Global exponential synchronization of a class of BAM neural networks with time-varying delays

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Abstract: In this paper, the global exponential synchronization problem is considered for a class of BAM neural networks with time-varying delays. By using Lyapunov functional method and analysis techniques, three sufficient conditions for the global exponential synchronization of the drive-response system are derived. Two numerical examples are given in the end to illustrated the effectiveness of our theoretical results.

Key–Words: BAM neural networks; Synchronization; Global exponential synchronization; Lyapunov functional; Halanay inequality; Time-varying delays

1 Introduction

Synchronization, as a typical nonlinear phenomena, first introduced by Pecora and Carrol in chaos systems [1], has been intensively studied because of its potential applications in many technological fields mainly including secure communications, chemical reactions, biological neural networks, human heartbeat regulation, chaos generators design, information science, image processing, harmonic oscillation generation, etc[2-9]. Up to now, various approaches have been proposed for the synchronization of nonlinear systems such as impulsive control method[10,11], adaptive design control [12,13], feedback control[14,15], fuzzy control [16,17] , periodically intermittent control [18,19] and so on.

On the other hand, the dynamics of delayed neural networks (DNNS) has attracted considerable attention due to the fact that there exist some complicated dynamics and even chaotic behaviours if the network's parameters and time delays are appropriately chosen. Some good sufficient conditions for the synchronization of the DNNS are presented [20-27].

However, all of the mentioned works mainly focus on the single-layer neural networks, such as Hopfield neural networks, competitive neural networks, cellular neural networks, Cohen-Grossherg neural networks , etc. For two-layer hetero associative neural networks, called bidirectional associative memory (BAM) neural networks, there are few authors to discuss the problem of synchronization except that Juhong Ge and Jian Xu [28] studied the synchronization and synchronized periodic solution in a simplified five-neuron BAM neural networks with delays. But the delays considered in [28] is constants. In fact, absolute constant delays is rarely in the process of signals transmission among neurons, which is an idealized approximations of varying delays. So it is more realistic to consider time-varying delays in neural networks than constant delays. Besides, BAM neural networks has been successfully applied to pattern recognition and artificial intelligence due to its generalization of the single-layer auto-associative Hebbian correlator to a two-layer pattern-matched hetero-associative circuit. Therefore, it is important and interest to investigate the synchronization of BAM neural networks with time-varying delays.

Inspired by the above discussion, in this paper, we consider the problem of synchronization for a class of BAM neural networks with varying-time delays. The organization of this paper is as follows. In Section 2, model description and some preliminaries are given. Three main theorems are obtained in Section 3 to ensure the global exponential synchronization of the BAM neural networks with varying-time delays. In Section 4, the effectiveness of the proposed theory is shown by two numerical examples.

2 Model and Preliminaries

In this paper, the BAM neural networks with varying-time delays takes the following form:

The drive system is

$$\begin{cases} \frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum\limits_{j=1}^{m} c_{ij} f_j(y_j(t)) \\ \qquad + \sum\limits_{j=1}^{m} d_{ij} f_j(y_j(t-\tau_{ij}(t))) + I_i, \\ \frac{dy_j(t)}{dt} = -b_j y_j(t) + \sum\limits_{i=1}^{n} p_{ji} g_i(x_i(t)) \\ \qquad + \sum\limits_{i=1}^{n} q_{ji} g_i(x_i(t-\sigma_{ji}(t))) + J_j, \end{cases} \quad (1)$$

and the response system is described by

$$\begin{cases} \frac{dz_i(t)}{dt} = -a_i z_i(t) + \sum\limits_{j=1}^{m} c_{ij} f_j(w_j(t)) \\ \qquad + \sum\limits_{j=1}^{m} d_{ij} f_j(w_j(t-\tau_{ij}(t))) + I_i - u_i(t), \\ \frac{dw_j(t)}{dt} = -b_j w_j(t) + \sum\limits_{i=1}^{n} p_{ji} g_i(z_i(t)) \\ \qquad + \sum\limits_{i=1}^{n} q_{ji} g_i(z_i(t-\sigma_{ji}(t))) + J_j - v_j(t), \end{cases} \quad (2)$$

where $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m, t \geq 0, x_i(t)$ and $y_j(t)$ are the state variables associated with the ith neuron and the jth neuron at time t, respectively; $a_i > 0$ and $b_j > 0$ represent the rate with which the i-th neurons and the jth neurons will reset its potential to the resting state in isolation when disconnected from the networks and external inputs, respectively; $c_{ij},\ p_{ji}, d_{ij},\ q_{ji}$ are constants, and denote the first-order connection weights of the neural networks ;$f_j,\ g_i$ are the activation functions of the jth neurons and ith neurons at time t respectively; I_i and J_i represent the ith and jth component of an external inputs source introduced from outside the network to the cell i and j, respectively; $u_i(t)$ and $v_j(t)$ denote the control inputs and will be appropriately designed to obtain a certain control objective. The varying-time delays $\tau_{ij}(t)$and $\sigma_{ji}(t)$ are bounded and correspond to finite speed of axonal signal transmission

Set

$$\tau^* = \max_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} \{\tau_{ij}(t)\}, \sigma^* = \max_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} \{\sigma_{ji}(t)\},$$

then the initial conditions of (1) and (2) can respectively be expressed as

$$\begin{array}{l} x_i(s) = \varphi_i(s), s \in [-\sigma^*, 0], y_j(s) = \phi_j(s), \\ s \in [-\tau^*, 0]\, i = 1, 2, \cdots, n, j = 1, 2, \cdots, m, \end{array} \quad (3)$$

and

$$\begin{array}{l} z_i(s) = \bar{\varphi}_i(s), s \in [-\sigma^*, 0], w_j(s) = \bar{\phi}_j(s), \\ s \in [-\tau^*, 0]\, i = 1, 2, \cdots, n, j = 1, 2, \cdots, m. \end{array} \quad (4)$$

Let $\alpha_i(t) = x_i(t) - z_i(t)$, $\beta_j(t) = y_j(t) - w_j(t)$, then we have the error system:

$$\begin{cases} \dot{\alpha}_i(t) = -a_i\alpha_i(t) + \sum\limits_{j=1}^{m} c_{ij}\left[f_j(y_j(t)) - f_j(w_j(t))\right] \\ \qquad + \sum\limits_{j=1}^{m} d_{ij}\left[f_j(y_j(t-\tau_{ij}(t)))\right. \\ \qquad \left. - f_j(w_j(t-\tau_{ij}(t)))\right] + u_i(t), \\ \dot{\beta}_j(t) = -b_j\beta_j(t) + \sum\limits_{i=1}^{n} p_{ji}\left[g_i(x_i(t)) - g_i(z_i(t))\right] \\ \qquad + \sum\limits_{i=1}^{n} q_{ji}\left[g_i(x_i(t-\sigma_{ji}(t)))\right. \\ \qquad \left. - g_i(z_i(t-\sigma_{ji}(t)))\right] + v_j(t), \end{cases} \quad (5)$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$. We denote $u(t) = (u_1(t), u_2(t), \cdots u_n(t))^T$, $v(t) = (v_1(t), v_2(t), \cdots v_m(t))^T$.

