# The Spectrum of A Parallel Repairable System with Warm Standby

Wenlong Wang
Zhejiang Yuexiu University of Foreign Language
College of Network Communication
Zhejiang Shaoxing, 312000
P. R. China
wwl@usx.edu.cn

Zhiying Li
Shaoxing University
Yuanpei College
Zhejiang Shaoxing, 312000
P. R. China
scorpiolzy@126.com

*Abstract:* In this paper, the spectrum of a parallel repairable system with warm standby is investigated. Firstly, we formulate the problem into a suitable Banach space. Then we carry out a detailed spectral analysis of the system operator. Based on the spectral analysis and $C_0$ semigroup theory, we prove the existence of positive solution and finite expansion of the solution corresponding to its eigenvector. As a consequence we get that its dynamic solutions converges exponentially to the stead-state solution. Finally, we obtain the finite expansion of solution and derive some reliability indices of the system.

*Key–Words:* $C_0$ semigroup theory; Spectrum; Parallel Repairable system; Stead-state; Availability;

## 1 Introduction

With the development of the modern technology and extensive use of electronic products, the reliability problem of the reparable systems has been a hot topic. It is well-known that reliability of a system is an important concept in engineering, it takes an essential rule in the plan, design and operation strategy of various complex systems. In order to increase the reliability of a system, a repair unit is necessary for increasing the performance and reducing the downtime or the maintenance. Therefore, repairable system is not only a kind of important system discussed in reliability theory but also one of the main objects studied in reliability mathematics. Many authors have worked in this field, including system modeling (see, [1][2][3]) and model analysis (see, [4][5][6][7][8]) and the references therein.

Different from the early study of reparable system, in which the key point emphases the reliability indices involving availability of the system, which usually were obtained by steady state, the issue is to obtain the time-dependent solution of the system govern by the partial differential equations. This is because we cannot wait for a long time in some cases, for example, the cases of [9] and [10]. The change of key point of the issue requires us to analyze completely the system including spectrum of the system operator and finite expansion of solution. From application point of view, the time we can observe the steady state of the system becomes obviously an important index, which is especial important in the investigation of human health problem or recovery. Therefore our task is mainly to solve the following questions:

(1) the system under consideration has a unique nonnegative time-dependent solution;

(2) approximate of solution;

(3) the system has a steady state, and the dynamic solution of the system converges to the steady state.

Let us recall the observation time issue. Let $S$ be a reparable system and $P(t)$ be the state vector, which describe the probability in the various states. Suppose that the system has a steady state $\hat{P}_0$. If there is a time $\tau_0$ such that $\|P(t) - \hat{P}_0\|\| \leq 0.25, t \geq \tau_0$, then it is said that the steady state of $S$ is observable at time $\tau_0$. Obviously, the observable time $\tau_0$ is a more valuable information in application. From the observation time issue we see that it is not only an issue of existence of the solution and steady state but also the quasi-exponential decay issue of the system. How to determine the decay rate of the dynamic solution, however, is hard work, which needs more detail spectral information of the operator determined by the system. In Ref.[11], Yuan and Xu investigated the spectrum of a two unit deteriorating standby system with repair and obtained batter results.

In the present paper, we mainly study the spectrum of the system operator, from which we can obtain an answer for the observe time issue. In the present paper, our model under consideration after certain assumptions is the same as the one in [12] although it has different background.

Redundancy plays an important role in enhancing system reliability. One of the commonly used form of redundancy is the steady system. In this case, one or more unit operate and the remaining redundant units are kept in their mode.

There have been several publications on human error analysis of redundant systems([13]-[17]). In Ref. [13]-[17], failed system repair rates are assumed to be constant, Ref. [12] consider the case when the failed system repair rates are time-dependent. A mathematical model of a repairable parallel system with standby units involving human and common-cause failures is present in Ref. [12], Markov and supplementary variable techniques are used to obtain the general expressions of the model. Furthermore, in Ref. [12], the general expression of the steady state availability is obtained, the Laplace transform technique is used to obtain the time-dependent availability, reliability and mean time to failure expressions.

The system of differential equations associated with this model is the following([12]):

$$\begin{cases} \frac{dP_0(t)}{dt} = -h_0P_0(t) + \mu_1P_1(t) + \mu_2P_2(t) \\ + \sum\limits_{j=4}^{5} \int_0^{+\infty} \mu_j(y)P_j(y,t)dy, \\ \frac{dP_1(t)}{dt} = 2\lambda_1P_0(t) - h_1P_1(t) + \mu_3P_3(t), \\ \frac{dP_2(t)}{dt} = \lambda_2P_0(t) - h_2P_2(t) + \mu_3P_3(t), \\ \frac{dP_3(t)}{dt} = 2\lambda_1P_1(t) + 2\lambda_1P_2(t) - h_3P_3(t), \\ \frac{\partial P_4(y,t)}{\partial t} + \frac{\partial P_4(y,t)}{\partial y} = -\mu_4(y)P_4(y,t), \\ \frac{\partial P_5(y,t)}{\partial t} + \frac{\partial P_5(y,t)}{\partial y} = -\mu_5(y)P_5(y,t), \end{cases} \quad (1)$$

