Some New Dynamic Inequalities and Their Applications in the Qualitative Analysis of Dynamic Equations

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Abstract: In this paper, we establish some new Gronwall-Bellman-type dynamic inequalities in two independent variables containing integration on infinite intervals on time scales, which can be used as a handy tool in the boundedness analysis for solutions to some certain dynamic equations containing integration on infinite intervals on time scales. The presented inequalities are of new forms so far in the literature to our best knowledge.

Key–Words: Gronwall-Bellman-type inequality; Time scales; Dynamic equation; Qualitative analysis; Quantitative analysis; Bound

1 Introduction

As is known to us, in the qualitative as well as quantitative analysis for solutions of differential equations, difference equations and dynamic equations on time scales, the Gronwall-Bellman inequality [1,2] play an important role as it provides explicit bounds for the unknown functions concerned. During the past few decades, various generalizations of the Gronwall-Bellman inequality have been developed (see [3-27] and the references therein). But we notice that in the analysis of boundedness for the solutions for some certain dynamic equations containing integration on infinite intervals on time scales, for example,

$$(u^p(x,y))^{\Delta\Delta}_{yx} = F\left(x, y, u(x,y), \int_y^\infty \int_x^\infty W(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta\right),$$

or

$$u^p(x,y) = C + \int_y^\infty \int_x^\infty F_1(s,t,u(s,t), \int_t^\infty \int_s^\infty W_1(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta)\Delta s\Delta t + \int_N^\infty \int_M^\infty F_2(s,t,u(s,t), \int_t^\infty \int_s^\infty W_2(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta)\Delta s\Delta t,$$

it is inadequate to obtain the bounds for their solutions by use of the existing results in the literature. So it is necessary to seek new approach to fulfil such analysis for them.

The aim of this paper is to establish some new Gronwall-Bellman type dynamic inequalities in two independent variables containing integration on infinite intervals on time scales, based on which some new bounds for the solutions for the two equations mentioned above are derived.

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0,\infty)$. $\mathbb{T}$ denotes an arbitrary time scale. $\mathbb{T}_0 = [x_0,\infty)\bigcap\mathbb{T}$, $\widetilde{\mathbb{T}}_0 = [y_0,\infty)\bigcap\mathbb{T}$, where $x_0,\ y_0 \in \mathbb{T}$. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma \in (\mathbb{T},\mathbb{T})$ and $\rho \in (\mathbb{T},\mathbb{T})$ such that $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$, $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$. The graininess $\mu \in (\mathbb{T},\mathbb{R}_+)$ is defined by $\mu(t) = \sigma(t) - t$.

Definition 1 *A function $f \in (\mathbb{T},\mathbb{R})$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1+\mu(t)f(t) \neq 0$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f | f \in \mathfrak{R},\ 1+\mu(t)f(t) > 0,\ \forall t \in \mathbb{T}\}$.*

Definition 2 *The cylinder transformation ξ_h is defined by*

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h}, & if\ h \neq 0\ (for\ z \neq -\frac{1}{h}), \\ z, & if\ h = 0, \end{cases}$$

where Log is the principal logarithm function.

Definition 3 *For $p(x,y) \in \Re$ with respect to y, the exponential function is defined by*

$$e_p(y,s) = exp(\int_s^y \xi_{\mu(\tau)}(p(x,\tau))\Delta\tau)$$

for $s, y \in \mathbb{T}$.

Theorem 4 $[28, Theorem 1.12]$*: If $p(x,y) \in \Re$ with respect to y, then the following conclusions hold*

(i) $e_p(y,y) \equiv 1$, and $e_0(s,y) \equiv 1$,
(ii) $e_p(s,\sigma(y)) = (1+\mu(y)p(x,y))e_p(s,y)$,
(iii) If $p \in \Re^+$ with respect to y, then $e_p(s,y) > 0$ for $\forall s,\ y \in \mathbb{T}$,
(iv) If $p \in \Re^+$ with respect to y, then $\ominus p \in \Re^+$,
(v) $e_p(s,y) = \frac{1}{e_p(y,s)} = e_{\ominus p}(y,s)$,
where $(\ominus p)(x,y) = -\dfrac{p(x,y)}{1+\mu(y)p(x,y)}$.

Theorem 5 $[28, Theorem 1.13]$*: If $p(x,y) \in \Re$ with respect to y, $y_0 \in \mathbb{T}$ is a fixed number, then the exponential function $e_p(y,y_0)$ is the unique solution of the following initial value problem*

$$\begin{cases} z_y^\Delta(x,y) = p(x,y)z(x,y), \\ z(x,y_0) = 1. \end{cases}$$

2 Main Results

Lemma 6 *Assume that $u(x,.),\ a(x,.),\ b(x,.),\ m(x,.) \in C_{rd}(\widetilde{\mathbb{T}}_0, \mathbb{R}_+)$ with respect to y, $\widetilde{m}(x,y) = -m(x,y)b(x,y)$ and $\widetilde{m}(x,.) \in \Re_+$ with respect to y. Then for any fixed $x \in \mathbb{T}_0$,*

$$u(x,y) \le a(x,y) + b(x,y)\int_y^\infty m(x,t)u(x,t)\Delta t, \quad y \in \widetilde{\mathbb{T}}_0 \tag{1}$$

implies

$$u(x,y) \le a(x,y) + b(x,y)\int_y^\infty e_{\widetilde{m}}(y,\sigma(t))m(x,t)a(x,t)\Delta t, \quad y \in \widetilde{\mathbb{T}}_0. \tag{2}$$

Proof: Denote $v(x,y) = \int_y^\infty m(x,t)u(x,t)\Delta t$. Then

$$u(x,y) \le a(x,y) + b(x,y)v(x,y),$$

and

$$\begin{aligned} v_y^\Delta(x,y) &= -m(x,y)u(x,y) \\ &\ge -m(x,y)b(x,y)v(x,y) - m(x,y)a(x,y) \\ &= \widetilde{m}(x,y)v(x,y) - m(x,y)a(x,y). \end{aligned}$$

Since $\widetilde{m} \in \Re^+$, then from Theorem 4(iv) we have $\ominus\widetilde{m} \in \Re^+$, and furthermore from Theorem 4(iii) we obtain $e_{\ominus\widetilde{m}}(y,\alpha) > 0$ for $\forall \alpha \in \widetilde{\mathbb{T}}_0$.