The control inputs associated with the state-feedback are designed as follows:

$$\begin{array}{l} u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} \sum\limits_{k=1}^{n} w_{1k}(x_k(t) - z_k(t)) \\ \sum\limits_{k=1}^{n} w_{2k}(x_k(t) - z_k(t)) \\ \vdots \\ \sum\limits_{k=1}^{n} w_{nk}(x_k(t) - z_k(t)) \end{bmatrix} \\ = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ & \cdots & \cdots & \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) - z_1(t) \\ x_2(t) - z_2(t) \\ \vdots \\ x_n(t) - z_n(t) \end{bmatrix} \\ = \Omega\alpha(t), \end{array} \quad (6)$$

$$\begin{array}{l} v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_m(t) \end{bmatrix} = \begin{bmatrix} \sum\limits_{l=1}^{m} \gamma_{1l}(y_l(t) - w_l(t)) \\ \sum\limits_{l=1}^{n} \gamma_{2l}(y_l(t) - w_l(t)) \\ \vdots \\ \sum\limits_{l=1}^{m} \gamma_{ml}(y_l(t) - w_l(t)) \end{bmatrix} \\ = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\ & \cdots & \cdots & \\ \gamma_{m1} & \gamma_{m2} & \cdots & \gamma_{mm} \end{bmatrix} \begin{bmatrix} y_1(t) - w_1(t) \\ y_2(t) - w_2(t) \\ \vdots \\ y_m(t) - w_m(t) \end{bmatrix} \\ = \Gamma\beta(t), \end{array} \quad (7)$$

where Ω and Γ are the gain matrixes to be determined for synchronizing both the drive system and the response system.

For simplicity, let

$$\begin{aligned} G_i(\alpha_i(t)) &= g_i(x_i(t)) - g_i(z_i(t)) \\ &= g_i(\alpha_i(t) + z_i(t)) - g_i(z_i(t)), \\ F_j(\beta_j(t)) &= f_j(y_j(t)) - f_j(w_j(t)) \\ &= f_j(\beta_j(t) + w_j(t)) - f_j(w_j(t)), \end{aligned}$$

then, error system (5) can be written as

$$\begin{cases} \dot{\alpha}_i(t) = -a_i\alpha_i(t) + \sum\limits_{j=1}^{m} c_{ij}F_j(\beta_j(t)) \\ \qquad + \sum\limits_{j=1}^{m} d_{ij}F_j(\beta_j(t-\tau_{ij}(t))) + u_i(t), \\ \dot{\beta}_j(t) = -b_j\beta_j(t) + \sum\limits_{i=1}^{n} p_{ji}G_i(\alpha_i(t)) \\ \qquad + \sum\limits_{i=1}^{n} q_{ji}G_i(\alpha_i(t-\sigma_{ji}(t))) + v_j(t), \end{cases} \tag{8}$$

where

$$u_i(t) = \sum_{k=1}^{n} w_{ik}(x_k(t) - z_k(t)) = \sum_{k=1}^{n} w_{ik}\alpha_k(t), \tag{9}$$

$$v_j(t) = \sum_{l=1}^{m} \gamma_{jl}(y_l(t) - w_l(t)) = \sum_{l=1}^{m} \gamma_{jl}\beta_j(t). \tag{10}$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$.

Throughout the paper, we have the following assumptions.

(H_0) There exists constants k_j and h_i such that

$$\begin{aligned} |f_j(x) - f_j(y)| &\le k_j|x-y|, j = 1, 2, \cdots, m, \\ |g_i(x) - g_i(y)| &\le h_i|x-y|, i = 1, 2, \cdots, n. \end{aligned}$$

(H_1) $\tau_{ij}(t)$ and $\sigma_{ji}(t)$ are differential on t and

$$0 < \tau = \max_{\substack{1\le i\le n \\ 1\le j\le m}} \{\dot{\tau}_{ij}(t)\} < 1,$$
$$0 < \sigma = \max_{\substack{1\le i\le n \\ 1\le j\le m}} \{\dot{\sigma}_{ji}(t)\} < 1.$$

Definition 1 *The norms used in this paper are defined as*

$$\begin{aligned} \|x(t) - z(t)\| &= \sum_{i=1}^{n} |x_i(t) - z_i(t)|^r, \\ \|y(t) - w(t)\| &= \sum_{j=1}^{m} |y_j(t) - w_j(t)|^r, \\ \|\varphi(s) - \bar{\varphi}(s)\| &= \sup_{-\sigma^*\le s\le 0} \sum_{i=1}^{n} |\varphi_i(s) - \bar{\varphi}_i(s)|^r, \\ \|\phi(s) - \bar{\phi}(s)\| &= \sup_{-\tau^*\le s\le 0} \sum_{i=1}^{n} |\phi_j(s) - \bar{\phi}_j(s)|^r, \end{aligned}$$

where $r = 1$ or 2, $\varphi(s) = (\varphi_1(s), \varphi_2(s), \cdots \varphi_n(s))^T$ and $\phi(s) = (\phi_1(s), \phi_2(s), \cdots \phi_m(s))^T$ are the initial conditions of the drive system (1), $\bar{\varphi}(s) = (\bar{\varphi}_1(s), \bar{\varphi}_2(s), \cdots \bar{\varphi}_n(s))^T$ and $\bar{\phi}(s) = (\bar{\phi}_1(s), \bar{\phi}_2(s), \cdots \bar{\phi}_m(s))^T$ are the initial conditions of the response system (2).

Definition 2 *The drive system (1) and the response system (2) are said to be globally exponentially synchronized if there are control inputs $u(t)$ and $v(t)$, and further there exists constants $M \ge 1$ and $\lambda > 0$ such that*

$$\begin{aligned} &\|x(t) - z(t)\| + \|y(t) - w(t)\| \\ &\le M\left[\|\varphi(s) - \phi(s)\| + \|\bar{\varphi}(s) - \bar{\phi}(s)\|\right]e^{-\lambda t}(t \ge 0). \end{aligned}$$

Lemma 3 *Assume that*

$$-a_i + \sum_{j=1}^{m}\left(|p_{ji}| + (1-\sigma)^{-1}|q_{ji}|\right)h_i + \sum_{k=1}^{n}|w_{ki}| < 0, \quad i = 1, 2, \cdots, n, \tag{11}$$

$$-b_j + \sum_{i=1}^{n}\left(|c_{ij}| + (1-\tau)^{-1}|d_{ij}|\right)k_j + \sum_{l=1}^{m}|\gamma_{lj}| < 0, \quad j = 1, 2, \cdots, m, \tag{12}$$

then there exists $\varepsilon > 0$ such that

$$\begin{aligned} &\varepsilon - a_i + \sum_{j=1}^{m}\left(|p_{ji}| + (1-\sigma)^{-1}e^{\varepsilon\sigma^*}|q_{ji}|\right)h_i \\ &+ \sum_{k=1}^{n} w_{ki} \le 0, i = 1, 2, \cdots, n, \\ &\varepsilon - b_j + \sum_{i=1}^{n}\left(|c_{ij}| + (1-\tau)^{-1}e^{\varepsilon\tau^*}|d_{ij}|\right)k_j \\ &+ \sum_{l=1}^{m} \gamma_{lj} \le 0, j = 1, 2, \cdots, m. \end{aligned}$$

The proof of Lemma 3 is similar to that of Lemma 2.4 of [29], here we omit it.

Lemma 4 *(Halanay inequality)*[30] *Let $\tau \ge 0$ be a constant, and $x(t)$ be a non-negative continuous function defined for $[t_0 - \tau, t)$ which satisfied*

$$\dot{x}(t) \le -ax(t) + b\bar{x}(t),$$

where $\bar{x}(t) = \sup\limits_{t-\tau\le s\le t} x(s)$, a and b are constant satisfying $a > b > 0$, then there exist constants $k > 0$, such that

$$x(t) \le \bar{x}(t_0)e^{-k(t_0-\tau)}, t \ge t_0.$$

3 Main results

In this section, we will derive some sufficient conditions to ensure the exponential stability of system (1) and (2).