with the boundary conditions and initial values

$$\begin{cases} P_4(0,t) = \lambda_1P_3(t), \\ P_5(0,t) = \sum\limits_{i=0}^{3} \lambda_{ci}P_i(t); \\ P_0(0) = 1, P_i(0) = 0, i = 1,2,3, \\ P_j(0,t) = 0, j = 4,5, \end{cases} \quad (2)$$

where $h_0 = 2\lambda_1+\lambda_{c0}+\lambda_2, h_1 = 2\lambda_1+\mu_1+\lambda_{c1}, h_2 = 2\lambda_1 + \mu_2 + \lambda_{c2}, h_3 = \lambda_1 + 2\mu_3 + \lambda_{c3}$.

The symbols in equations have the following meaning:

$\lambda_1$: constant failure rate of a unit;

$\lambda_2$: constant failure rate of switching mechanism or standby itself;

$\mu_1$: constant repair rate when one of the parallel units is disabled;

$\mu_2$: constant repair rate for the switching mechanism or the standby itself;;

$\mu_3$: constant repair rate when two units have been disabled;

$\mu_j(y)$: time-dependent system repair rate when the system is in state $j$ and has an elapsed repair time of $y$, for $j = 4,5$;

$\lambda_{c0}$ : constant critical common-cause failure rate;

$\lambda_{c1}$ : constant common-cause failure rate of the system when one of the parallel units has failed;

$\lambda_{c2}$ : constant common-cause failure rate of the system when switching mechanism or standby itself is disabled;

$\lambda_{c3}$ : constant common-cause failure rate when two units have failed;

$P_i(t)$ : the probability that the system is in state $i$ at time $t$, for $i = 0,1,2,3$;

$P_j(y,t)$ : the probability that the failed system is in state $j$ and has an elapse repair time of $y$, for $j = 4,5$.

In Ref.[18], Hu has investigated the well-posedness and the asymptotic stability of system (1)(2). The rest of this paper is organized as follows. In section 2, we formulate the problem into a suitable Banach space. In section 3, we carry out a detailed spectral analysis of the system operator. In section 4, based on the spectral analysis and $C_0$ semigroup theory, we prove the linear stability and the exponential stability of system. In section 5, we get that finite expansion of the solution corresponding to its eigenvector and its dynamic solutions converges exponentially to the stead-state solution. In section 6, we derive some reliability indices of the system.

## 2 Formulation of the system

In the following, we denote by $\mathbb{R}$ the real number set, $\mathbb{R}_+$ the non-negative real number set. Let $X = \mathbb{R}^4 \times (L^1(\mathbb{R}^+))^2$ equipped the norm

$$\| P \| = |P_0|+|P_1|+|P_2|+|P_3|+\|P_4(y)\|_1+\|P_5(y)\|_1$$

for $(P_0, P_1, P_2, P_3, P_4(y), P_5(y)) \in X$. It is easily to see that $X$ is a Banach space.

Before we define the system operator, we make the following assumptions:

(1) The general distributions

$$A_j(y) = 1 - e^{-\int_0^y \mu_j(s)ds}, j = 4,5, \quad (3)$$

where $\mu_j(y)$ are nonnegative and local integrable on $[0, +\infty)$, and

$$\sup_{y\geq 0} \mu_j(y) < +\infty, j = 4,5. \quad (4)$$

(2) The functions $\mu_j(y)$ satisfy

$$\int_0^{+\infty} \mu_j(y)dy = +\infty, j = 4,5. \quad (5)$$

We define the operator $\mathcal{A}$ by

$$\mathcal{A}\begin{pmatrix} P_0 \\ \\ \\ P_1 \\ P_2 \\ P_3 \\ P_4(y) \\ P_5(y) \end{pmatrix} = \begin{pmatrix} -h_0P_0 + \mu_1 P_1 + \mu_2 P_2 \\ +\sum_{j=4}^{5}\int_0^{+\infty}\mu_j(y)P_j(y)dy \\ 2\lambda_1 P_0 - h_1 P_1 + \mu_3 P_3 \\ \lambda_2 P_0 - h_2 P_2 + \mu_3 P_3 \\ 2\lambda_1 P_1 + 2\lambda_1 P_2 - h_3 P_3 \\ -P_4'(y) - \mu_4(y)P_4(y) \\ -P_5'(y) - \mu_5(y)P_5(y) \end{pmatrix} \quad (6)$$

with domain

$$D(\mathcal{A}) = \{(P_0, P_1, P_2, P_3, P_4(y), P_5(y)) \in X : \\ P_j'(y), \mu_j(y)P_j(y) \in L^1(R^+), P_j(y) \\ \text{is an absolutly continuous function}, j = 4, 5; \\ P_4(0) = \lambda_1 P_3, P_5(0) = \sum_{i=0}^{3}\lambda_{ci}P_i\}.$$