Moreover,

$$\begin{aligned} &[v(x,y)e_{\ominus\widetilde{m}}(y,\alpha)]_y^\Delta \\ &= [e_{\ominus\widetilde{m}}(y,\alpha)]_y^\Delta v(x,y) + e_{\ominus\widetilde{m}}(\sigma(y),\alpha)v_y^\Delta(x,y). \end{aligned} \tag{3}$$

On the other hand, from Theorem 5 we have

$$[e_{\ominus\widetilde{m}}(y,\alpha)]_y^\Delta = (\ominus\widetilde{m})(x,y)e_{\ominus\widetilde{m}}(y,\alpha). \tag{4}$$

A combination of (3), (4) and Theorem 4 yields

$$\begin{aligned} &[v(x,y)e_{\ominus\widetilde{m}}(y,\alpha)]_y^\Delta = \\ &(\ominus\widetilde{m})(x,y)e_{\ominus\widetilde{m}}(y,\alpha)v(x,y) \\ &\quad + e_{\ominus\widetilde{m}}(\sigma(y),\alpha)v_y^\Delta(x,y) \\ &= e_{\ominus\widetilde{m}}(\sigma(y),\alpha) \times \\ &\quad [\frac{(\ominus\widetilde{m})(x,y)}{1+\mu(y)(\ominus\widetilde{m})(x,y)}v(x,y) + v_y^\Delta(x,y)] \\ &= e_{\ominus\widetilde{m}}(\sigma(y),\alpha)[v_y^\Delta(x,y) - \widetilde{m}(x,y)v(x,y)] \end{aligned} \tag{5}$$

Substituting y with t, and an integration for (5) with respect to t from α to ∞ yields

$$\begin{aligned} &v(x,\infty)e_{\ominus\widetilde{m}}(\infty,\alpha) - v(x,\alpha)e_{\ominus\widetilde{m}}(\alpha,\alpha) \\ &= \int_\alpha^\infty e_{\ominus\widetilde{m}}(\sigma(t),\alpha)[v_y^\Delta(x,t) - \widetilde{m}(x,t)v(x,t)]\Delta t. \end{aligned} \tag{6}$$

Considering $v(x,\infty) = 0$, $e_{\ominus\widetilde{m}}(\alpha,\alpha) = 1$, from (1) and (6) we have

$$\begin{aligned} -v(x,\alpha) &\ge -\int_\alpha^\infty e_{\ominus\widetilde{m}}(\sigma(t),\alpha)m(x,t)a(x,t)\Delta t \\ &= -\int_\alpha^\infty e_{\widetilde{m}}(\alpha,\sigma(t))m(x,t)a(x,t)\Delta t, \end{aligned}$$

which is followed by

$$v(x,\alpha) \le \int_\alpha^\infty e_{\widetilde{m}}(\alpha,\sigma(t))m(x,t)a(x,t)\Delta t. \tag{7}$$

Since $\alpha \in \widetilde{\mathbb{T}}_0$ is arbitrary, after substituting α with y we obtain the desired inequality.

Lemma 7 *Under the conditions of Lemma 6, furthermore, assume $a(x,y)$ is nonincreasing in y for every fixed x, $b(x,y) \equiv 1$. Then we have*

$$u(x,y) \le a(x,y)e_{-m}(y,\infty).$$

Proof: Since $b(x,y) \equiv 1$, and $a(x,y)$ is nonincreasing on $\widetilde{\mathbb{T}}_0$ with respect to y, then $\widetilde{m} = -m$, and

$$\begin{aligned} u(x,y) &\le a(x,y) + \int_y^\infty e_{-m}(y,\sigma(t))a(x,t)m(x,t)\Delta t \\ &\le a(x,y)[1 + \int_y^\infty e_{-m}(y,\sigma(t))m(x,t)\Delta t]. \end{aligned}$$

From [29, Theorem 2.39 and 2.36 (i)], we have

$$\int_y^\infty e_{-m}(y,\sigma(y))m(x,t)\Delta t$$
$$= \lim_{\varepsilon\to\infty}\int_\varepsilon^y e_{-m}(y,\sigma(t))(-m(x,t))\Delta t$$
$$= \lim_{\varepsilon\to\infty} e_{-m}(y,\varepsilon) - e_{-m}(y,y) = e_{-m}(y,\infty)-1,$$

Combining the above information we can obtain the desired inequality. □

Lemma 8 [30] *Assume that $a \ge 0, p \ge q \ge 0$, and $p \ne 0$, then for any $K > 0$,*

$$a^{\frac{q}{p}} \le \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}}.$$

Theorem 9 *Suppose $u, f, g, h, a, b \in C_{rd}(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0, \mathbb{R}_+)$, and a, b are nonincreasing. p, q, r, m are constants, and $p \ge q \ge 0$, $p \ge r \ge 0$, $p \ge m \ge 0$, $p \ne 0$. If for $(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality:*

$$u^p(x,y) \le a(x,y) + b(x,y)\times \int_y^\infty\int_x^\infty [f(s,t)u^q(s,t)+g(s,t)u^r(s,t)]\Delta s\Delta t + b(x,y)\times \int_y^\infty\int_x^\infty\int_t^\infty\int_s^\infty h(\xi,\eta)u^m(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t, \tag{8}$$

then

$$u(x,y) \le [B_1(x,y) + b(x,y)\times \int_y^\infty e_{\overline{B}_2}(y,\sigma(t))B_2(x,t)B_1(x,t)\Delta t]^{\frac{1}{p}}, \quad (x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0, \tag{9}$$

provided that $\overline{B}_2(x,.)\in\mathfrak{R}^+$, where

$$B_1(x,y) = a(x,y) + b(x,y)\times \int_y^\infty\int_x^\infty [f(s,t)\frac{p-q}{p}K^{\frac{q}{p}} + g(s,t)\frac{p-r}{p}K^{\frac{r}{p}}]\Delta s\Delta t + b(x,y)\times \int_y^\infty\int_x^\infty\int_t^\infty\int_s^\infty h(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta\Delta s\Delta t, \tag{10}$$

$$B_2(x,y) = \int_x^\infty [f(s,y)\frac{q}{p}K^{\frac{q-p}{p}} + g(s,y)\frac{r}{p}K^{\frac{r-p}{p}} + \int_y^\infty\int_s^\infty h(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\Delta\xi\Delta\eta]\Delta s, \forall K>0, \tag{11}$$

and

$$\overline{B}_2(x,y) = -b(x,y)B_2(x,y). \tag{12}$$

Proof: Denote the right side of (8) by $v(x,y)$. Then we have

$$u(x,y) \le v^{\frac{1}{p}}(x,y),\ (x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0. \tag{13}$$

Fix $X\in\mathbb{T}_0$. Then it follows that

$$v(X,y) = a(X,y) + b(X,y) \int_y^\infty\int_X^\infty [f(s,t)u^q(s,t) + g(s,t)u^r(s,t)]\Delta s\Delta t + b(X,y)\int_y^\infty\int_X^\infty\int_t^\infty\int_s^\infty h(\xi,\eta)u^m(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t$$
$$\le a(X,y) + b(X,y)\int_y^\infty\int_X^\infty [f(s,t)v^{\frac{q}{p}}(s,t) + g(s,t) v^{\frac{r}{p}}(s,t) + \int_t^\infty\int_s^\infty h(\xi,\eta)v^{\frac{m}{p}}(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t. \tag{14}$$