Theorem 5 *The drive system (1) and the response system (2) are globally exponentially synchronized if* (H_0) *and* (H_1) *hold, moreover,*

$$-2a_i + \sum_{j=1}^{m} (|c_{ij}| + |d_{ij}|) k_j + \sum_{k=1}^{n} |w_{ik}| + \sum_{k=1}^{n} |w_{ki}| + \sum_{j=1}^{m} \left(|p_{ji}| + (1-\sigma)^{-1} |q_{ji}|\right) h_i < 0, i = 1, 2, \cdots, n, \tag{13}$$

$$-2b_j + \sum_{i=1}^{n} (|p_{ji}| + |q_{ji}|) h_i + \sum_{l=1}^{m} |\gamma_{jl}| + \sum_{l=1}^{m} |\gamma_{lj}| + \sum_{i=1}^{n} \left(|c_{ij}| + (1-\tau)^{-1} |d_{ij}|\right) k_j < 0, j = 1, 2, \cdots, m, \tag{14}$$

hold.

Proof. From (8)-(10), it is not difficult to find that

$$\begin{cases} \frac{d\alpha_i^2(t)}{dt} = -2a_i\alpha_i^2(t) + 2\alpha_i(t) \sum_{j=1}^{m} c_{ij}F_j(\beta_j(t)) \\ \qquad +2\alpha_i(t) \sum_{j=1}^{m} d_{ij}F_j(\beta_j(t-\tau_{ij}(t))) \\ \qquad +2\alpha_i(t) \sum_{k=1}^{n} w_{ik}\alpha_k(t), \\ \frac{d\beta_j^2(t)}{dt} = -2b_j\beta_j(t) + 2\beta_j(t) \sum_{i=1}^{n} p_{ji}G_i(\alpha_i(t)) \\ \qquad +2\beta_j(t) \sum_{i=1}^{n} q_{ji}G_i(\alpha_i(t-\sigma_{ji}(t))) \\ \qquad +2\beta_j(t) \sum_{l=1}^{m} \gamma_{jl}\beta_j(t), \end{cases} \tag{15}$$

Consider the following Lyapunov functional

$$V(t) = \sum_{i=1}^{n} e^{\varepsilon t}\alpha_i^2(t) + \sum_{j=1}^{n} e^{\varepsilon t}\beta_j^2(t) +(1-\tau)^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} |d_{ij}|k_j \int_{t-\tau_{ij}(t)}^{t} e^{\varepsilon(s+\tau_{ij}(s))}\beta_j^2(s)ds +(1-\sigma)^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} |q_{ji}|h_i \int_{t-\sigma_{ji}(t)}^{t} e^{\varepsilon(s+\sigma_{ji}(s))}\alpha_i^2(s)ds, \tag{16}$$

then

$$\begin{aligned} D^+V(t) &\le e^{\varepsilon t} \sum_{i=1}^{n} \Big[\varepsilon\alpha_i^2(t) - 2a_i\alpha_i^2(t) \\ &+2 \sum_{j=1}^{m} |c_{ij}| k_j |\alpha_i(t)| |\beta_j(t)| \\ &+2 \sum_{j=1}^{m} |d_{ij}| k_j |\alpha_i(t)| |\beta_j(t-\tau_{ij}(t))|\Big] \\ &+2e^{\varepsilon t} \sum_{i=1}^{n}\sum_{k=1}^{n} |w_{ik}| |\alpha_i(t)| |\alpha_k(t)| \\ &+e^{\varepsilon t} \sum_{j=1}^{m} \Big[\varepsilon\beta_j^2(t) - 2b_j\beta_j^2(t) \\ &+2 \sum_{i=1}^{n} |p_{ji}| h_i |\beta_j(t)| |\alpha_i(t)|\Big] \\ &+2e^{\varepsilon t} \sum_{j=1}^{m} \Big[\sum_{i=1}^{n} |q_{ji}| h_i |\beta_j(t)| |\alpha_i(t-\sigma_{ji}(t))| \\ &+ \sum_{l=1}^{m} |\gamma_{jl}| |\beta_j(t)| |\beta_l(t)|\Big] \\ &+e^{\varepsilon t}(1-\tau)^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} |d_{ij}|k_j \Big[e^{\varepsilon\tau_{ij}(t)}\beta_j^2(t) \\ &-\beta_j^2(t-\tau_{ij}(t))(1-\dot{\tau}_{ij}(t))\Big] \\ &+e^{\varepsilon t}(1-\sigma)^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} |q_{ji}|h_i \big[e^{\varepsilon\sigma_{ji}(t)}\alpha_i^2(t)) \\ &-\alpha_i^2(t-\sigma_{ji}(t))(1-\dot{\sigma}_{ji}(t)\big] \\ &\le e^{\varepsilon t} \sum_{i=1}^{n} \Big[\varepsilon\alpha_i^2(t) - 2a_i\alpha_i^2(t) \\ &+ \sum_{j=1}^{m} |c_{ij}| k_j \left(\alpha_i^2(t) + \beta_j^2(t)\right) \\ &+ \sum_{j=1}^{m} |d_{ij}| k_j \left(\alpha_i^2(t) + \beta_j^2(t-\tau_{ij}(t))\right)\Big] \\ &+e^{\varepsilon t} \sum_{i=1}^{n} \Big[\sum_{k=1}^{n} |w_{ik}| \left(\alpha_i^2(t) + \alpha_k^2(t)\right)\Big] \\ &+e^{\varepsilon t} \sum_{j=1}^{m} \Big[\varepsilon\beta_j^2(t) - 2b_j\beta_j^2(t) \\ &+ \sum_{i=1}^{n} |p_{ji}| h_i \left(\beta_j^2(t) + \alpha_i^2(t)\right)\Big] \\ &+e^{\varepsilon t} \sum_{j=1}^{m} \Big[\sum_{i=1}^{n} |q_{ji}| h_i \left(\beta_j^2(t) + \alpha_i^2(t-\sigma_{ji}(t))\right) \\ &+ \sum_{l=1}^{m} |\gamma_{jl}| \left(\beta_j^2(t) + \beta_l^2(t)\right)\Big] \\ &+e^{\varepsilon t}(1-\tau)^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} |d_{ij}|k_j \Big[e^{\varepsilon\tau_{ij}(t)}\beta_j^2(t) \\ &-\beta_j^2(t-\tau_{ij}(t))(1-\dot{\tau}_{ij}(t))\Big] \\ &+e^{\varepsilon t}(1-\sigma)^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} |q_{ji}|h_i \big[e^{\varepsilon\sigma_{ji}(t)}\alpha_i^2(t) \\ &-\alpha_i^2(t-\sigma_{ji}(t))(1-\dot{\sigma}_{ji}(t))\big] \\ &\le e^{\varepsilon t} \sum_{i=1}^{n} \Big[\varepsilon - 2a_i + \sum_{j=1}^{m} (|c_{ij}| + |d_{ij}|) k_j \\ &+ \sum_{k=1}^{n} |w_{ik}| + \sum_{k=1}^{n} |w_{ki}| \\ &+h_i \sum_{j=1}^{m} \left(|p_{ji}| + (1-\sigma)^{-1}e^{\varepsilon\sigma^*} |q_{ji}|\right)\Big] \alpha_i^2(t) \\ &+e^{\varepsilon t} \sum_{j=1}^{m} [\varepsilon - 2b_j + \sum_{i=1}^{n} (|p_{ji}| + |q_{ji}|) h_i \\ &+ \sum_{l=1}^{m} |\gamma_{jl}| + \sum_{l=1}^{m} |\gamma_{lj}| \\ &+k_j \sum_{i=1}^{n} \left(|c_{ij}| + (1-\tau)^{-1}e^{\varepsilon\tau^*} |d_{ij}|\right)\Big] \beta_j^2(t), \end{aligned}$$

By (13), (14) and Lemma 3, we can find that $D^+V(t) \le 0$, and so $V(t) \le V(0)$ for all $t \ge 0$.

