Interval Oscillation Criteria For A Class Of Nonlinear Fractional Differential Equations

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Abstract: In this work, some new interval oscillation criteria for solutions of a class of nonlinear fractional differential equations are established by using a generalized Riccati function and inequality technique. For illustrating the validity of the established results, we also present some applications for them.

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1 Introduction

Recently, research for oscillation of various equations including differential equations, difference equations and dynamic equations on time scales etc. has been a hot topic in the literature, and much effort has been done to establish new oscillation criteria for these equations so far (for example, see [1-27], and the references therein). In these investigations, we notice that very little attention is paid to oscillation of fractional differential equations. Recent results in this direction only include Chen's work [28].

In this paper, we are concerned with oscillation of solutions of the nonlinear fractional differential equation of the following form:

$$\begin{aligned}&(a(t)[(r(t)D_-^\alpha x(t))']^\gamma)'\\&-q(t)f(\textstyle\int_t^\infty(\xi-t)^{-\alpha}x(\xi)d\xi)=0,\\&t\in[t_0,\infty),\end{aligned}\tag{1}$$

where $a\in C^1([t_0,\infty),\mathbb{R}_+)$, $r\in C^2([t_0,\infty),\mathbb{R}_+)$, $q\in C([t_0,\infty),\mathbb{R}_+)$, $f\in C(\mathbb{R},\mathbb{R})$ satisfying $xf(x)>0$, $\frac{f(x)}{x^\gamma}\geq L>0$ for $x\neq 0$, γ is a quotient of two odd positive integers, $\alpha\in(0,1)$, $D_-^\alpha x(t)$ denotes the Liouville right-sided fractional derivative of order α of x, and $D_-^\alpha x(t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^\infty(\xi-t)^{-\alpha}x(\xi)d\xi$.

A solution of Eq. (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Eq. (1) is said to be oscillatory in case all its solutions are oscillatory.

Motivated by the idea in [29], we will establish some new interval oscillation criteria for Eq. (1) by a generalized Riccati function and inequality technique in Section 2, and present some applications for our results in Section 3. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+=(0,\infty)$. For more details about the theory of fractional differential equations, we refer the reader to [30-32].

2 Main Results

For the sake of convenience, in the rest of this paper, we set

$$X(t)=\int_t^\infty(\xi-t)^{-\alpha}x(\xi)d\xi,$$

$$\delta_1(t,a)=\int_a^t\frac{1}{a^{\frac{1}{\gamma}}(s)}ds,\ \delta_2(t,a)=\int_a^t\frac{\delta_1(s,a)}{r(s)}ds.$$

Lemma 1 *Assume x is a solution of Eq. (1). Then $X'(t)=-\Gamma(1-\alpha)D_-^\alpha x(t)$.*

Lemma 2 *Assume x is a eventually positive solution of Eq. (1), and*

$$\int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{\gamma}}(s)} ds = \infty, \tag{2}$$

$$\int_{t_0}^{\infty} \frac{1}{r(s)} ds = \infty, \tag{3}$$

$$\int_{t_0}^{\infty} \frac{1}{r(\xi)} \int_{\xi}^{\infty} [\frac{1}{a(\tau)} \int_{\tau}^{\infty} q(s)ds]^{\frac{1}{\gamma}} d\tau d\xi = \infty, \tag{4}$$

Then there exists a sufficiently large T such that

$$(r(t)D_{-}^{\alpha}x(t))' < 0$$

on $[T, \infty)$, and either $D_{-}^{\alpha}x(t) < 0$ on $[T, \infty)$ or $\lim_{t\to\infty} X(t) = 0$.

Proof. Since x is a eventually positive solution of (1), there exists t_1 such that $x(t) > 0$ on $[t_1, \infty)$. So $X(t) > 0$ on $[t_1, \infty)$, and we have

$$\begin{aligned} &(a(t)[(r(t)D_{-}^{\alpha}x(t))']^{\gamma})' = q(t)f(X(t)) \\ &\geq Lq(t)X^{\gamma}(t) > 0. \end{aligned} \tag{5}$$

Then $a(t)[(r(t)D_{-}^{\alpha}x(t))']^{\gamma}$ is strictly increasing on $[t_1, \infty)$, and thus $(r(t)D_{-}^{\alpha}x(t))'$ is eventually of one sign. We claim $(r(t)D_{-}^{\alpha}x(t))' < 0$ on $[t_2, \infty)$, where $t_2 > t_1$ is sufficiently large. Otherwise, assume there exists a sufficiently large $t_3 > t_2$ such that $(r(t)D_{-}^{\alpha}x(t))' > 0$ on $[t_3, \infty)$. Then for $t \in [t_3, \infty)$, we have

$$\begin{aligned} &r(t)D_{-}^{\alpha}x(t) - r(t_3)D_{-}^{\alpha}x(t_3) \\ &= \int_{t_3}^{t} \frac{a^{\frac{1}{\gamma}}(s)(r(s)D_{-}^{\alpha}x(s))'}{a^{\frac{1}{\gamma}}(s)} ds \\ &\geq a^{\frac{1}{\gamma}}(t_3)(r(t_3)D_{-}^{\alpha}x(t_3))' \int_{t_3}^{t} \frac{1}{a^{\frac{1}{\gamma}}(s)} ds. \end{aligned}$$

By (2), we have

$$\lim_{t\to\infty} r(t)D_{-}^{\alpha}x(t) = \infty,$$

which implies for some sufficiently large $t_4 > t_3$, $D_{-}^{\alpha}x(t) > 0,\ t \in [t_4, \infty)$. By Lemma 1, we have

$$\begin{aligned} &X(t) - X(t_4) = \int_{t_4}^{t} X'(s)ds \\ &= -\Gamma(1-\alpha) \int_{t_4}^{t} D_{-}^{\alpha}x(s)ds \\ &= -\Gamma(1-\alpha) \int_{t_4}^{t} \frac{r(s)D_{-}^{\alpha}x(s)}{r(s)} ds \\ &\leq -\Gamma(1-\alpha)r(t_4)D_{-}^{\alpha}x(t_4) \int_{t_4}^{t} \frac{1}{r(s)} ds. \end{aligned}$$

By (3), we obtain $\lim_{t\to\infty} X(t) = -\infty$, which contradicts $X(t) > 0$ on $[t_1, \infty)$. So $(r(t)D_{-}^{\alpha}x(t))' < 0$ on $[t_2, \infty)$. Thus $D_{-}^{\alpha}x(t)$ is eventually of one sign.