Combining (14) with Lemma 8 we obtain

$$v(X,y) \le a(X,y) + b(X,y) \int_y^\infty\int_X^\infty f(s,t)(\frac{q}{p}K^{\frac{q-p}{p}}v(s,t) + \frac{p-q}{p}K^{\frac{q}{p}})\Delta s\Delta t + b(X,y)\int_y^\infty\int_X^\infty g(s,t)(\frac{r}{p}K^{\frac{r-p}{p}}v(s,t) + \frac{p-r}{p}K^{\frac{r}{p}}) \Delta s\Delta t + b(X,y)\int_y^\infty\int_X^\infty\int_t^\infty\int_s^\infty h(\xi,\eta) (\frac{m}{p}K^{\frac{m-p}{p}}v(\xi,\eta) + \frac{p-m}{p}K^{\frac{m}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$
$$\le a(X,y) + b(X,y)\int_y^\infty\int_X^\infty [f(s,t)\frac{p-q}{p}K^{\frac{q}{p}} + g(s,t)\frac{p-r}{p}K^{\frac{r}{p}} + \int_t^\infty\int_s^\infty h(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t + b(X,y)\int_y^\infty\{\int_X^\infty [f(s,t)\frac{q}{p}K^{\frac{q-p}{p}} + g(s,t)\frac{r}{p}K^{\frac{r-p}{p}} + \int_t^\infty\int_s^\infty h(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\Delta\xi\Delta\eta]\Delta s\}v(X,t)\Delta t$$
$$= B_1(X,y) + b(X,y)\int_y^\infty B_2(X,t)v(X,t)\Delta t, \tag{15}$$

By use of Lemma 6, we obtain

$$v(X,y) \le B_1(X,y) + b(X,y) \int_y^\infty e_{\overline{B}_2}(y,\sigma(t))B_2(X,t)B_1(X,t)\Delta t,\ y\in\widetilde{\mathbb{T}}_0, \tag{16}$$

Since $X\in\mathbb{T}_0$ is arbitrary, then in fact (16) holds for $\forall x\in\mathbb{T}_0$, that is,

$$v(x,y) \le B_1(x,y) + b(x,y)$$

$$\int_y^\infty e_{\overline{B}_2}(y,\sigma(t))B_2(x,t)B_1(x,t)\Delta t,\ (x,y)\in(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0). \tag{17}$$

Combining (13), (17) we obtain the desired inequality. □

If we apply Lemma 7 instead of Lemma 6 at the end of the proof of Theorem 9, then we obtain the following theorem.

Theorem 10 *Suppose u, f, g, h, a, p, q, r, m are defined as in Theorem 2.1. If for $(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality:*

$$u^p(x,y)\le a(x,y)+\int_y^\infty\int_x^\infty[f(s,t)u^q(s,t)$$

$$+g(s,t)u^r(s,t)+\int_t^\infty\int_s^\infty h(\xi,\eta)u^m(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t,$$

then

$$u(x,y)\le[B_1(x,y)e_{-B_2}(y,\infty)]^{\frac{1}{p}},\ \ (x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0,$$

provided that $-B_2(x,.)\in\mathfrak{R}^+$, where

$$B_1(x,y)=a(x,y)+\int_y^\infty\int_x^\infty[f(s,t)\frac{p-q}{p}K^{\frac{q}{p}}$$

$$+g(s,t)\frac{p-r}{p}K^{\frac{r}{p}}]\Delta s\Delta t+$$

$$\int_y^\infty\int_x^\infty\int_t^\infty\int_s^\infty h(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta\Delta s\Delta t,$$

$$B_2(x,y)=\int_x^\infty[f(s,y)\frac{q}{p}K^{\frac{q-p}{p}}+g(s,y)\frac{r}{p}K^{\frac{r-p}{p}}$$

$$+\int_y^\infty\int_s^\infty h(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\Delta\xi\Delta\eta]\Delta s,\forall K>0.$$

Theorem 11 *Suppose u, f, g, h are defined as in Theorem 9. If for $(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality:*

$$u(x,y)\le\int_y^\infty\int_x^\infty[f(s,t)u(s,t)+g(s,t)u(s,t)$$

$$+\int_t^\infty\int_s^\infty h(\xi,\eta)u(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t,$$

then $u(x,y)\equiv 0$.

The proof for Theorem 11 is similar to Theorem 9, and we omit it here.

Based on Theorem 9, we establish a Gronwall-Bellman-Volterra-Fredholm type inequality containing integration on infinite intervals on time scales as follows.

Theorem 12 *Suppose u, f_i, g_i, $h_i\in C_{rd}(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0,\mathbb{R}_+)$, $i=1,2$, a, p, q, r, m are defined as in Theorem 9, and $M\in\mathbb{T}_0$, $N\in\widetilde{\mathbb{T}}_0$ are two fixed numbers. If for $(x,y)\in([M,\infty)\bigcap\mathbb{T})\times([N,\infty)\bigcap\mathbb{T})$, $u(x,y)$ satisfies the following inequality:*

$$u^p(x,y)\le a(x,y)+\int_y^\infty\int_x^\infty[f_1(s,t)u^q(s,t)$$

$$+g_1(s,t)u^r(s,t)]\Delta s\Delta t$$

$$+\int_y^\infty\int_x^\infty\int_t^\infty\int_s^\infty h_1(\xi,\eta)u^m(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t$$

$$+\int_N^\infty\int_M^\infty[f_2(s,t)u^q(s,t)+g_2(s,t)u^r(s,t)]\Delta s\Delta t$$

$$+\int_N^\infty\int_M^\infty\int_t^\infty\int_s^\infty h_2(\xi,\eta)u^m(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t, \tag{18}$$

then we have

$$u(x,y)\le\{[\frac{\lambda+\widetilde{B}_6}{1-\widetilde{B}_5}]\widetilde{B}_3(x,y)+\widetilde{B}_4(x,y)\}^{\frac{1}{p}},$$

$$(x,y)\in([M,\infty)\bigcap\mathbb{T})\times([N,\infty)\bigcap\mathbb{T}), \tag{19}$$

provided that $\widetilde{B}_5<1$, and $-\widetilde{B}_2(x,.)\in\mathfrak{R}^+$, where

$$\lambda=\int_N^\infty\int_M^\infty[f_2(s,t)\frac{p-q}{p}K^{\frac{q}{p}}+g_2(s,t)\frac{p-r}{p}K^{\frac{r}{p}}$$

$$+\int_t^\infty\int_s^\infty h_2(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t, \tag{20}$$