From (16), it is easy to see that

$$e^{\varepsilon t}\left[\sum_{i=1}^{n}\alpha_i^2(s)+\sum_{j=1}^{n}\beta_j^2(s)\right]\leq\sum_{i=1}^{n}\alpha_i^2(0)+\sum_{j=1}^{n}\beta_j^2(0)$$
$$+(1-\tau)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}|d_{ij}|k_j\int_{-\tau_{ij}(0)}^{0}\beta_j^2(s)ds$$
$$+(1-\sigma)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}|q_{ji}|h_i\int_{-\sigma_{ji}(0)}^{0}\alpha_i^2(s)ds$$
$$\leq\sum_{i=1}^{n}|\alpha_i(0)|^2+\sum_{j=1}^{n}\left|\beta_j(0)\right|^2$$
$$+(1-\tau)^{-1}\sum_{i=1}^{n}\max_{1\leq j\leq m}\left(|d_{ij}|\,k_j\right)\int_{-\tau^*}^{0}\sum_{j=1}^{m}\beta_j^2(s)ds$$
$$+(1-\sigma)^{-1}\sum_{i=1}^{n}\max_{1\leq i\leq n}\left(|q_{ji}|\,h_i\right)\int_{-\sigma^*}^{0}\sum_{i=1}^{n}\alpha_i^2(s)ds,$$
$$\leq\left[1+(1-\sigma)^{-1}\sigma^*\sum_{i=1}^{n}\max_{1\leq i\leq n}\left(|q_{ji}|\,h_i\right)\right]$$
$$\cdot\sup_{-\sigma^*\leq s\leq 0}\sum_{i=1}^{n}|\alpha_i(s)|^2$$
$$+\left[1+(1-\tau)^{-1}\tau^*\sum_{i=1}^{n}\max_{1\leq j\leq m}\left(|d_{ij}|\,k_j\right)\right]$$
$$\cdot\sup_{-\tau^*\leq s\leq 0}\sum_{j=1}^{m}|\beta_j(s)|^2$$
$$\leq M_1\left[\|\varphi(s)-\phi(s)\|+\left\|\bar{\varphi}(s)-\bar{\phi}(s)\right\|\right]\quad t\geq 0,$$

where

$$M_1=\max\left\{1+(1-\tau)^{-1}\tau^*\sum_{i=1}^{n}\max_{1\leq j\leq m}\left(|d_{ij}|\,k_j\right),\right.$$
$$\left.1+(1-\sigma)^{-1}\sigma^*\sum_{i=1}^{n}\max_{1\leq i\leq n}\left(|q_{ji}|\,h_i\right)\right\}\geq 1.$$

That is

$$\|x(t)-z(t)\|+\|y(t)-w(t)\|$$
$$\leq M\left[\|\varphi(s)-\phi(s)\|+\left\|\bar{\varphi}(s)-\bar{\phi}(s)\right\|\right]e^{-\lambda t}(t\geq 0).$$

This completes the proof. □

Theorem 6 *The drive system (1) and the response system (2) are globally exponentially synchronized if* (H_0),(H_1) *(11) and (12) hold.*

Proof. Construct the following Lyapunov functional

$$V(t)=\sum_{i=1}^{n}e^{\varepsilon t}|\alpha_i(t)|+\sum_{j=1}^{n}e^{\varepsilon t}|\beta_j(t)|$$
$$+(1-\tau)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}|d_{ij}|k_j$$
$$\cdot\int_{t-\tau_{ij}(t)}^{t}e^{\varepsilon(s+\tau_{ij}(s))}|\beta_j(s)|ds \tag{17}$$
$$+(1-\sigma)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}|q_{ji}|h_i$$
$$\cdot\int_{t-\sigma_{ji}(t)}^{t}e^{\varepsilon(s+\sigma_{ji}(s))}|\alpha_i(s)|ds.$$

Then

$$D^+V(t)=\varepsilon e^{\varepsilon t}\left[\sum_{i=1}^{n}|\alpha_i(t)|+\sum_{j=1}^{m}|\beta_j(t)|\right]$$
$$+e^{\varepsilon t}\sum_{i=1}^{n}\dot{\alpha}_i(t)\ sign\left(\alpha_i(t)\right)$$
$$+e^{\varepsilon t}\sum_{j=1}^{m}\dot{\beta}_j(t)\ sign\left(\beta_i(t)\right)$$
$$+(1-\tau)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}|d_{ij}|k_j\left[e^{\varepsilon(t+\tau_{ij}(t))}|\beta_j(t)|\right.$$
$$\left.-e^{\varepsilon t}|\beta_j(t-\tau_{ij}(t))|\left(1-\dot{\tau}_{ij}(t)\right)\right]$$
$$+(1-\sigma)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}|q_{ji}|h_i\left[e^{\varepsilon(t+\sigma_{ji}(t))}|\alpha_i(t)|\right.$$
$$\left.-e^{\varepsilon t}|\alpha_i(t-\sigma_{ji}(t))|\left(1-\dot{\sigma}_{ji}(t)\right)\right],$$
$$\leq e^{\varepsilon t}\sum_{i=1}^{n}\left[\varepsilon|\alpha_i(t)|-a_i|\alpha_i(t)|\right.$$
$$+\sum_{j=1}^{m}|c_{ij}|\ |F_j\left(\beta_j(t)\right)|$$
$$+\sum_{j=1}^{m}|d_{ij}|\ |F_j\left(\beta_j(t-\tau_{ij}(t))\right)|$$
$$\left.+sign(\alpha_i(t))u_i(t)\right]$$
$$+(1-\tau)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}|d_{ij}|k_j\left[e^{\varepsilon(t+\tau_{ij}(t))}|\beta_j(t)|\right.$$
$$\left.-e^{\varepsilon t}|\beta_j(t-\tau_{ij}(t))|\left(1-\dot{\tau}_{ij}(t)\right)\right]$$
$$+e^{\varepsilon t}\sum_{j=1}^{m}\left[\varepsilon|\beta_j(t)|-b_j|\beta_j(t)|\right.$$
$$+\sum_{i=1}^{n}|p_{ji}|\ |G_i\left(\alpha_i(t)\right)|$$
$$+\sum_{i=1}^{n}|q_{ji}|\ |G_i\left(\alpha_i(t-\sigma_{ji}(t))\right)|$$
$$\left.+sign(\beta_j(t))v_j(t)\right]$$
$$+(1-\sigma)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}|q_{ji}|h_i\left[e^{\varepsilon(t+\sigma_{ji}(t))}|\alpha_i(t)|\right.$$
$$\left.-e^{\varepsilon t}|\alpha_i(t-\sigma_{ji}(t))|\left(1-\dot{\sigma}_{ji}(t)\right)\right).$$