Now we assume $D_{-}^{\alpha}x(t) > 0,\ t \in [t_5, \infty)$ for some sufficiently $t_5 > t_4$. Then by Lemma 1, $X'(t) < 0$ for $t \in [t_5, \infty)$. Since $X(t) > 0$, furthermore we have $\lim_{t\to\infty} X(t) = \beta \geq 0$. We claim $\beta = 0$. Otherwise, assume $\beta > 0$. Then $X(t) \geq \beta$ on $[t_5, \infty)$, and for $t \in [t_5, \infty)$, by (5) we have

$$\begin{aligned} &(a(t)[(r(t)D_{-}^{\alpha}x(t))']^{\gamma})' \geq Lq(t)X^{\gamma}(t) \\ &\geq L\beta^{\gamma}q(t). \end{aligned} \tag{6}$$

Substituting t with s in (6), an integration for (6) with respect to s from t to ∞ yields

$$\begin{aligned} &-a(t)[(r(t)D_{-}^{\alpha}x(t))']^{\gamma} \\ &\geq -\lim_{t\to\infty} a(t)[(r(t)D_{-}^{\alpha}x(t))']^{\gamma} + L\beta^{\gamma} \int_{t}^{\infty} q(s)ds \\ &> L\beta^{\gamma} \int_{t_5}^{t} q(s)ds, \end{aligned}$$

which means

$$(r(t)D_{-}^{\alpha}x(t))' < -L^{\frac{1}{\gamma}}\beta[\frac{1}{a(t)} \int_{t}^{\infty} q(s)ds]^{\frac{1}{\gamma}}. \tag{7}$$

Substituting t with τ in (7), an integration for (7) with respect to τ from t to ∞ yields

$$\begin{aligned} &-r(t)D_{-}^{\alpha}x(t) \\ &< -\lim_{t\to\infty} r(t)D_{-}^{\alpha}x(t) - L^{\frac{1}{\gamma}}\beta \int_{t}^{\infty} [\frac{1}{a(\tau)} \int_{\tau}^{\infty} q(s)ds]^{\frac{1}{\gamma}} d\tau \\ &< -L^{\frac{1}{\gamma}}\beta \int_{t}^{\infty} [\frac{1}{a(\tau)} \int_{\tau}^{\infty} q(s)ds]^{\frac{1}{\gamma}} d\tau, \end{aligned}$$

that is,

$$X'(t) < -L^{\frac{1}{\gamma}}\Gamma(1-\alpha)\frac{\beta}{r(t)} \int_{t}^{\infty} [\frac{1}{a(\tau)} \int_{\tau}^{\infty} q(s)ds]^{\frac{1}{\gamma}} d\tau \ . \tag{8}$$

Substituting t with ξ in (8), an integration for (8) with respect to ξ from t_5 to t yields

$$\begin{aligned} &X(t) - X(t_5) \\ &< -L^{\frac{1}{\gamma}}\Gamma(1-\alpha)\beta \int_{t_5}^{t} \frac{1}{r(\xi)} \int_{\xi}^{\infty} [\frac{1}{a(\tau)} \int_{\tau}^{\infty} q(s)ds]^{\frac{1}{\gamma}} d\tau d\xi. \end{aligned}$$

By (4), one can see $\lim_{t\to\infty} X(t) = -\infty$, which is a contradiction. So the proof is complete.

Lemma 3 *Assume that x is a eventually positive solution of Eq. (1) such that*

$$(r(t)D_{-}^{\alpha}x(t))' < 0,\ D_{-}^{\alpha}x(t) < 0$$

on $[t_1, \infty)_{\mathbb{T}}$,
where $t_1 \geq t_0$ is sufficiently large. Then we have

$$X'(t) \geq -\frac{\Gamma(1-\alpha)\delta_1(t, t_1)a^{\frac{1}{\gamma}}(t)(r(t)D_{-}^{\alpha}x(t))'}{r(t)},$$

and

$$X(t) \geq -\Gamma(1-\alpha)\delta_2(t, t_1)a^{\frac{1}{\gamma}}(t)(r(t)D_{-}^{\alpha}x(t))'.$$

Proof. By Lemma 2, we have $a(t)[(r(t)D_-^\alpha x(t))']^\gamma$ is strictly increasing on $[t_1,\infty)$. So

$$\begin{aligned} &r(t)D_-^\alpha x(t) \le r(t)D_-^\alpha x(t) - r(t_1)D_-^\alpha x(t_1)\\ &= \int_{t_1}^t \frac{a^{\frac{1}{\gamma}}(s)[r(s)D_-^\alpha x(s)]'}{a^{\frac{1}{\gamma}}(s)}ds\\ &\le a^{\frac{1}{\gamma}}(t)(r(t)D_-^\alpha x(t))'\int_{t_1}^t \frac{1}{a^{\frac{1}{\gamma}}(s)}ds\\ &= \delta_1(t,t_1)a^{\frac{1}{\gamma}}(t)(r(t)D_-^\alpha x(t))'. \end{aligned}$$

Using Lemma 1 we obtain that

$$X'(t) \ge -\frac{\Gamma(1-\alpha)\delta_1(t,t_1)a^{\frac{1}{\gamma}}(t)(r(t)D_-^\alpha x(t))'}{r(t)}.$$