$$\widetilde{B}_1(x,y)=a(x,y)+\int_y^\infty\int_x^\infty[f_1(s,t)\frac{p-q}{p}K^{\frac{q}{p}}$$

$$+g_1(s,t)\frac{p-r}{p}K^{\frac{r}{p}}]\Delta s\Delta t+$$

$$\int_y^\infty\int_x^\infty\int_t^\infty\int_s^\infty h_1(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta\Delta s\Delta t, \tag{21}$$

$$\widetilde{B}_2(x,y)=\int_x^\infty[f_1(s,y)\frac{q}{p}K^{\frac{q-p}{p}}+g_1(s,y)\frac{r}{p}K^{\frac{r-p}{p}}$$

$$+\int_y^\infty\int_s^\infty h_1(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\Delta\xi\Delta\eta]\Delta s,\ \ \forall K>0, \tag{22}$$

$$\widetilde{B}_3(x,y)=1+\int_y^\infty e_{-\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(x,t)\Delta t, \tag{23}$$

$$\widetilde{B}_4(x,y)=\widetilde{B}_1(x,y)$$

$$+\int_y^\infty e_{-\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(x,t)\widetilde{B}_1(x,t)\Delta t, \tag{24}$$

$$\widetilde{B}_5=\int_N^\infty\int_M^\infty[f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}\widetilde{B}_3(s,t)$$
$$+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}\widetilde{B}_3(s,t)]\Delta s\Delta t+\int_N^\infty\int_M^\infty$$
$$\int_t^\infty\int_s^\infty h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\widetilde{B}_3(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t,\quad (25)$$
$$\widetilde{B}_6=\int_N^\infty\int_M^\infty[f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}\widetilde{B}_4(s,t)$$
$$+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}\widetilde{B}_4(s,t)]\Delta s\Delta t+\int_N^\infty\int_M^\infty$$
$$\int_t^\infty\int_s^\infty h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\widetilde{B}_4(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t.\quad (26)$$

Proof: Let the right side of (18) be $v(x,y)$, and

$$\mu=\int_N^\infty\int_M^\infty[f_2(s,t)u^q(s,t)+g_2(s,t)u^r(s,t)]\Delta s\Delta t$$
$$+\int_N^\infty\int_M^\infty\int_t^\infty\int_s^\infty h_2(\xi,\eta)u^m(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t.\quad (27)$$

Then

$$u(x,y)\le v^{\frac{1}{p}}(x,y),$$
$$(x,y)\in([M,\infty)\bigcap\mathbb{T})\times([N,\infty)\bigcap\mathbb{T}),\quad (28)$$

Fix $X\in[M,\infty)\bigcap\mathbb{T}$. Then

$$v(X,y)=a(X,y)+\mu$$
$$+\int_y^\infty\int_X^\infty[f_1(s,t)u^q(s,t)+g_1(s,t)u^r(s,t)]\Delta s\Delta t$$
$$+\int_y^\infty\int_X^\infty\int_t^\infty\int_s^\infty h_1(\xi,\eta)u^m(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t$$
$$\le a(X,y)+\mu+$$
$$\int_y^\infty\int_X^\infty[f_1(s,t)v^{\frac{q}{p}}(s,t)+g_1(s,t)v^{\frac{r}{p}}(s,t)]\Delta s\Delta t$$
$$+\int_y^\infty\int_X^\infty\int_t^\infty\int_s^\infty h_1(\xi,\eta)v^{\frac{m}{p}}(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t.\quad (29)$$

Considering the structure of (29) is similar to (14), then following in a same manner as the process of (14)-(17) we can deduce that for $y\in[N,\infty)\bigcap\mathbb{T}$,

$$v(X,y)\le\mu+\widetilde{B}_1(X,y)$$
$$+\int_y^\infty e_{-\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(X,t)(\mu+\widetilde{B}_1(X,t))\Delta t$$
$$=\mu[1+\int_y^\infty e_{-\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(X,t)\Delta t]+\widetilde{B}_1(X,y)$$
$$+\int_y^\infty e_{-\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(X,t)\widetilde{B}_1(X,t)\Delta t.\quad (30)$$

Since X is selected from $[M,\infty)\bigcap\mathbb{T}$ arbitrarily, then in fact (30) holds for $\forall x\in[M,\infty)\bigcap\mathbb{T}$, that is

$$v(x,y)\le\mu[1+\int_y^\infty e_{-\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(x,t)\Delta t]+$$
$$\widetilde{B}_1(x,y)+\int_y^\infty e_{-\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(x,t)\widetilde{B}_1(x,t)\Delta t$$
$$=\mu\widetilde{B}_3(x,y)+\widetilde{B}_4(x,y),$$
$$(x,y)\in([M,\infty)\bigcap\mathbb{T})\times([N,\infty)\bigcap\mathbb{T}),\quad (31)$$

On the other hand, from Lemma 8, (27) and (28) we obtain

$$\mu\le\int_N^\infty\int_M^\infty[f_2(s,t)v^{\frac{q}{p}}(s,t)+g_2(s,t)v^{\frac{r}{p}}(s,t)$$
$$+\int_t^\infty\int_s^\infty h_2(\xi,\eta)v^{\frac{m}{p}}(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t$$
$$\le\int_N^\infty\int_M^\infty[f_2(s,t)(\frac{q}{p}K^{\frac{q-p}{p}}v(s,t)+\frac{p-q}{p}K^{\frac{q}{p}})$$
$$+g_2(s,t)(\frac{r}{p}K^{\frac{r-p}{p}}v(s,t)+\frac{p-r}{p}K^{\frac{r}{p}})]\Delta s\Delta t$$
$$+\int_N^\infty\int_M^\infty\int_t^\infty\int_s^\infty h_2(\xi,\eta)(\frac{m}{p}K^{\frac{m-p}{p}}v(\xi,\eta)$$
$$+\frac{p-m}{p}K^{\frac{m}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$
$$=\lambda+\int_N^\infty\int_M^\infty[f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}v(s,t)$$
$$+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}v(s,t)]\Delta s\Delta t+\int_N^\infty\int_M^\infty$$
$$\int_t^\infty\int_s^\infty h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}v(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t,\quad (32)$$

Then using (31) in (32) yields

$$\mu\le\lambda+\int_N^\infty\int_M^\infty\{f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}[\mu\widetilde{B}_3(s,t)+\widetilde{B}_4$$
$$(s,t)]+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}[\mu\widetilde{B}_3(s,t)+\widetilde{B}_4(s,t)]\}\Delta s\Delta t$$
$$+\int_N^\infty\int_M^\infty\int_t^\infty\int_s^\infty h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}[\mu\widetilde{B}_3(\xi,\eta)$$
$$+\widetilde{B}_4(\xi,\eta)]\Delta\xi\Delta\eta\Delta s\Delta t$$
$$=\lambda+\mu\{\int_N^\infty\int_M^\infty[f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}\widetilde{B}_3(s,t)$$

$$+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}\widetilde{B}_3(s,t)]\Delta s\Delta t$$
$$+\int_N^\infty\int_M^\infty\int_t^\infty\int_s^\infty h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}$$
$$\widetilde{B}_3(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t\}$$
$$+\int_N^\infty\int_M^\infty[f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}\widetilde{B}_4(s,t)$$
$$+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}\widetilde{B}_4(s,t)]\Delta s\Delta t+\int_N^\infty\int_M^\infty$$
$$\int_t^\infty\int_s^\infty h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\widetilde{B}_4(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t$$
$$=\lambda+\mu\widetilde{B}_5+\widetilde{B}_6, \tag{33}$$

which is followed by

$$\mu\le\frac{\lambda+\widetilde{B}_6}{1-\widetilde{B}_5}. \tag{34}$$

Combining (28), (31) and (34) we can obtain the desired inequality (19).