By assumptions (H_0) and (H_1), we have

$$D^+V(t)\leq e^{\varepsilon t}\sum_{i=1}^{n}\left[\varepsilon|\alpha_i(t)|-a_i|\alpha_i(t)|\right.$$
$$+\sum_{j=1}^{m}\left(|c_{ij}|\,k_j+(1-\tau)^{-1}e^{\varepsilon\tau^*}|d_{ij}|\,k_j\right)|\beta_j(t)|$$
$$\left.+sign(\alpha_i(t))u_i(t)\right]+e^{\varepsilon t}\sum_{j=1}^{m}\left[\varepsilon|\beta_j(t)|-b_j|\beta_j(t)|\right.$$
$$+\sum_{i=1}^{n}\left(|p_{ji}|\,h_i+(1-\sigma)^{-1}e^{\varepsilon\sigma^*}|q_{ji}|\,h_i\right)|\alpha_i(t)|$$
$$\left.+sign(\beta_j(t))v_j(t)\right]$$
$$=e^{\varepsilon t}\sum_{i=1}^{n}\left\{\left[\varepsilon-a_i+h_i\sum_{j=1}^{m}\left(|p_{ji}|+(1-\sigma)^{-1}e^{\varepsilon\sigma^*}|q_{ji}|\right)\right]\right.$$
$$\left.\cdot|\alpha_i(t)|+sign(\alpha_i(t))u_i(t)\right\}$$
$$+e^{\varepsilon t}\sum_{j=1}^{m}\left\{\left[\varepsilon-b_j+k_j\sum_{i=1}^{n}\left(|c_{ij}|+(1-\tau)^{-1}e^{\varepsilon\tau^*}|d_{ij}|\right)\right]\right.$$
$$\left.\cdot|\beta_j(t)|+sign(\beta_j(t))v_j(t)\right\}$$

$$= e^{\varepsilon t}\sum_{i=1}^{n}\left[\varepsilon - a_i + h_i\sum_{j=1}^{m}\left(|p_{ji}| + (1-\sigma)^{-1}e^{\varepsilon\sigma^*}|q_{ji}|\right) + \sum_{k=1}^{n}|w_{ki}|\right]|\alpha_i(t)| + e^{\varepsilon t}\sum_{j=1}^{m}\left[\varepsilon - b_j + k_j\sum_{i=1}^{n}\left(|c_{ij}| + (1-\tau)^{-1}e^{\varepsilon\tau^*}|d_{ij}|\right) + \sum_{l=1}^{m}|\gamma_{lj}|\right]|\beta_j(t)|.$$

By conditions (11), (12) and Lemma 3, we can find that $D^+V(t) \le 0$, and so $V(t) \le V(0)$, for all $t \ge 0$.

From (17), it follows that

$$\begin{aligned}
&e^{\varepsilon t}\left[\sum_{i=1}^{n}|\alpha_i(t)| + \sum_{j=1}^{n}|\beta_j(t)|\right]\\
&\le \sum_{i=1}^{n}|\alpha_i(0)| + \sum_{j=1}^{n}|\beta_j(0)|\\
&+(1-\tau)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}|d_{ij}|k_j\\
&\cdot\int_{-\tau_{ij}(0)}^{0}e^{\varepsilon(s+\tau_{ij}(s))}|\beta_j(s)|ds\\
&+(1-\sigma)^{-1}\sum_{j=1}^{m}\sum_{i=1}^{n}|q_{ji}|h_i\\
&\cdot\int_{-\sigma_{ji}(0)}^{0}e^{\varepsilon(s+\sigma_{ji}(s))}|\alpha_i(s)|ds,\\
&\le \sum_{i=1}^{n}|\alpha_i(0)| + \sum_{j=1}^{n}|\beta_j(0)|\\
&+(1-\tau)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}|d_{ij}|k_j\int_{-\tau^*}^{0}e^{\varepsilon(s+\tau^*)}|\beta_j(s)|ds\\
&+(1-\sigma)^{-1}\sum_{j=1}^{m}\sum_{i=1}^{n}|q_{ji}|h_i\int_{-\sigma^*}^{0}e^{\varepsilon(s+\sigma^*)}|\alpha_i(s)|ds\\
&\le \sup_{-\sigma^*\le s\le 0}\sum_{i=1}^{n}|\alpha_i(s)| + \sup_{-\tau^*\le s\le 0}\sum_{j=1}^{m}|\beta_j(s)|\\
&+(1-\tau)^{-1}\sup_{-\tau^*\le s\le 0}\sum_{j=1}^{m}|\beta_j(s)|\\
&\cdot\sum_{i=1}^{n}\max_{1\le j\le m}(|d_{ij}|\,k_j)\int_{-\tau^*}^{0}e^{\varepsilon(s+\tau^*)}ds\\
&+(1-\sigma)^{-1}\sup_{-\sigma^*\le s\le 0}\sum_{i=1}^{n}|\alpha_i(s)|\\
&\cdot\sum_{i=1}^{n}\max_{1\le i\le n}(|q_{ji}|\,h_i)\int_{-\sigma^*}^{0}e^{\varepsilon(s+\sigma^*)}ds\\
&\le \left[1 + (1-\sigma)^{-1}\sum_{j=1}^{m}\max_{1\le i\le n}(|q_{ji}|\,h_i)\frac{e^{\varepsilon\sigma^*}}{\varepsilon}\right]\\
&\cdot\sup_{-\sigma^*\le s\le 0}\sum_{i=1}^{n}|\alpha_i(s)|\\
&+\left[1 + (1-\tau)^{-1}\sum_{i=1}^{n}\max_{1\le j\le m}(|d_{ij}|\,k_j)\frac{e^{\varepsilon\tau^*}}{\varepsilon}\right]\\
&\cdot\sup_{-\tau^*\le s\le 0}\sum_{j=1}^{m}|\beta_j(s)|\\
&\le M\left[\|\varphi(s) - \phi(s)\| + \left\|\bar\varphi(s) - \bar\phi(s)\right\|\right] \quad t \ge 0,
\end{aligned}$$

where

$$M = \max\left\{1 + (1-\sigma)^{-1}\sum_{i=1}^{n}\max_{1\le i\le n}(|q_{ji}|\,h_i)\frac{e^{\varepsilon\sigma^*}}{\varepsilon},\ 1 + (1-\tau)^{-1}\sum_{i=1}^{n}\max_{1\le j\le m}(|d_{ij}|\,k_j)\frac{e^{\varepsilon\tau^*}}{\varepsilon}\right\} \ge 1.$$

That is

$$\begin{aligned}
&\|x(t) - z(t)\| + \|y(t) - w(t)\|\\
&\le M\left[\|\varphi(s) - \phi(s)\| + \left\|\bar\varphi(s) - \bar\phi(s)\right\|\right]e^{-\lambda t}(t \ge 0).
\end{aligned}$$

By Definition 2, the drive system (1) and the response system (2) are globally exponentially synchronized.□

Theorem 7 *The drive system (1) and the response system (2) are globally exponentially synchronized if $(H_0)$and (H_1) hold, moreover,*

$$-\delta + \max\{A_1, B_1\} + \max\{C_1, D_1\} < 0, \quad (18)$$

where $\delta = \max_{\substack{1\le i\le n\\1\le j\le m}}\{w_{ii} - a_i,\ \gamma_{jj} - b_j\} > 0,$

$$\begin{aligned}
A_1 &= \sum_{j=1}^{m}\max_{1\le i\le n}\{|p_{ji}|\,h_i\} + \sum_{k=1}^{n}\max_{\substack{1\le i\le n\\ i\ne k}}\{|w_{ki}|\},\\
B_1 &= \sum_{i=1}^{n}\max_{1\le j\le m}\{|c_{ij}|\,k_j\} + \sum_{l=1}^{m}\max_{\substack{1\le j\le m\\ j\ne l}}\{|\gamma_{lj}|\},\\
C_1 &= \sum_{j=1}^{m}\max_{1\le i\le m}\{|q_{ji}|\,h_i\},\\
D_1 &= \sum_{i=1}^{m}\max_{1\le j\le m}\{|d_{ij}|\,k_j\}.
\end{aligned}$$