Then

$$\begin{aligned} &X(t) \ge X(t) - X(t_1)\\ &\ge -\int_{t_1}^t \frac{\Gamma(1-\alpha)\delta_1(s,t_1)a^{\frac{1}{\gamma}}(s)(r(t)D_-^\alpha x(s))'}{r(s)}ds\\ &\ge -\Gamma(1-\alpha)a^{\frac{1}{\gamma}}(t)(r(t)D_-^\alpha x(t))'\int_{t_1}^t \frac{\delta_1(s,t_1)}{r(s)}ds\\ &= -\Gamma(1-\alpha)\delta_2(t,t_1)a^{\frac{1}{\gamma}}(t)(r(t)D_-^\alpha x(t))'. \end{aligned}$$

Lemma 4 *[33, Theorem 41]. Assume that A and B are nonnegative real numbers. Then*

$$\lambda AB^{\lambda-1} - A^\lambda \le (\lambda-1)B^\lambda$$

for all $\lambda > 1$.

Theorem 5 *Assume (2)-(4) hold, and there exists two functions $\phi \in C^1([t_0,\infty),\mathbb{R}_+)$ and $\varphi \in C^1([t_0,\infty),[0,\infty))$ such that*

$$\begin{aligned} &\int_T^\infty \{L\phi(s)q(s) - \phi(s)\varphi'(s)\\ &\quad + \frac{\phi(s)\Gamma(1-\alpha)\delta_1(s,T)\varphi^{1+\frac{1}{\gamma}}(s)}{r(s)}\\ &-\frac{[(\gamma+1)\varphi^{\frac{1}{\gamma}}(s)\phi(s)\Gamma(1-\alpha)\delta_1(s,T)+r(s)\phi'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(s)\delta_1(s,T)]^\gamma r(s)}\}ds\\ &= \infty, \end{aligned} \tag{9}$$

for all sufficiently large T. Then every solution of Eq. (1) is oscillatory or satisfies $\lim_{t\to\infty} X(t) = 0$.

Proof. Assume (1) has a nonoscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 2 we have $(r(t)D_-^\alpha x(t))' < 0$, $t \in [t_2,\infty)$, where $t_2 > t_1$ is sufficiently large, and either $D_-^\alpha x(t) < 0$ on $[t_2,\infty)$ or $\lim_{t\to\infty} X(t) = 0$. Define the generalized Riccati function:

$$\omega(t) = \phi(t)\{-\frac{a(t)[(r(t)D_-^\alpha x(t))']^\gamma}{X^\gamma(t)} + \varphi(t)\}.$$

Then for $t \in [t_2,\infty)$, we have

$$\begin{aligned} &\omega'(t) = -\phi'(t)\frac{a(t)[(r(t)D_-^\alpha x(t))']^\gamma}{X^\gamma(t)}\\ &+\phi(t)\{-\frac{a(t)[(r(t)D_-^\alpha x(t))']^\gamma}{X^\gamma(t)}\}' + \phi'(t)\varphi(t) + \phi(t)\varphi'(t)\\ &= -\phi(t)\{\frac{(a(t)[(r(t)D_-^\alpha x(t))']^\gamma)'}{X^\gamma(t)}\}\\ &+\frac{\gamma\phi(t)X'(t)a(t)[(r(t)D_-^\alpha x(t))']^\gamma}{X^{\gamma+1}(t)} + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t)\\ &= -\frac{\phi(t)q(t)f(X(t))}{X^\gamma(t)} + \frac{\gamma\phi(t)X'(t)a(t)[(r(t)D_-^\alpha x(t))']^\gamma}{X^{\gamma+1}(t)}\\ &+\frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t). \end{aligned} \tag{10}$$

By Lemma 3 and the definition of f we get that

$$\begin{aligned} &\omega'(t) \le -L\phi(t)q(t) - \gamma\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)\times\\ &\frac{a^{\frac{1}{\gamma}}(t)(r(t)D_-^\alpha x(t))'a(t)[(r(t)D_-^\alpha x(t))']^\gamma}{r(t)X^{\gamma+1}(t)}\\ &\qquad + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t)\\ &= -L\phi(t)q(t) - \frac{\gamma\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)}{r(t)}\times\\ &[-(\frac{\omega(t)}{\phi(t)} - \varphi(t))]^{1+\frac{1}{\gamma}} + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t)\\ &= -L\phi(t)q(t) - \frac{\gamma\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)}{r(t)}\times\\ &[\frac{\omega(t)}{\phi(t)} - \varphi(t)]^{1+\frac{1}{\gamma}} + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t). \end{aligned} \tag{11}$$

Using the following inequality (see [34, Eq. (18)]):

$$(u-v)^{1+\frac{1}{\gamma}} \ge u^{1+\frac{1}{\gamma}} + \frac{1}{\gamma}v^{1+\frac{1}{\gamma}} - (1+\frac{1}{\gamma})v^{\frac{1}{\gamma}}u,$$

we obtain

$$\begin{aligned} &[\frac{\omega(t)}{\phi(t)} - \varphi(t)]^{1+\frac{1}{\gamma}}\\ &\ge \frac{\omega^{1+\frac{1}{\gamma}}(t)}{\phi^{1+\frac{1}{\gamma}}(t)} + \frac{1}{\gamma}\varphi^{1+\frac{1}{\gamma}}(t) - (1+\frac{1}{\gamma})\frac{\varphi^{\frac{1}{\gamma}}(t)\omega(t)}{\phi(t)}. \end{aligned} \tag{12}$$