In the proof of Theorem 12, if we let the right-hand side of (18) be $a(x,y)+v(x,y)$ in the first statement, then following in a similar process as in Theorem 12 we obtain another bound of the function $u(x,y)$, which is shown in the following theorem.

Theorem 13 *Under the conditions of Theorem 12, if for $(x,y)\in([M,\infty)\bigcap\mathbb{T})\times([N,\infty)\bigcap\mathbb{T})$, $u(x,y)$ satisfies (18), then we have*

$$u(x,y)\le\{a(x,y)+\frac{\widetilde{\mu}+J_1(M,N)}{1-\widetilde{\lambda}}e_{-J_2}(y,\infty)\}^{\frac{1}{p}},$$
$$(x,y)\in([M,\infty)\bigcap\mathbb{T})\times([N,\infty)\bigcap\mathbb{T}),$$

provided that $\widetilde{\lambda}<1$ and $-J_2(x,.)\in\mathfrak{R}^+$, where

$$\widetilde{\lambda}=\int_N^\infty\int_M^\infty[f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}e_{-J_2}(t,\infty)$$
$$+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}e_{-J_2}(t,\infty)]\Delta s\Delta t$$
$$+\int_N^\infty\int_M^\infty\int_t^\infty\int_s^\infty h_2(\xi,\eta)$$
$$\frac{m}{p}K^{\frac{m-p}{p}}e_{-J_2}(\eta,\infty)\Delta\xi\Delta\eta\Delta s\Delta t,$$
$$\widetilde{\mu}=\int_N^\infty\int_M^\infty[f_2(s,t)(\frac{q}{p}K^{\frac{q-p}{p}}a(s,t)+\frac{p-q}{p}K^{\frac{q}{p}})$$
$$+g_2(s,t)(\frac{r}{p}K^{\frac{r-p}{p}}a(s,t)+\frac{p-r}{p}K^{\frac{r}{p}})$$
$$+\int_t^\infty\int_s^\infty h_2(\xi,\eta)(\frac{m}{p}K^{\frac{m-p}{p}}(a(\xi,\eta)+v(\xi,\eta))$$
$$+\frac{p-m}{p}K^{\frac{m}{p}})\Delta\xi\Delta\eta]\Delta s\Delta t,$$
$$J_1(x,y)=\int_y^\infty\int_x^\infty[f_1(s,t)(\frac{q}{p}K^{\frac{q-p}{p}}a(s,t)+\frac{p-q}{p}$$
$$K^{\frac{q}{p}})+g_1(s,t)(\frac{r}{p}K^{\frac{r-p}{p}}a(s,t)+\frac{p-r}{p}K^{\frac{r}{p}})]\Delta s\Delta t$$
$$+\int_y^\infty\int_X^\infty\int_t^\infty\int_s^\infty h_1(\xi,\eta)(\frac{m}{p}K^{\frac{m-p}{p}}a(\xi,\eta)$$
$$+\frac{p-m}{p}K^{\frac{m}{p}})\Delta\xi\Delta\eta\Delta s\Delta t,$$
$$J_2(x,y)=\int_x^\infty[f_1(s,y)\frac{q}{p}K^{\frac{q-p}{p}}+g_1(s,y)\frac{r}{p}K^{\frac{r-p}{p}}$$
$$+\int_y^\infty\int_s^\infty h_1(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\Delta\xi\Delta\eta]\Delta s.$$

Finally, we establish a more general inequality than in Theorems 2.4-2.5. Consider the following inequality:

$$u^p(x,y)\le a(x,y)+\int_y^\infty\int_x^\infty[L(s,t,u(s,t))$$
$$+\int_t^\infty\int_s^\infty h_1(\xi,\eta)u^q(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t$$
$$+\int_N^\infty\int_M^\infty[L(s,t,u(s,t))$$
$$+\int_t^\infty\int_s^\infty h_2(\xi,\eta)u^q(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t, \tag{35}$$

where u, a, p, q are defined as in Theorem 9, $M\in\mathbb{T}_0$, $N\in\widetilde{\mathbb{T}}_0$ are two fixed numbers, $L\in(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0\times\mathbb{R}_+,\mathbb{R}_+)$, and $0\le L(s,t,x)-L(s,t,y)\le A(s,t,y)(x-y)$ for $x\ge y\ge 0$, where $A\in(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0\times\mathbb{R}_+,\mathbb{R}_+)$.

Theorem 14 *If for $(x,y)\in([M,\infty)\bigcap\mathbb{T})\times([N,\infty)\bigcap\mathbb{T})$, $u(x,y)$ satisfies (35), then the following inequality holds.*

$$u(x,y)\le\{[\frac{\widehat{\lambda}+\widehat{B}_6}{1-\widehat{B}_5}]\widehat{B}_3(x,y)+\widehat{B}_4(x,y)\}^{\frac{1}{p}},$$
$$(x,y)\in([M,\infty)\bigcap\mathbb{T})\times([N,\infty)\bigcap\mathbb{T}), \tag{36}$$

provided that $\widehat{B}_5<1$, and $-\widehat{B}_2(x,.)\in\mathfrak{R}^+$, where

$$\widehat{\lambda}=\int_N^\infty\int_M^\infty[L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})$$

$$+\int_t^\infty\int_s^\infty h_2(\xi,\eta)\frac{p-q}{p}K^{\frac{q}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t, \quad (37)$$

$$\widehat{B}_1(x,y)=a(x,y)+\int_y^\infty\int_x^\infty[L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})$$

$$+\int_t^\infty\int_s^\infty h_1(\xi,\eta)\frac{p-q}{p}K^{\frac{q}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t, \quad (38)$$

$$\widehat{B}_2(x,y)=\int_x^\infty[A(s,y,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}$$

$$+\int_y^\infty\int_s^\infty h_1(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\Delta\xi\Delta\eta]\Delta s, \quad \forall K>0, \quad (39)$$

$$\widehat{B}_3(x,y)=1+\int_y^\infty e_{-\widehat{B}_2}(y,\sigma(t))\widehat{B}_2(x,t)\Delta t, \quad (40)$$

$$\widehat{B}_4(x,y)=\widehat{B}_1(x,y)$$

$$+\int_y^\infty e_{-\widehat{B}_2}(y,\sigma(t))\widehat{B}_2(x,t)\widehat{B}_1(x,t)\Delta t, \quad (41)$$

$$\widehat{B}_5=\int_N^\infty\int_M^\infty[A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}\widehat{B}_3(s,t)$$

$$+\int_t^\infty\int_s^\infty h_2(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\widehat{B}_3(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t, \quad (42)$$

$$\widehat{B}_6=\int_N^\infty\int_M^\infty[A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}\widehat{B}_4(s,t)$$

$$+\int_t^\infty\int_s^\infty h_2(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\widehat{B}_4(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t. \quad (43)$$