Proof. From (5), (6) and (7), it follows that

$$\begin{aligned}
\dot\alpha_i(t) = &-a_i\alpha_i(t) + \sum_{j=1}^{m}c_{ij}F_j(\beta_j(t))\\
&+\sum_{j=1}^{m}d_{ij}F_j(\beta_j(t - \tau_{ij}(t)))\\
&+w_{ii}\alpha_i(t) + \sum_{\substack{k=1\\k\ne i}}^{n}w_{ik}\alpha_k(t),
\end{aligned} \quad (19)$$

$$\begin{aligned}
\dot\beta_j(t) = &-b_j\beta_j(t) + \sum_{i=1}^{n}p_{ji}G_i(\alpha_i(t))\\
&+\sum_{i=1}^{n}q_{ji}G_i(\alpha_i(t - \sigma_{ji}(t)))\\
&+\gamma_{jj}\beta_j(t) + \sum_{\substack{l=1\\l\ne j}}^{n}\gamma_{jl}\beta_l(t).
\end{aligned} \quad (20)$$

From (19), we have

$$\begin{aligned}
&\dot\alpha_i(t) + (a_i - w_{ii})\alpha_i(t) = \sum_{j=1}^{m}c_{ij}F_j(\beta_j(t))\\
&+\sum_{j=1}^{m}d_{ij}F_j(\beta_j(t - \tau_{ij}(t))) + \sum_{\substack{k=1\\k\ne i}}^{n}w_{ik}\alpha_k(t).
\end{aligned} \quad (21)$$

Multiplying both sides of (21) with $e^{(a_i-w_{ii})t}$, yields

$$e^{(a_i-w_{ii})t}\left[\dot{\alpha}_i(t)+(a_i-w_{ii})\alpha_i(t)\right] = e^{(a_i-w_{ii})t}\left[\sum_{j=1}^{m} c_{ij}F_j\left(\beta_j(t)\right) + \sum_{j=1}^{m} d_{ij}F_j\left(\beta_j(t-\tau_{ij}(t))\right)+\sum_{\substack{k=1\\k\neq i}}^{n} w_{ik}\alpha_k(t)\right]. \tag{22}$$

Integrate both sides of (22), we have

$$e^{(a_i-w_{ii})t}\alpha_i(t)=\alpha_i(0)+\int_0^t e^{(a_i-w_{ii})s} \cdot\sum_{j=1}^{m}\left[c_{ij}F_j\left(\beta_j(s)\right)+\sum_{j=1}^{m} d_{ij}F_j\left(\beta_j(s-\tau_{ij}(s))\right) + \sum_{\substack{k=1\\k\neq i}}^{n} w_{ik}\alpha_k(s)\right]ds,$$

$$\alpha_i(t)=e^{(w_{ii}-a_i)t}\alpha_i(0)+\int_0^t e^{(a_i-w_{ii})(s-t)} \cdot\sum_{j=1}^{m}\left[c_{ij}F_j\left(\beta_j(s)\right)+\sum_{j=1}^{m} d_{ij}F_j\left(\beta_j(s-\tau_{ij}(s))\right) + \sum_{\substack{k=1\\k\neq i}}^{n} w_{ik}\alpha_k(s)\right]ds,$$

then

$$\sum_{i=1}^{n}|\alpha_i(t)|\le\sum_{i=1}^{n}\Big\{|\alpha_i(0)|\,e^{(w_{ii}-a_i)t} + \int_0^t e^{(a_i-w_{ii})(s-t)}\left[\sum_{j=1}^{m}|c_{ij}|\,k_j\,|\beta_j(s)| + \sum_{j=1}^{m}|d_{ij}|\,k_j\,|\beta_j(s-\tau_{ij}(s))| + \sum_{\substack{k=1\\k\neq i}}^{n}|w_{ik}|\,|\alpha_k(s)|\right]ds\Big\}. \tag{23}$$

Similar to the above deductions, from (20), we obtain

$$\sum_{j=1}^{m}|\beta_j(t)|\le\sum_{j=1}^{m}\Big\{|\beta_j(0)|\,e^{(\gamma_{jj}-b_j)t} + \int_0^t e^{(b_j-\gamma_{jj})(s-t)}\left[\sum_{i=1}^{n}p_{ji}h_i\,|\alpha_i(s)| + \sum_{i=1}^{n}q_{ji}h_i\,|\alpha_i(s-\sigma_{ji}(s))| + \sum_{\substack{l=1\\l\neq j}}^{m}|\gamma_{jl}|\,|\beta_l(s)|\right]ds\Big\}. \tag{24}$$

Let

$$A=\sum_{i=1}^{n}\Big\{|\alpha_i(0)|\,e^{(w_{ii}-a_i)t} + \int_0^t e^{(a_i-w_{ii})(s-t)}\left[\sum_{j=1}^{m}|c_{ij}|\,k_j\,|\beta_j(s)| + \sum_{j=1}^{m}|d_{ij}|\,k_j\,|\beta_j(s-\tau_{ij}(s))| + \sum_{\substack{k=1\\k\neq i}}^{n}|w_{ik}|\,|\alpha_k(s)|\right]ds\Big\},$$

$$B=\sum_{j=1}^{m}\Big\{|\beta_j(0)|\,e^{(\gamma_{jj}-b_j)t} + \int_0^t e^{(b_j-\gamma_{jj})(s-t)}\left[\sum_{i=1}^{n}p_{ji}h_i\,|\alpha_i(s)| + \sum_{i=1}^{n}q_{ji}h_i\,|\alpha_i(s-\sigma_{ji}(s))| + \sum_{\substack{l=1\\l\neq j}}^{m}|\gamma_{jl}|\,|\beta_l(s)|\right]ds\Big\}.$$

Then, we have $\sum_{i=1}^{n}|\alpha_i(t)|+\sum_{j=1}^{m}|\beta_j(t)|\le A+B$.

Let $P(t)=A+B$, $\eta=\max_{\substack{1\le i\le n,\\1\le j\le m}}\{\tau^*,\sigma^*\}$, $\bar{P}(t)=\sup_{t-\eta\le s\le t}P(s)$. Then

$$P(t)\ge\sum_{i=1}^{n}|\alpha_i(t)|+\sum_{j=1}^{m}|\beta_j(t)|, \tag{25}$$

and

$$\bar{P}(t)\ge\sup_{t-\eta\le s\le t}\left\{\sum_{i=1}^{n}|\alpha_i(s)|+\sum_{j=1}^{m}|\beta_j(s)|\right\},$$

$$\dot{P}(t)\le-\delta P(t)+\sum_{i=1}^{n}\left\{\sum_{j=1}^{m}|c_{ij}|\,k_j\,|\beta_j(t)| + \sum_{j=1}^{m}|d_{ij}|\,k_j\,|\beta_j(t-\tau_{ij}(t))|+\sum_{\substack{k=1\\k\neq i}}^{n}|w_{ik}|\,|\alpha_k(t)|\right\} + \sum_{j=1}^{m}\left\{\sum_{i=1}^{n}|p_{ji}|\,h_i\,|\alpha_i(t)| + \sum_{i=1}^{n}|q_{ji}|\,h_i\,|\alpha_i(t-\sigma_{ji}(t))|+\sum_{\substack{l=1\\l\neq j}}^{m}|\gamma_{jl}|\,|\beta_l(t)|\right\}, \tag{26}$$