A combination of (11) and (12) yields:

$$\begin{aligned} &\omega'(t) \le -L\phi(t)q(t) + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t)\\ &-\frac{\gamma\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)}{r(t)}\times\\ &[\frac{\omega^{1+\frac{1}{\gamma}}(t)}{\phi^{1+\frac{1}{\gamma}}(t)} + \frac{1}{\gamma}\varphi^{1+\frac{1}{\gamma}}(t) - (1+\frac{1}{\gamma})\frac{\varphi^{\frac{1}{\gamma}}(t)\omega(t)}{\phi(t)}]\\ &= -L\phi(t)q(t) + \phi(t)\varphi'(t)\\ &-\frac{\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)\varphi^{1+\frac{1}{\gamma}}(t)}{r(t)}\\ &-\frac{\gamma\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)}{r(t)}\frac{\omega^{1+\frac{1}{\gamma}}(t)}{\phi^{1+\frac{1}{\gamma}}(t)}\\ &+\frac{(\gamma+1)\varphi^{\frac{1}{\gamma}}(t)\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)+r(t)\phi'(t)}{r(t)\phi(t)}\omega(t). \end{aligned} \tag{13}$$

Setting

$$\lambda = 1+\frac{1}{\gamma},\ A^{\lambda} = \frac{\gamma\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)}{r(t)}\frac{\omega^{1+\frac{1}{\gamma}}(t)}{\phi^{1+\frac{1}{\gamma}}(t)},$$
$$B^{\lambda-1} = \gamma^{\frac{1}{\gamma+1}}\frac{(\gamma+1)\varphi^{\frac{1}{\gamma}}(t)\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)+r(t)\phi'(t)}{(\gamma+1)[\Gamma(1-\alpha)\phi(t)\delta_1(t,t_2)]^{\frac{\gamma}{\gamma+1}}r^{\frac{1}{\gamma+1}}(t)},$$

Using Lemma 4 in (13) we get that

$$\begin{aligned}&\omega'(t) \le -L\phi(t)q(t)+\phi(t)\varphi'(t)\\ &-\frac{\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)\varphi^{1+\frac{1}{\gamma}}(t)}{r(t)}\\ &+\frac{[(\gamma+1)\varphi^{\frac{1}{\gamma}}(t)\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)+r(t)\phi'(t)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(t)\delta_1(t,t_2)]^{\gamma}r(t)}.\end{aligned} \tag{14}$$

Substituting t with s in (14), an integration for (14) with respect to s from t_2 to t yields

$$\begin{aligned}&\int_{t_2}^{t}\{L\phi(s)q(s)-\phi(s)\varphi'(s)\\ &\quad+\frac{\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)\varphi^{1+\frac{1}{\gamma}}(s)}{r(s)}\\ &-\frac{[(\gamma+1)\varphi^{\frac{1}{\gamma}}(s)\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)+r(s)\phi'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(s)\delta_1(s,t_2)]^{\gamma}r(s)}\}ds\\ &\le \omega(t_2)-\omega(t)\le\omega(t_2)<\infty,\end{aligned}$$

which contradicts (9). So the proof is complete.

Theorem 6 *Assume (2)-(4) hold, and for all sufficiently large T,*

$$\begin{aligned}&\int_{T}^{\infty}\{L\phi(s)q(s)-\phi(s)\varphi'(s)+\\ &\frac{\gamma\phi(s)[\Gamma(1-\alpha)]^{\gamma}\delta_1(s,T)\delta_2^{\gamma-1}(s,T)\varphi^2(s)}{r(s)}-\\ &\frac{\{2\gamma\phi(s)\varphi(s)[\Gamma(1-\alpha)]^{\gamma}\delta_1(s,T)\delta_2^{\gamma-1}(s,T)+r(s)\phi'(s)\}^2}{4\gamma[\Gamma(1-\alpha)]^{\gamma}\delta_1(s,T)\delta_2^{\gamma-1}(s,T)r(s)\phi(s)}\}ds\\ &=\infty,\end{aligned} \tag{15}$$

where ϕ, φ are defined as in Theorem 5. Then every solution of Eq. (1) is oscillatory or satisfies $\lim_{t\to\infty} X(t) = 0$.

Proof. Assume (1) has a nonoscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_2,\infty)$, where t_2 is sufficiently large. By Lemma 2 we have $(r(t)D_{-}^{\alpha}x(t))' < 0,\ t\in[t_2,\infty)$, where $t_2 > t_2$ is sufficiently large, and either $D_{-}^{\alpha}x(t) < 0$ on $[t_2,\infty)$ or $\lim_{t\to\infty} X(t) = 0$. Let $\omega(t)$ be defined as in Theorem 5. Proceeding as in Theorem 5, we obtain (10). By Lemma 3, we have the following observation:

$$\begin{aligned}&\frac{X'(t)}{X(t)} \ge -\frac{\Gamma(1-\alpha)\delta_1(t,t_2)a^{\frac{1}{\gamma}}(t)(r(t)D_{-}^{\alpha}x(t))'}{r(t)X(t)}\\ &= -\frac{\Gamma(1-\alpha)\delta_1(t,t_2)a^{\frac{1}{\gamma}}(t)(r(t)D_{-}^{\alpha}x(t))'}{r(t)X^{\gamma}(t)}X^{\gamma-1}(t)\\ &\ge -\frac{\Gamma(1-\alpha)\delta_1(t,t_2)a^{\frac{1}{\gamma}}(t)(r(t)D_{-}^{\alpha}x(t))'}{r(t)X^{\gamma}(t)}\times\\ &\{-\Gamma(1-\alpha)\delta_2(t,t_2)a^{\frac{1}{\gamma}}(t)(r(t)D_{-}^{\alpha}x(t))'\}^{\gamma-1}(t)\\ &= -\frac{[\Gamma(1-\alpha)]^{\gamma}\delta_1(t,t_2)\delta_2^{\gamma-1}(t,t_2)}{r(t)}\times\{\frac{a(t)[(r(t)D_{-}^{\alpha}x(t))']^{\gamma}}{X^{\gamma}(t)}\}\end{aligned} \tag{16}$$

