,

or,M is isomorphic to $M[K_3] \cup M[K_3]' \bigcup e$, where e isn't incident with any two vertices of each triangle at the same time;

(3) M is isomorphic to $M[K_{1,r_1}] \cup M[K_{1,s}]$,

or, M is isomorphic to $M[K_{1,r_1}] \cup M[K_3]'$,

or,M is isomorphic to $M[K_3] \cup [K_{1,s}]$,

or,M is isomorphic to $M[K_3] \cup M[K_3]'$.

Above all, they satisfies $1 \leq r_1, r_2 \leq m$, $1 \leq s \leq m + 1$, $e(M[K_{1,r_1}]) = e(M[K_{1,r_2}]) = e(M[K_3]) = e(M[K_3]') = m$, and $e(M[K_{1,s}]) = e(M[K_3]') = m + 1$.

Proof: We can easily verify that the multigraphs described in the theorem are all M_2-equipackable.

Let M be M_2-equipackable, then we have two cases:

Case 1: None of K_3, $K_{1,3}$ or $S^{1,2}$ is contained in M as a submultigraph.

That is, M is a union of simple odd paths, simple even paths, simple odd circles or simple even circles, in which the number of all odd circles and odd paths is odd. Then for any edge of M, say e, there does always exist an edge, say f, and the two edges form an M_2. Delete them, and go on taking the same action, until only disjoint edges of odd number or a P_4 are remained in M. Obviously, M is M_2-equipackable.

Case 2: At least one copy of K_3, $K_{1,3}$ or $S^{1,2}$ is contained in M as a submultigraph.

By Lemma 39, for any submultigraph H which is isomorphic to K_3, $K_{1,3}$ or $S^{1,2}$, $M - H$ is not M_2-decomposable, so by Theorem 20, $\omega_1(M - H) > m - 1$. Then we have $m \leq \omega_1(M - H) \leq \omega_1(M) \leq m+1$. So, $\omega_1(M)$ has two possible values. When $\omega_1(M) = m$, we get the condition (2). When $\omega_1(M) = m + 1$, we get the condition (3). □

Hence, we have characterized all the M_2-equipackable multigraphs by Theorem 37, Theorem 38 and Theorem 41.

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