$$
\begin{aligned}
&\dot{P}(t) \le -\delta P(t) + \sum_{j=1}^{m} \max_{1\le i\le n} \{|p_{ji}|\, h_i\} \sum_{i=1}^{n} |\alpha_i(t)| \\
&+ \sum_{i=1}^{n} \max_{1\le j\le m} \{|c_{ij}|\, k_j\} \sum_{j=1}^{m} |\beta_j(t)| \\
&+ \sum_{j=1}^{m} \max_{1\le i\le n} \{|q_{ji}|\, h_i\} \sum_{i=1}^{n} |\alpha_i(t-\sigma_{ji}(t))| \\
&+ \sum_{i=1}^{n} \max_{1\le j\le m} \{|d_{ij}|\, k_j\} \sum_{j=1}^{m} |\beta_j(t-\tau_{ij}(t))| \\
&+ \sum_{k=1}^{n} \max_{\substack{1\le i\le n,\\ i\ne k}} \{|w_{ki}|\} \sum_{i=1}^{n} |\alpha_i(t)| \\
&+ \sum_{l=1}^{m} \max_{\substack{1\le j\le n,\\ i\ne l}} \{|\gamma_{lj}|\} \sum_{j=1}^{m} |\beta_j(t)| \\
&= -\delta P(t) + A_1 \sum_{i=1}^{n} |\alpha_i(t)| + B_1 \sum_{j=1}^{m} |\beta_j(t)| \\
&+ C_1 \sum_{i=1}^{n} |\alpha_i(t-\sigma_{ji}(t))| + D_1 \sum_{j=1}^{m} |\beta_j(t-\tau_{ij}(t))| \\
&\le -\delta P(t) + \max\{A_1, B_1\}\, P(t) \\
&+ \max\{C_1, D_1\}\, \bar{P}(t) \\
&= -[\delta - \max\{A_1, B_1\}]\, P(t) \\
&+ \max\{C_1, D_1\}\, \bar{P}(t),
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\dot{P}(t) \le& -[\delta - \max\{A_1, B_1\}]\, P(t) \\
&+ \max\{C_1, D_1\}\, \bar{P}(t),
\end{aligned}
\tag{27}
$$

By Lemma 4, from (27), there exists $k > 0$, such that

$$P(t) \le \bar{P}(0) e^{-kt}, t \ge 0, \tag{28}$$

and

$$\bar{P}(0) = \sup_{-\eta\le s\le 0} P(0) \le \sum_{i=1}^{n} |\alpha_i(0)| + \sum_{j=1}^{m} |\beta_j(0)|\,. \tag{29}$$

Substitute (29) into (28), and by (25), we have

$$
\begin{aligned}
&\|x(t)-z(t)\| + \|y(t)-w(t)\| \\
&\le M\left[\|\varphi(s)-\phi(s)\| + \left\|\bar{\varphi}(s)-\bar{\phi}(s)\right\|\right] e^{-\lambda t} (t \ge 0),
\end{aligned}
$$

where $M = 1, \lambda = k > 0$.

By Definition 2, we have that the drive system (1) and the response system (2) are globally exponentially synchronized. □

Remark 8 *Theorem 5-7 are presented by the use of different Lemmas and various analysis techniques. They provide three sufficient conditions to ensure the exponential synchronization of system (1) and (2). Compare Theorem 5 with Theorem 6, we find that conditions (11) and (12) in Theorem 6 are more simple to be verified than conditions (13) and (14) in Theorem 5. So, from the point of practical application, Theorem 6 is more suitable for the exponential synchronization of BAM neural networks with varying delays.*

Remark 9 *From conditions (11) and (12), we get that*

$$
\begin{aligned}
a_i >& \sum_{j=1}^{m} \left(|p_{ji}| + (1-\sigma)^{-1} |q_{ji}|\right) h_i \\
&+ \sum_{k=1}^{n} |w_{ki}|, i = 1, 2, \cdots n,
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
b_j >& \sum_{i=1}^{n} \left(|c_{ij}| + (1-\tau)^{-1} |d_{ij}|\right) k_j \\
&+ \sum_{l=1}^{m} |\gamma_{lj}|\,, j = 1, 2, \cdots m,
\end{aligned}
\tag{31}
$$

while from condition (18), it follows that

$$
\begin{aligned}
&a_i < w_{ii} - \max\{A_1, B_1\} - \max\{C_1, D_1\}\,, \\
&\quad i = 1, 2, \cdots n,
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
&b_j < \gamma_{jj} - \max\{A_1, B_1\} - \max\{C_1, D_1\}\,, \\
&\quad j = 1, 2, \cdots m,
\end{aligned}
\tag{33}
$$

where A_1, B_1, C_1, D_1 are the same as that in Theorem 7.

Then, we get the following two Corollaries.

Corollary 10 *The drive system (1) and the response system (2) are globally exponentially synchronized if (H_0), (H_1) (30) and (31) hold.*

Corollary 11 *The drive system (1) and the response system (2) are globally exponentially synchronized if (H_0), (H_1) (32) and (32) hold.*

Remark 12 *From Corollary 10 and 11, if parameters $a_i(i = 1, 2, \cdots, n)$, $b_j(j = 1, 2, \cdots, m)$ are larger than certain values, then we can judge the synchronization of (1) and (2) by Theorem 5-6 or Corollary 10. On the other hand, if parameters $a_i(i = 1, 2, \cdots, n)$, $b_j(j = 1, 2, \cdots, m)$ are smaller than certain values, then we can use Theorem 7 or Corollary 11 to decide the synchronization of (1) and (2). Therefore, Theorem 5-7 can be used to different systems to ensure the synchronization between the master system and the slave system. In the following, we will give two examples to show the usefulness of Theorem 5-7.*

4 Numerical Examples

In this section, we will give two examples to illustrate our results.

Example 1 Consider the following second-order BAM neural networks with time-varying delays

$$\begin{cases} \frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum\limits_{j=1}^{2} c_{ij} f_j\left(y_j(t)\right) \\ + \sum\limits_{j=1}^{2} d_{ij} f_j\left(y_j(t-\tau_{ij}(t))\right) + I_i, \\ \frac{dy_j(t)}{dt} = -b_j y_j(t) + \sum\limits_{i=1}^{2} p_{ji} g_i\left(x_i(t)\right) \\ + \sum\limits_{i=1}^{2} q_{ji} g_i\left(x_i(t-\sigma_{ji}(t))\right) + J_j, \end{cases} \tag{34}$$

where $f_1(x) = f_2(x) = g_1(x) = g_2(x) = \sin\left(\frac{x}{2}\right)$, we select $k_1 = k_2 = h_1 = h_2 = 0.5$, The delays $\tau_{ij}(t) = 0.4(1-\cos t), \sigma_{ji}(t) = 0.6(1-\cos t)$, and satisfy

$0 \le \tau_{ij}(t) \le 2 = \tau^*, \dot{\tau}_{ij}(t) \le 0.4 = \tau, 0 \le \sigma_{ji}(t) \le 2 = \sigma^*, \dot{\sigma}_{ji}(t) \le 0.6 = \sigma, i,j = 1,2;$