Using (16) in (10) we get that

$$\begin{aligned}&\omega'(t) \le -L\phi(t)q(t)-\frac{\gamma\phi(t)[\Gamma(1-\alpha)]^{\gamma}\delta_1(t,t_2)\delta_2^{\gamma-1}(t,t_2)}{r(t)}\times\\ &\{\frac{a(t)[(r(t)D_{-}^{\alpha}x(t))']^{\gamma}}{X^{\gamma}(t)}\}^2+\frac{\phi'(t)}{\phi(t)}\omega(t)+\phi(t)\varphi'(t)\\ &= -L\phi(t)q(t)-\frac{\gamma\phi(t)[\Gamma(1-\alpha)]^{\gamma}\delta_1(t,t_2)\delta_2^{\gamma-1}(t,t_2)}{r(t)}\times\\ &[\frac{\omega(t)}{\phi(t)}-\varphi(t)]^2+\frac{\phi'(t)}{\phi(t)}\omega(t)+\phi(t)\varphi'(t)\\ &= -L\phi(t)q(t)-\frac{\gamma\phi(t)[\Gamma(1-\alpha)]^{\gamma}\delta_1(t,t_2)\delta_2^{\gamma-1}(t,t_2)\varphi^2(t)}{r(t)}\\ &-\frac{\gamma[\Gamma(1-\alpha)]^{\gamma}\delta_1(t,t_2)\delta_2^{\gamma-1}(t,t_2)}{r(t)\phi(t)}\omega^2(t)+\phi(t)\varphi'(t)\\ &+\frac{2\gamma\phi(t)\varphi(t)[\Gamma(1-\alpha)]^{\gamma}\delta_1(t,t_2)\delta_2^{\gamma-1}(t,t_2)+r(t)\phi'(t)}{r(t)\phi(t)}\omega(t)\\ &\le -L\phi(t)q(t)+\phi(t)\varphi'(t)\\ &-\frac{\gamma\phi(t)[\Gamma(1-\alpha)]^{\gamma}\delta_1(t,t_2)\delta_2^{\gamma-1}(t,t_2)\varphi^2(t)}{r(t)}\\ &+\frac{\{2\gamma\phi(t)\varphi(t)[\Gamma(1-\alpha)]^{\gamma}\delta_1(t,t_2)\delta_2^{\gamma-1}(t,t_2)+r(t)\phi'(t)\}^2}{4\gamma[\Gamma(1-\alpha)]^{\gamma}\delta_1(t,t_2)\delta_2^{\gamma-1}(t,t_2)r(t)\phi(t)}.\end{aligned} \tag{17}$$

Substituting t with s in (17), an integration for (17) with respect to s from t_2 to t yields

$$\begin{aligned}&\int_{t_2}^{t}\{L\phi(s)q(s)-\phi(s)\varphi'(s)+\\ &\frac{\gamma\phi(s)[\Gamma(1-\alpha)]^{\gamma}\delta_1(s,t_2)\delta_2^{\gamma-1}(s,t_2)\varphi^2(s)}{r(s)}\\ &-\frac{\{2\gamma\phi(s)\varphi(s)[\Gamma(1-\alpha)]^{\gamma}\delta_1(s,t_2)\delta_2^{\gamma-1}(s,t_2)+r(s)\phi'(s)\}^2}{4\gamma[\Gamma(1-\alpha)]^{\gamma}\delta_1(s,t_2)\delta_2^{\gamma-1}(s,t_2)r(s)\phi(s)}\}ds\\ &\le\omega(t_2)-\omega(t)\le\omega(t_2)<\infty,\end{aligned}$$

which contradicts (15). So the proof is complete.

Based on Theorems 5 and 6, next we present two Philos-type oscillatory criteria for Eq. (1) in the following two theorems.

Theorem 7 *Define $\mathbb{D} = \{(t,s)|t \ge s \ge t_0\}$. Assume (2)-(4) hold, and there exists a function $H \in C^1(\mathbb{D},\mathbb{R})$ such that*

$$H(t,t) = 0,\ for\ t \ge t_0,\ \ H(t,s) > 0,\ for\ t > s \ge t_0,$$

and H has a nonpositive continuous partial *derivative $H'_s(t,s)$, and*

$$\lim_{t\to\infty}\sup \frac{1}{H(t,t_0)}\{\int_{t_0}^{t} H(t,s)\{L\phi(s)q(s)-\phi(s)\varphi'(s) + \frac{\phi(s)\Gamma(1-\alpha)\delta_1(s,T)\varphi^{1+\frac{1}{\gamma}}(s)}{r(s)} - \frac{[(\gamma+1)\varphi^{\frac{1}{\gamma}}(s)\phi(s)\Gamma(1-\alpha)\delta_1(s,T)+r(s)\phi'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(s)\delta_1(s,T)]^{\gamma}r(s)}\}ds\} = \infty, \tag{18}$$

for all sufficiently large T, where ϕ, φ are defined as in Theorem 5. Then every solution of Eq. (1) is oscillatory or satisfies $\lim_{t\to\infty} X(t)=0$.