Proof: Let the right side of (35) be $v(x,y)$, and

$$\widehat{\mu}=\int_N^\infty\int_M^\infty[L(s,t,u(s,t))$$

$$+\int_t^\infty\int_s^\infty h_2(\xi,\eta)u^q(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t. \quad (44)$$

Then

$$u(x,y)\le v^{\frac{1}{p}}(x,y),$$

$$(x,y)\in([M,\infty)\bigcap\mathbb{T})\times([N,\infty)\bigcap\mathbb{T}), \quad (45)$$

Fix $X\in[M,\infty)\bigcap\mathbb{T}$. Then

$$v(X,y)=a(X,y)+\widehat{\mu}+\int_y^\infty\int_X^\infty[L(s,t,u(s,t))$$

$$+\int_t^\infty\int_s^\infty h_1(\xi,\eta)u^q(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t$$

$$\le a(X,y)+\widehat{\mu}+\int_y^\infty\int_X^\infty[L(s,t,v^{\frac{1}{p}}(s,t))$$

$$+\int_t^\infty\int_s^\infty h_1(\xi,\eta)v^{\frac{q}{p}}(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t. \quad (46)$$

Combining with Lemma 8 we have

$$v(X,y)\le a(X,y)+\widehat{\mu}+$$

$$\int_y^\infty\int_X^\infty L(s,t,\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)+\frac{p-1}{p}K^{\frac{1}{p}})\Delta s\Delta t$$

$$+\int_y^\infty\int_X^\infty\int_t^\infty\int_s^\infty h_1(\xi,\eta)(\frac{q}{p}K^{\frac{q-p}{p}}v(\xi,\eta)$$

$$+\frac{p-q}{p}K^{\frac{q}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$

$$=a(X,y)+\widehat{\mu}$$

$$+\int_y^\infty\int_X^\infty[L(s,t,\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)+\frac{p-1}{p}K^{\frac{1}{p}})$$

$$-L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})+L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})]\Delta s\Delta t$$

$$+\int_y^\infty\int_X^\infty\int_t^\infty\int_s^\infty h_1(\xi,\eta)(\frac{q}{p}K^{\frac{q-p}{p}}v(\xi,\eta)$$

$$+\frac{p-q}{p}K^{\frac{q}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$

$$\le a(X,y)+\widehat{\mu}+$$

$$\int_y^\infty\int_X^\infty[A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)$$

$$+L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})]\Delta s\Delta t+\int_y^\infty\int_X^\infty$$

$$[\int_t^\infty\int_s^\infty h_1(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\Delta\xi\Delta\eta]v(X,t)\Delta s\Delta t$$

$$+\int_y^\infty\int_X^\infty\int_t^\infty\int_s^\infty h_1(\xi,\eta)\frac{p-q}{p}K^{\frac{q}{p}}\Delta\xi\Delta\eta\Delta s\Delta t$$

$$\le a(X,y)+\widehat{\mu}+$$

$$\int_y^\infty[\int_X^\infty A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}\Delta s]v(X,t)\Delta t$$

$$+\int_y^\infty\int_X^\infty L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\Delta s\Delta t+\int_y^\infty$$

$$[\int_X^\infty\int_t^\infty\int_s^\infty h_1(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\Delta\xi\Delta\eta\Delta s]v(X,t)\Delta t$$

$$+\int_y^\infty\int_X^\infty\int_t^\infty\int_s^\infty h_1(\xi,\eta)\frac{p-q}{p}K^{\frac{q}{p}}\Delta\xi\Delta\eta\Delta s\Delta t$$

$$=\widehat{\mu}+\widehat{B}_1(X,y)+\int_y^\infty\widehat{B}_2(X,t)v(X,t)\Delta t, \quad (47)$$

We notice the structure of (47) is similar to (15). So following in a same manner as (15)-(17) we obtain

$$v(x,y)\le\widehat{\mu}[1+\int_y^\infty e_{-\widehat{B}_2}(y,\sigma(t))\widehat{B}_2(x,t)\Delta t]+$$

$$\widehat{B}_1(x,y)+\int_y^{\infty} e_{-\widehat{B}_2}(y,\sigma(t))\widehat{B}_2(x,t)\widehat{B}_1(x,t)\Delta t$$

$$=\widehat{\mu}\widehat{B}_3(x,y)+\widehat{B}_4(x,y),$$

$$(x,y)\in([M,\infty)\bigcap\mathbb{T})\times([N,\infty)\bigcap\mathbb{T}), \quad (48)$$

On the other hand, from Lemma 8, (44) and (45) we have

$$\widehat{\mu}\le\int_N^{\infty}\int_M^{\infty}[L(s,t,v^{\frac{1}{p}})$$

$$+\int_t^{\infty}\int_s^{\infty}h_2(\xi,\eta)v^{\frac{q}{p}}(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t$$

$$\le\int_N^{\infty}\int_M^{\infty}[L(s,t,\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)+\frac{p-1}{p}K^{\frac{1}{p}})+$$

$$\int_t^{\infty}\int_s^{\infty}h_2(\xi,\eta)(\frac{q}{p}K^{\frac{q-p}{p}}v(\xi,\eta)$$

$$+\frac{p-q}{p}K^{\frac{q}{p}})\Delta\xi\Delta\eta]\Delta s\Delta t$$

$$=\int_N^{\infty}\int_M^{\infty}[L(s,t,\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)+\frac{p-1}{p}K^{\frac{1}{p}})$$

$$-L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})+L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})]\Delta s\Delta t$$

$$+\int_N^{\infty}\int_M^{\infty}\int_t^{\infty}\int_s^{\infty}h_2(\xi,\eta)(\frac{q}{p}K^{\frac{q-p}{p}}v(\xi,\eta)$$

$$+\frac{p-q}{p}K^{\frac{q}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$