Let $a_1 = 3, a_2 = 4, b_1 = 3, b_2 = 4, p_{11} = 0.3, p_{21} = -0.7, p_{12} = -0.2, p_{22} = 0.8; q_{11} = -0.1, q_{21} = 0.3, q_{12} = 0.2, q_{22} = -0.2, c_{11} = -0.8, c_{21} = 1.2, c_{12} = 1, c_{22} = -1; d_{11} = 0.2, d_{21} = 0.4, d_{12} = -0.3, d_{22} = -0.3$; and the response system is designed by

$$\begin{cases} \frac{dz_i(t)}{dt} = -a_i z_i(t) + \sum\limits_{j=1}^{2} c_{ij} f_j\left(w_j(t)\right) \\ + \sum\limits_{j=1}^{2} d_{ij} f_j\left(w_j(t-\tau_{ij}(t))\right) + I_i - u_i(t), \\ \frac{dw_j(t)}{dt} = -b_j w_j(t) + \sum\limits_{i=1}^{2} p_{ji} g_i\left(z_i(t)\right) \\ + \sum\limits_{i=1}^{2} q_{ji} g_i\left(z_i(t-\sigma_{ji}(t))\right) + J_j - v_j(t), \end{cases} \tag{35}$$

$$u_i(t) = \sum_{k=1}^{2} w_{ik}\left(x_k(t) - z_k(t)\right) = \sum_{k=1}^{2} w_{ik}\alpha_k(t),$$

$$v_j(t) = \sum_{l=1}^{2} \gamma_{jl}\left(y_l(t) - w_l(t)\right) = \sum_{l=1}^{2} \gamma_{jl}\beta_j(t),$$

$i,j = 1,2$,the controller gain coefficients are chosen as $w_{11} = -1, w_{21} = 0.5, w_{12} = 2, w_{22} = -1; \quad \gamma_{11} = -0.5, \gamma_{21} = 0.5, \gamma_{12} = 1, \gamma_{22} = -1.$

By simple computation, it is not difficult to find that all the above parameters satisfy (11) and (12). Thus, by Theorem 6, system (34) and (35) are exponential synchronization. Simulation results with 10 random initial points are shown in Fig.1-Fig.3.

Example 2 For system (34) and (35), let $f_1(x) = f_2(x) = g_1(x) = g_2(x) = \tanh x$, we select $k_1 = k_2 = h_1 = h_2 = 1$. The delays $\tau_{ij}(t) = 0.5(1+\sin t), \sigma_{ji}(t) = 0.3(1+\sin t)$, and satisfy$0 \le$

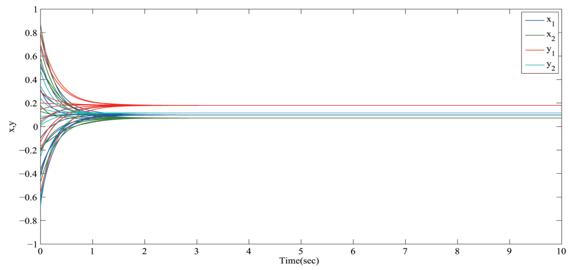

Figure 1: The state diagram of system (34) in example 1 with 10 random initial points.

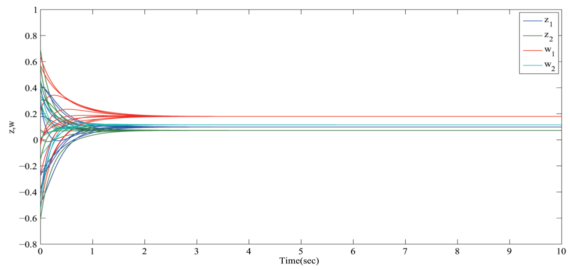

Figure 2: The state diagram of system (35) in example 1 with 10 random initial points.

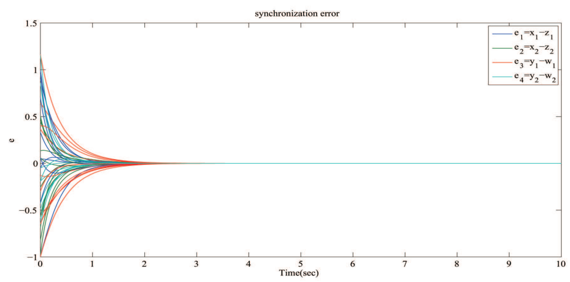

Figure 3: The synchronization error of (34) and (35) in example 1 with 10 random initial points.

$\tau_{ij}(t) \leq 1 = \tau^*, \dot{\tau}_{ij}(t) \leq 0.5 = \tau, 0 \leq \sigma_{ji}(t) \leq 0.6 = \sigma^*, \dot{\sigma}_{ji}(t) \leq 0.3 = \sigma, i, j = 1, 2;$

The other parameters $a_i, b_j, c_{ij,} d_{ij,} p_{ji}, q_{ji}, i, j = 1, 2$ are the same as those in Example 1.

Select the controller gain coefficients are chosen as $w_{11} = 4, w_{21} = -1.5, w_{12} = -1.2, w_{22} = 5;$ $\gamma_{11} = 4, \gamma_{21} = -0.8, \gamma_{12} = -1.2, \gamma_{22} = 5.0$ By simple computation, we find that all the above parameters satisfy (18). Thus, by Theorem 7, system (34) and (35) are exponential synchronization. Simulation results with 10 random initial points are shown in Fig.4-Fig.6.

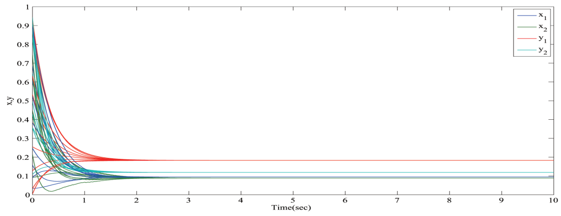

Figure 4: The state diagram of system (34) in example 2 with 10 random initial points.

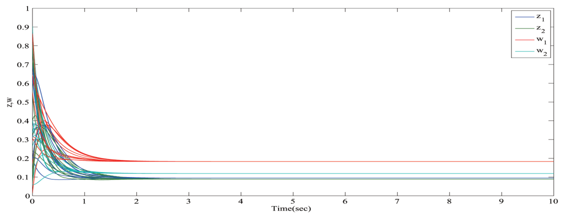

Figure 5: The state diagram of system (35) in example 2 with 10 random initial points.

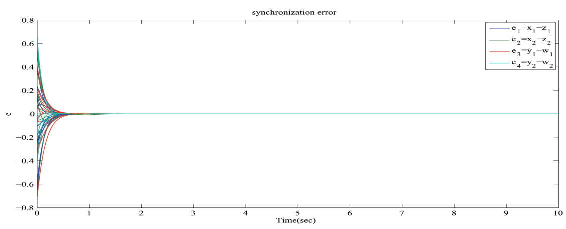

Figure 6: The synchronization error of (34) and (35) in example 2 with 10 random initial points.

Remark 13 *In Example 1, since* $w_{ii} - a_i < 0, \gamma_{jj} - b_j < 0, i, j = 1, 2,$ *then* $\delta = \min_{\substack{1 \leq i \leq n, \\ 1 \leq j \leq m}} \{w_{ii} - a_i,\ \gamma_{jj} - b_j\} < 0,$ *which contradict to condition (18). Therefore, the exponential synchronization of (34) and (35) in Example 1 can't be obtained by Theorem 7. On the other hand, In Example 2, since* $-a_i + w_{ii} > 0, \quad -b_j + \gamma_{jj} > 0, i, j = 1, 2,$ *then conditions (11) and (12) aren't satisfied. Thus, Theorem 6 aren't suitable for the exponential synchronization of (34) and (35) in Example 2. The examples 1 and 2 show that all the Theorems 5-7 in this paper can be used to different problems.*

5 Conclusions

Based on Lyapunov functional, analysis techniques and differential equation theory, three sufficient global exponential synchronization conditions are derived for a class of BAM neural networks with time-varying delays. The given algebra conditions are easily verifiable and useful in different models. Two examples and synchronization simulations with 10 random initial values are given to illustrate the effectiveness of our results.

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