Proof. Assume (1) has a nonoscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 2 we have $D_-^{\alpha}x(t)<0$ on $[t_2,\infty)$ for some sufficiently large $t_2>t_1$. Let $\omega(t)$ be defined as in Theorem 5. By (14) we have

$$L\phi(t)q(t)-\phi(t)\varphi'(t)+\frac{\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)\varphi^{1+\frac{1}{\gamma}}(t)}{r(t)} - \frac{[(\gamma+1)\varphi^{\frac{1}{\gamma}}(t)\phi(t)\Gamma(1-\alpha)\delta_1(t,t_2)+r(t)\phi'(t)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(t)\delta_1(t,t_2)]^{\gamma}r(t)} \le -\omega'(t). \tag{19}$$

Substituting t with s in (19), multiplying both sides by $H(t,s)$ and then integrating with respect to s from t_2 to t yields

$$\begin{aligned}&\int_{t_2}^{t} H(t,s)\{L\phi(s)q(s)-\phi(s)\varphi'(s) + \frac{\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)\varphi^{1+\frac{1}{\gamma}}(s)}{r(s)} - \frac{[(\gamma+1)\varphi^{\frac{1}{\gamma}}(s)\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)+r(s)\phi'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(s)\delta_1(s,t_2)]^{\gamma}r(s)}\}ds\\ &\le -\int_{t_2}^{t} H(t,s)\omega'(s)ds\\ &= H(t,t_2)\omega(t_2)+\int_{t_2}^{t} H'_s(t,s)\omega(s)\Delta s\\ &\le H(t,t_2)\omega(t_2)\le H(t,t_0)\omega(t_2).\end{aligned}$$

Then

$$\begin{aligned}&\int_{t_0}^{t} H(t,s)\{L\phi(s)q(s)-\phi(s)\varphi'(s)+ \frac{\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)\varphi^{1+\frac{1}{\gamma}}(s)}{r(s)} - \frac{[(\gamma+1)\varphi^{\frac{1}{\gamma}}(s)\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)+r(s)\phi'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(s)\delta_1(s,t_2)]^{\gamma}r(s)}\}ds\\ &= \int_{t_0}^{t_2} H(t,s)\{L\phi(s)q(s)-\phi(s)\varphi'(s+) \frac{\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)\varphi^{1+\frac{1}{\gamma}}(s)}{r(s)} - \frac{[(\gamma+1)\varphi^{\frac{1}{\gamma}}(s)\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)+r(s)\phi'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(s)\delta_1(s,t_2)]^{\gamma}r(s)}\}ds\\ &+\int_{t_2}^{t} H(t,s)\{L\phi(s)q(s)-\phi(s)\varphi'(s)+ \frac{\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)\varphi^{1+\frac{1}{\gamma}}(s)}{r(s)} - \frac{[(\gamma+1)\varphi^{\frac{1}{\gamma}}(s)\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)+r(s)\phi'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(s)\delta_1(s,t_2)]^{\gamma}r(s)}\}ds\\ &\le H(t,t_0)\omega(t_2)+H(t,t_0)\int_{t_0}^{t_2} |L\phi(s)q(s)-\phi(s)\varphi'(s) + \frac{\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)\varphi^{1+\frac{1}{\gamma}}(s)}{r(s)} - \frac{[(\gamma+1)\varphi^{\frac{1}{\gamma}}(s)\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)+r(s)\phi'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(s)\delta_1(s,t_2)]^{\gamma}r(s)}|ds.\end{aligned}$$

So

$$\begin{aligned}&\lim_{t\to\infty}\sup \frac{1}{H(t,t_0)}\{\int_{t_0}^{t} H(t,s)\{L\phi(s)q(s)-\phi(s)\varphi'(s) + \frac{\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)\varphi^{1+\frac{1}{\gamma}}(s)}{r(s)} - \frac{[(\gamma+1)\varphi^{\frac{1}{\gamma}}(s)\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)+r(s)\phi'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(s)\delta_1(s,t_2)]^{\gamma}r(s)}\}ds\}\\ &\le \omega(t_2)+\int_{t_0}^{t_2} |L\phi(s)q(s)-\phi(s)\varphi'(s) + \frac{\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)\varphi^{1+\frac{1}{\gamma}}(s)}{r(s)} - \frac{[(\gamma+1)\varphi^{\frac{1}{\gamma}}(s)\phi(s)\Gamma(1-\alpha)\delta_1(s,t_2)+r(s)\phi'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(s)\delta_1(s,t_2)]^{\gamma}r(s)}|ds\\ &<\infty,\end{aligned}$$

which contradicts (18). So the proof is complete.

Theorem 8 *Let H, ϕ, φ be defined as in Theorem 7. If (2)-(4) hold, and*

$$\lim_{t\to\infty}\sup \frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\{L\phi(s)q(s)-\phi(s)\varphi'(s) + \frac{\gamma\phi(s)[\Gamma(1-\alpha)]^{\gamma}\delta_1(s,T)\delta_2^{\gamma-1}(s,T)\varphi^2(s)}{r(s)} - \frac{\{2\gamma\phi(s)\varphi(s)[\Gamma(1-\alpha)]^{\gamma}\delta_1(s,T)\delta_2^{\gamma-1}(s,T)+r(s)\phi'(s)\}^2}{4\gamma[\Gamma(1-\alpha)]^{\gamma}\delta_1(s,T)\delta_2^{\gamma-1}(s,T)r(s)\phi(s)}\}ds\} = \infty, \tag{20}$$

for all sufficiently large T. Then every solution of Eq. (1) is oscillatory or satisfies $\lim_{t\to\infty} X(t)=0$.

Proof. Assume (1) has a nonoscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t)>0$ on $[t_2,\infty)$, where t_2 is sufficiently large. By Lemma 2 we have $D_-^{\alpha}x(t)<0$ on $[t_2,\infty)$ for some sufficiently large $t_2>t_2$. Let $\omega(t)$ be defined as in Theorem 5. By (17) we have

$$L\phi(t)q(t)-\phi(t)\varphi'(t) + \frac{\gamma\phi(t)[\Gamma(1-\alpha)]^{\gamma}\delta_1(t,t_2)\delta_2^{\gamma-1}(t,t_2)\varphi^2(t)}{r(t)} - \frac{\{2\gamma\phi(t)\varphi(t)[\Gamma(1-\alpha)]^{\gamma}\delta_1(t,t_2)\delta_2^{\gamma-1}(t,t_2)+r(t)\phi'(t)\}^2}{4\gamma[\Gamma(1-\alpha)]^{\gamma}\delta_1(t,t_2)\delta_2^{\gamma-1}(t,t_2)r(t)\phi(t)} \le -\omega'(t). \tag{21}$$