$$\le\int_N^{\infty}\int_M^{\infty}[A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)$$

$$+L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})]\Delta s\Delta t$$

$$+\int_N^{\infty}\int_M^{\infty}\int_t^{\infty}\int_s^{\infty}h_2(\xi,\eta)(\frac{q}{p}K^{\frac{q-p}{p}}v(\xi,\eta)$$

$$+\frac{p-q}{p}K^{\frac{q}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$

$$=\widehat{\lambda}+\int_N^{\infty}\int_M^{\infty}[A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)$$

$$+\int_t^{\infty}\int_s^{\infty}h_2(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}v(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t. \quad (49)$$

where $\widehat{\lambda}$ is defined in (37). Then using (48) in (49) yields

$$\widehat{\mu}\le\widehat{\lambda}+\int_N^{\infty}\int_M^{\infty}A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}$$

$$[\widehat{\mu}\widehat{B}_3(s,t)+\widehat{B}_4(s,t)]\Delta s\Delta t$$

$$+\int_N^{\infty}\int_M^{\infty}\int_t^{\infty}\int_s^{\infty}h_2(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}$$

$$[\widehat{\mu}\widehat{B}_3(\xi,\eta)+\widehat{B}_4(\xi,\eta)]\Delta\xi\Delta\eta\Delta s\Delta t$$

$$=\widehat{\lambda}+\widehat{\mu}\{\int_N^{\infty}\int_M^{\infty}A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}\widehat{B}_3(s,t)$$

$$+\int_t^{\infty}\int_s^{\infty}h_2(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\widehat{B}_3(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t\}$$

$$+\int_N^{\infty}\int_M^{\infty}[A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}\widehat{B}_4(s,t)$$

$$+\int_t^{\infty}\int_s^{\infty}h_2(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\widehat{B}_4(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t$$

$$=\widehat{\lambda}+\widehat{\mu}\widehat{B}_5+\widehat{B}_6, \quad (50)$$

which is followed by

$$\widehat{\mu}\le\frac{\widehat{\lambda}+\widehat{B}_6}{1-\widehat{B}_5}. \quad (51)$$

Combining (45), (48) and (51) we can obtain the desired result.

Remark 15 *In [31-32], the authors researched some Gronwall-Bellman type inequalities in two independent variables on time scales. We note that the presented inequalities in (8), (18) and (35) established here are of different forms from the main results in [31-32].*

3 Some Applications

In this section, we present some applications for the results established above. New explicit bounds for solutions for certain dynamic equations are derived in the first two examples, while the quantitative property of solutions is concerned in the final example.

Example 1: Consider the following dynamic differential equation

$$(u^p(x,y))^{\Delta\Delta}_{yx}=F(x,y,u(x,y),$$

$$\int_y^{\infty}\int_x^{\infty}W(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta),\ (x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0, \quad (52)$$

with the initial condition $u^p(\infty,y))^{\Delta}_y=b^{\Delta}(y)$, $u^p(x,\infty)=a(x)$, where $u\in C_{rd}(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0,\mathbb{R})$, $a\in C_{rd}(\mathbb{T}_0,\mathbb{R})$, $b\in C_{rd}(\widetilde{\mathbb{T}}_0,\mathbb{R})$, b is *delta differential*, and $b(\infty)=0$, $k\in C_{rd}(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0,\mathbb{R}_+)$, $p>0$ is a constant, $F\in(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0\times\mathbb{R}^2,\mathbb{R})$, $W\in(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0\times\mathbb{R},\mathbb{R})$.

Theorem 16 *Suppose $u(x,y)$ is a solution of (52), $|a(x)+b(y)| \le k(x,y)$, and $|F(x,y,u,v)| \le f(x,y)|u|^q + |v|$, $|W(\xi,\eta,u)| \le h(\xi,\eta)|u|^m$, where f, h, q, m are defined as in Theorem 9. Then*

$$|u(x,y)| \le [B_1(x,y) + \int_y^\infty e_{-B_2}(y,\sigma(t))$$

$$B_2(x,t)B_1(x,t)\Delta t]^{\frac{1}{p}},\ (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \quad (53)$$

where

$$B_1(x,y) = k(x,y) + \int_y^\infty \int_x^\infty [f(s,t)\frac{p-q}{p}K^{\frac{q}{p}}$$

$$+\int_t^\infty \int_s^\infty h(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t, \forall K > 0,$$

and $B_2(x,y)$ is defined as in Theorem 9 (with $g(x,y) \equiv 0$).

Proof: The equivalent integral form of (52) can be denoted by

$$u^p(x,y) = a(x) + b(y) + \int_y^\infty \int_x^\infty F(s,t,u(s,t),$$

$$\int_t^\infty \int_s^\infty W(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta)\Delta s\Delta t. \quad (54)$$

Then

$$|u^p(x,y)| \le k(x,y) + \int_y^\infty \int_x^\infty |F(s,t,u(s,t),$$

$$\int_t^\infty \int_s^\infty W(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta)|\Delta s\Delta t$$

$$\le k(x,y) + \int_y^\infty \int_x^\infty [f(s,t)|u(s,t)|^q$$

$$+|\int_t^\infty \int_s^\infty W(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta|]\Delta s\Delta t$$

$$\le k(x,y) + \int_y^\infty \int_x^\infty [f(s,t)|u(s,t)|^q$$

$$+\int_t^\infty \int_s^\infty h(\xi,\eta)|u(\xi,\eta)|^m\Delta\xi\Delta\eta]\Delta s\Delta t, \quad (55)$$

and a suitable application of Theorem 9 to (55) yields the desired inequality (53).

Theorem 17 *Under the conditions of Theorem 16, furthermore, we have*

$$|u(x,y)| \le [B_1(x,y)e_{-B_2}(y,\infty)]^{\frac{1}{p}},\ (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \quad (56)$$

where B_1, B_2 are defined as in Theorem 16.

Proof: The desired inequality can be obtained by an application of Theorem 10 to (55).

Example 2: Consider the following dynamic integral equation

$$u^p(x,y) = C + \int_y^\infty \int_x^\infty F_1(s,t,u(s,t),$$

$$\int_t^\infty \int_s^\infty W_1(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta)\Delta s\Delta t$$

$$+\int_N^\infty \int_M^\infty F_2(s,t,u(s,t),$$

$$\int_t^\infty \int_s^\infty W_2(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta)\Delta s\Delta t,$$

$$(x,y) \in ([M,\infty)\bigcap\mathbb{T}) \times ([N,\infty)\bigcap\mathbb{T}), \quad (57)$$

where $u \in C_{rd}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \mathbb{R})$, $p > 0$ is a constant, $C = u^p(\infty,\infty)$, $M \in \mathbb{T}_0$, $N \in \widetilde{\mathbb{T}}_0$ are two fixed numbers, $F_i \in (\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0 \times \mathbb{R}^2, \mathbb{R})$, $W_i \in (\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0 \times \mathbb{R}, \mathbb{R})$, $i = 1, 2$.