Substituting t with s in (21), multiplying both sides by $H(t,s)$ and then integrating with respect to s from t_2 to t yields

$$\begin{aligned}&\int_{t_2}^{t} H(t,s)\{L\phi(s)q(s)-\phi(s)\varphi'(s) + \frac{\gamma\phi(s)[\Gamma(1-\alpha)]^{\gamma}\delta_1(s,t_2)\delta_2^{\gamma-1}(s,t_2)\varphi^2(s)}{r(s)} - \frac{\{2\gamma\phi(s)\varphi(s)[\Gamma(1-\alpha)]^{\gamma}\delta_1(s,t_2)\delta_2^{\gamma-1}(s,t_2)+r(s)\phi'(t)\}^2}{4\gamma[\Gamma(1-\alpha)]^{\gamma}\delta_1(s,t_2)\delta_2^{\gamma-1}(s,t_2)r(s)\phi(s)}\}ds\\ &\le -\int_{t_2}^{t} H(t,s)\omega'(s)ds\end{aligned}$$

$$= H(t,t_2)\omega(t_2) + \int_{t_2}^{t} H'_s(t,s)\omega(s)\Delta s$$
$$\le H(t,t_2)\omega(t_2) \le H(t,t_0)\omega(t_2).$$

Then similar to the process of Theorem 7, we get that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\{\int_{t_0}^{t} H(t,s)\{L\phi(s)q(s) - \phi(s)\varphi'(s) + \frac{\gamma\phi(s)[\Gamma(1-\alpha)]^\gamma\delta_1(s,t_2)\delta_2^{\gamma-1}(s,t_2)\varphi^2(s)}{r(s)} - \frac{\{2\gamma\phi(s)\varphi(s)[\Gamma(1-\alpha)]^\gamma\delta_1(s,t_2)\delta_2^{\gamma-1}(s,t_2)+r(s)\phi'(s)\}^2}{4\gamma[\Gamma(1-\alpha)]^\gamma\delta_1(s,t_2)\delta_2^{\gamma-1}(s,t_2)r(s)\phi(s)}\}ds\} < \infty,$$

which contradicts (20). So the proof is complete.

Remark 9 *In Theorems 7 and 8, if we take $H(t,s)$ for some special functions such as $(t-s)^m$ or $\ln\frac{t}{s}$, then we can obtain some corollaries, which are omitted here.*

Remark 10 *The established oscillation criteria for Eq. (1) above are new results so far in the literature to our best knowledge.*

3 Applications

In this section, we will present some applications for the established results above.

Example 11 *Consider equation*

$$(t^{\frac{5}{3}}[(D_-^\alpha x(t))']^{\frac{5}{3}})' - t^{-\frac{8}{3}}[M + e^{\int_t^\infty(\xi-t)^{-\alpha}x(\xi)d\xi}]\times (\int_t^\infty(\xi-t)^{-\alpha}x(\xi)d\xi)^{\frac{5}{3}} = 0,\ t\in[2,\infty), \tag{22}$$

where $M > 0$ is a constant.

We have in (1) $\gamma = \frac{5}{3}$, $a(t) = t^{\frac{5}{3}}$, $q(t) = t^{-\frac{8}{3}}$, $f(x) = x^{\frac{5}{3}}[e^x + M]$, $r(t) = 1$, $t_0 = 2$. Then $\frac{f(x)}{x^\gamma} \ge M = L$. Then we have

$$\int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{\gamma}}(s)}ds = \int_2^\infty \frac{1}{s}ds = \infty,$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(s)}ds = \infty.$$

Furthermore,

$$\int_{t_0}^{\infty}\frac{1}{r(\xi)}\int_{\xi}^{\infty}[\frac{1}{a(\tau)}\int_{\tau}^{\infty} q(s)ds]^{\frac{1}{\gamma}}d\tau d\xi$$
$$= \int_2^\infty\int_\xi^\infty[\frac{1}{\tau^{\frac{5}{3}}}\int_\tau^\infty s^{-\frac{8}{3}}ds]^{\frac{3}{5}}d\tau d\xi$$
$$= (\frac{3}{5})^{\frac{3}{5}}\int_2^\infty[\int_\xi^\infty \frac{1}{\tau^2}d\tau]d\xi$$
$$= (\frac{3}{5})^{\frac{3}{5}}\int_2^\infty \frac{1}{\xi}d\xi = \infty.$$

On the other hand, for a sufficiently large T, we have

$$\delta_1(t,T) = \int_T^t \frac{1}{a^{\frac{1}{\gamma}}(s)}ds = \int_T^t \frac{1}{s}ds \to \infty.$$

So we can take $T^* > T$ such that $\delta_1(t,T) > 1$ for $t\in[T^*,\infty)$. Taking $\phi(t) = t^{\frac{5}{3}}$, $\varphi(t) = 0$ in (9), we get that

$$\int_T^\infty\{L\phi(s)q(s) - \frac{[r(s)\phi'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(s)\delta_1(s,T)]^\gamma r(s)}\}ds$$
$$= \int_T^\infty\{M - (\frac{5}{8})^{\frac{8}{3}}\frac{1}{[\Gamma(1-\alpha)\delta_1(s,T)]^{\frac{5}{3}}}\}\frac{1}{s}ds$$
$$= \int_T^{T^*}\{M - (\frac{5}{8})^{\frac{8}{3}}\frac{1}{[\Gamma(1-\alpha)\delta_1(s,T)]^{\frac{5}{3}}}\}\frac{1}{s}ds + \int_{T^*}^\infty\{M - (\frac{5}{8})^{\frac{8}{3}}\frac{1}{[\Gamma(1-\alpha)\delta_1(s,T)]^{\frac{5}{3}}}\}\frac{1}{s}ds$$
$$\ge \int_T^{T^*}\{M - (\frac{5}{8})^{\frac{8}{3}}\frac{1}{[\Gamma(1-\alpha)\delta_1(s,T)]^{\frac{5}{3}}}\}\frac{1}{s}ds + \int_{T^*}^t\{M - (\frac{5}{8})^{\frac{8}{3}}\frac{1}{[\Gamma(1-\alpha)]^{\frac{5}{3}}}\}\frac{1}{s}ds \to \infty,$$

provided that $M > (\frac{5}{8})^{\frac{8}{3}}\frac{1}{[\Gamma(1-\alpha)]^{\frac{5}{3}}}$. So (2)-(4) and (9) all hold, and by Theorem 5 we deduce that every solution of Eq. (22) is oscillatory or satisfies $\lim_{t\to\infty} X(t) = 0$ under the condition $M > (\frac{5}{8})^{\frac{8}{3}}\frac{1}{[\Gamma(1-\alpha)]^{\frac{5}{3}}}$.

Example 12 *Consider equation*

$$(t^3[(D_-^\alpha x(t))']^3)' - Mt^{-4}[\int_t^\infty(\xi-t)^{-\alpha}x(\xi)d\xi]^3 = 0,\ t\in[2,\infty), \tag{23}$$

where $\alpha\in(0,1)$, and $M > 0$ is a constant.

We have in (1) $\gamma = 3$, $a(t) = t^3$, $q(t) = t^{-4}$, $f(x) = Mx^3$, $r(t) = 1$, $t_0 = 2$. Then $\frac{f(x)}{x^\gamma} \ge M = L$. Then we have

$$\int_{t_0}^\infty \frac{1}{a^{\frac{1}{\gamma}}(s)}ds = \int_2^\infty \frac{1}{a^{\frac{1}{3}}(s)}ds$$
$$= \int_2^\infty \frac{1}{s}ds = \infty,$$

and

$$\int_{t_0}^\infty \frac{1}{r(s)}ds = \infty.$$

Furthermore,

$$\int_{t_0}^\infty\frac{1}{r(\xi)}\int_\xi^\infty[\frac{1}{a(\tau)}\int_\tau^\infty q(s)ds]^{\frac{1}{\gamma}}d\tau d\xi$$
$$= \int_2^\infty\int_\xi^\infty[\frac{1}{\tau^3}\int_\tau^\infty s^{-4}ds]^{\frac{1}{3}}d\tau d\xi$$
$$= \frac{1}{\sqrt[3]{3}}\int_2^\infty[\int_\xi^\infty\frac{1}{\tau^2}d\tau]d\xi$$
$$= \frac{1}{\sqrt[3]{3}}\int_2^\infty\frac{1}{\xi}d\xi = \infty.$$

On the other hand, for a sufficiently large T, we have

$$\delta_1(t,T) = \int_T^t \frac{1}{a^{\frac{1}{\gamma}}(s)}ds = \int_T^t \frac{1}{s}ds \to \infty.$$

So we can take $T^* > T$ such that $\delta_1(t,T) > 1$ for $t \in [T^*,\infty)$. Taking $\phi(t) = t^3$, $\varphi(t) = 0$, $H(t,s) = t - s$ in (18), we get that

$$\begin{aligned}
&\lim_{t\to\infty}\sup \tfrac{1}{t-t_0}\{\int_{t_0}^{t}(t-s)\{L\phi(s)q(s) \\
&-\tfrac{[r(s)\phi'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\phi(s)\delta_1(s,T)]^{\gamma}r(s)}\}ds\} \\
&= \lim_{t\to\infty}\sup \tfrac{1}{t-2}\{\int_{2}^{t}(t-s)\times \\
&\ \{M-(\tfrac{3}{4})^4\tfrac{1}{[\Gamma(1-\alpha)\delta_1(s,T)]^3}\}\tfrac{1}{s}ds\} \\
&= \lim_{t\to\infty}\sup \tfrac{1}{t-2}\{\int_{2}^{T^*}(t-s)\times \\
&\ \{M-(\tfrac{3}{4})^4\tfrac{1}{[\Gamma(1-\alpha)\delta_1(s,T)]^3}\}\tfrac{1}{s}ds \\
&+\int_{T^*}^{t}(t-s)\{M-(\tfrac{3}{4})^4\tfrac{1}{[\Gamma(1-\alpha)\delta_1(s,T)]^3}\}\tfrac{1}{s}ds\} \\
&\geq \lim_{t\to\infty}\sup \tfrac{1}{t-2}\{\int_{2}^{T^*}(t-s)\times \\
&\ \{M-(\tfrac{3}{4})^4\tfrac{1}{[\Gamma(1-\alpha)]^3}\}\tfrac{1}{s}ds \\
&+\int_{T^*}^{t}(t-s)\{M-(\tfrac{3}{4})^4\tfrac{1}{[\Gamma(1-\alpha)]^3}\}\tfrac{1}{s}ds\} = \infty,
\end{aligned}$$

provided that $M > (\frac{3}{4})^4\frac{1}{[\Gamma(1-\alpha)]^3}$. So (2)-(4) and (18) all hold, and by Theorem 7 we deduce that every solution of Eq. (23) is oscillatory or satisfies $\lim_{t\to\infty} X(t) = 0$ under the condition $M > (\frac{3}{4})^4\frac{1}{[\Gamma(1-\alpha)]^3}$.

4 Conclusion

We have established some new interval oscillation criteria for a class of nonlinear fractional differential equations by a generalized Riccati function and inequality technique. The presented examples show that our results are effective in the oscillation analysis of some special fractional differential equations. The present method can be applied to obtain oscillation criteria for other fractional differential equations such as the nonlinear fractional differential equations with damping term, which are supposed to further research.

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