Theorem 18 *Suppose $u(x,y)$ is a solution of (57), and $|F_i(x,y,u,v)| \le L(x,y,|u|) + |v|$, $|W_i(\xi,\eta,u)| \le h_i(\xi,\eta)|u|^q$, $i = 1,2$, where L, h_i, $i = 1,2$, q are defined as in Theorem 14. Then the following inequality holds.*

$$|u(x,y)| \le \{[\frac{\widehat{\lambda} + \widehat{B}_6}{1 - \widehat{B}_5}]\widehat{B}_3(x,y) + \widehat{B}_4(x,y)\}^{\frac{1}{p}},$$

$$(x,y) \in ([M,\infty)\bigcap\mathbb{T}) \times ([N,\infty)\bigcap\mathbb{T}), \quad (58)$$

provided that $\widehat{B}_5 < 1$, where $\widehat{\lambda}$, $\widehat{B}_2(x,y)$, $\widehat{B}_3(x, y)$, $\widehat{B}_4(x,y)$, $\widehat{B}_5$, $\widehat{B}_6$ are defined the same as in Theorem 14, and

$$\widehat{B}_1(x,y) = |C| + \int_y^\infty \int_x^\infty [L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})$$

$$+\int_t^\infty \int_s^\infty h_1(\xi,\eta)\frac{p-q}{p}K^{\frac{q}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t.$$

Proof: From (57) we have

$$|u^p(x,y)| \le |C| + \int_y^\infty \int_x^\infty |F_1(s,t,u(s,t),$$

$$\int_t^\infty \int_s^\infty W_1(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta)|\Delta s\Delta t$$

$$+\int_N^\infty \int_M^\infty |F_2(s,t,u(s,t),$$

$$\int_t^\infty \int_s^\infty W_2(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta)|\Delta s\Delta t$$
$$\le |C| + \int_y^\infty \int_x^\infty [L(s,t,|u(s,t|)$$
$$+|\int_t^\infty \int_s^\infty W_1(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta|]\Delta s\Delta t$$
$$+\int_N^\infty \int_M^\infty [L(s,t,|u(s,t|)$$
$$+|\int_t^\infty \int_s^\infty W_2(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta|]\Delta s\Delta t$$
$$\le |C| + \int_y^\infty \int_x^\infty [L(s,t,|u(s,t|)$$
$$+\int_t^\infty \int_s^\infty h_1(\xi,\eta)|u(\xi,\eta)|^q\Delta\xi\Delta\eta]\Delta s\Delta t$$
$$+\int_N^\infty \int_M^\infty [L(s,t,|u(s,t|)$$
$$+\int_t^\infty \int_s^\infty h_2(\xi,\eta)|u(\xi,\eta)|^q\Delta\xi\Delta\eta]\Delta s\Delta t.$$

So by use of Theorem 14 we can obtain the desired inequality (58).

Example 3: Consider the following dynamic integral equation

$$u(x,y) = C + \int_y^\infty \int_x^\infty F(s,t,u(s,t),$$
$$\int_t^\infty \int_s^\infty W(\xi,\eta,u(\xi,\eta))\Delta\xi\Delta\eta)\Delta s\Delta t, \quad (59)$$

where $u \in C_{rd}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \mathbb{R})$, $C = u^p(\infty,\infty)$, $F \in (\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0 \times \mathbb{R}^2, \mathbb{R})$, $W \in (\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0 \times \mathbb{R}, \mathbb{R})$.

Theorem 19 *Assume* $|F(s,t,u_1,v_1) - F(s,t,u_2,v_2)| \le f(s,t)|u_1 - u_2| + |v_1 - v_2|$, $|W(s,t,u_1) - W(s,t,u_2)| \le h(s,t)|u_1 - u_2|$, *where* f, h *are defined as in Theorem 9, and furthermore, assume* $\tau_1(x) \ge x_0$, $\tau_2(y) \ge y_0$, *then Eq. (62) has at most one solution.*

Proof: Suppose $u_1(x,y)$, $u_2(x,y)$ are two solutions of (59). Then we have

$$|u_1(x,y) - u_2(x,y)| \le |\int_y^\infty \int_x^\infty [F(s,t,u_1(s,t)$$
$$\int_t^\infty \int_s^\infty W(\xi,\eta,u_1(\xi,\eta))\Delta\xi\Delta\eta) - F(s,t,u_2(s,t)$$
$$\int_t^\infty \int_s^\infty W(\xi,\eta,u_2(\xi,\eta))\Delta\xi\Delta\eta)]\Delta s\Delta t|$$
$$\le \int_y^\infty \int_x^\infty |F(s,t,u_1(s,t) \int_t^\infty \int_s^\infty W(\xi,\eta,u_1(\xi,\eta))$$
$$\Delta\xi\Delta\eta) - F(s,t,u_2(s,t) \int_t^\infty \int_s^\infty W(\xi,\eta,u_2(\xi,\eta))$$
$$\Delta\xi\Delta\eta)|\Delta s\Delta t$$
$$\le \int_y^\infty \int_x^\infty f(s,t)|u_1(s,t) - u_2(s,t)|\Delta s\Delta t$$
$$+\int_y^\infty \int_x^\infty \int_t^\infty \int_s^\infty |W(\xi,\eta,u_1(\xi,\eta))$$
$$-W(\xi,\eta,u_2(\xi,\eta))|\Delta\xi\Delta\eta\Delta s\Delta t$$
$$\le \int_y^\infty \int_x^\infty f(s,t)|u_1(s,t) - u_2(s,t)|\Delta s\Delta t + \int_y^\infty$$
$$\int_x^\infty \int_t^\infty \int_s^\infty h(\xi,\eta)|u_1(\xi,\eta) - u_2(\xi,\eta)|\Delta\xi\Delta\eta\Delta s\Delta t, \quad (60)$$

A suitable application of Theorem 11 yields $|u_1(x,y) - u_2(x,y)| \le 0$, that is, $u_1(x,y) \equiv u_2(x,y)$, and the proof is complete.

4 Conclusions

We have established some new Gronwall-Bellman-type dynamic inequalities in two independent variables containing integration on infinite intervals on time scales. As applications, we apply the results established to research boundedness and quantitative property for the solutions to some certain dynamic equations on time scales. In fact, the motive to establish Gronwall-Bellman type inequalities with new forms mostly comes from the research for the properties of solutions to various differential equations, difference equations, and dynamic equations on time scales. It is worth to note that in order to fulfill analysis for the properties of solutions to some fractional differential equations, it is necessary to investigate how to establish new Gronwall-Bellman type fractional inequalities, which are supposed to further research.

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