Then the equations system (1)(2) can be rewritten as an abstract Cauchy problem in $X$:

$$\begin{cases} \frac{dP(t)}{dt} = \mathcal{A}P(t), t > 0 \\ P(0) = \tilde{P}_0 \end{cases} \quad (7)$$

where $P(t) = (P_0(t), P_1(t), P_2(t), P_3(t), P_4(t), P_5(t))$, $\tilde{P}_0 = (1, 0, 0, 0, 0, 0)$.

In Ref.[18] Hu has obtained the following results.

**Theorem 1** (1) $\gamma_0 = 0$ *is a simple eigenvalue of* $\mathcal{A}$ *and there exists a corresponding positive eigenvector;* (2) *The operator* $\mathcal{A}$ *generates a positive* $C_0$ *contractive semigroup on* $X$.

## 3 Spectral analysis of $\mathcal{A}$

In this section we shall carry out a spectral analysis of $\mathcal{A}$. In what follows we always regard $X$ as a complex Banach space.

Let $z \in \mathbb{C}$, for any

$$P = (P_0, P_1, P_2, P_3, P_4(y), P_5(y)),$$

$$F = (f_0, f_1, f_2, f_3, f_4(y), f_5(y)) \in X,$$

we consider the resolvent equation $(zI - \mathcal{A})P = F$. That is

$$\begin{cases} zP_0 + h_0P_0 - \mu_1 P_1 - \mu_2 P_2 \\ -\sum_{j=4}^{5}\int_0^{+\infty}\mu_j(y)P_j(y)dy = f_0, \\ zP_1 - 2\lambda_1 P_0 + h_1 P_1 - \mu_3 P_3 = f_1, \\ zP_2 - \lambda_2 P_0 + h_2 P_2 - \mu_3 P_3 = f_2, \\ zP_3 - 2\lambda_1 P_1 - 2\lambda_1 P_2 + h_3 P_3 = f_3, \\ zP_4(y) + P_4'(y) + \mu_4(y)P_4(y) = f_4(y), \\ zP_5(y) + P_5'(y) + \mu_5(y)P_5(y) = f_5(y), \end{cases} \quad (8)$$

therefore we get

$$\begin{cases} P_4(y) = e^{-\int_0^y[z+\mu_4(s)]ds}\{P_4(0) \\ +\int_0^y f_4(r)e^{\int_0^r[z+\mu_4(s)ds]}dr\}, \\ P_5(y) = e^{-\int_0^y[z+\mu_5(s)]ds}\{P_5(0) \\ +\int_0^y f_5(r)e^{\int_0^r[z+\mu_5(s)ds]}dr\}. \end{cases} \quad (9)$$

In order that $P_j(y) \in L^1(\mathbb{R}^+), j = 4, 5$, it must hold that

$$\int_0^y f_j(r)e^{-\int_r^y[z+\mu_j(s)ds]}dr \in L^1(\mathbb{R}^+),$$

$$e^{-\int_0^y[z+\mu_j(s)]ds} \in L^1(\mathbb{R}^+), j = 4, 5.$$

These imply that $z$ must satisfy conditions

$$\sup_{r\geq 0}\int_r^{+\infty} e^{-\int_r^y[\Re z+\mu_j(s)]ds}dy < +\infty, j = 4, 5.$$

Thus we define non-negative real numbers $\alpha_j$ and $\alpha$ as follows

$$\alpha_j = \sup_{\eta\geq 0, r\geq 0}\int_0^{+\infty} e^{\eta y-\int_0^y\mu_j(s+r)ds}dy, \quad (10)$$

$$\alpha = \min\{\alpha_j < +\infty, j = 4, 5\}. \quad (11)$$

Obviously, if $\eta < \alpha_j$, then the integral for any $r \geq 0$,

$$\int_r^{+\infty} e^{-\int_r^y[\mu_j(s)-\eta]ds}dy \\ = \int_0^{+\infty} e^{-\int_0^y[\mu_j(s+r)-\eta]ds}dy < +\infty,$$

$j = 4, 5$, while $\eta > \alpha_j$,

$$\int_r^{+\infty} e^{-\int_r^y[\mu_j(s)-\eta]ds}dy = +\infty, j = 4, 5.$$

Note that real number $\alpha_j (j = 4, 5)$ are the measure of essential repair rate of the system.

Obviously, if $\Re z < -\alpha$, then at least one of $P_4(y), P_5(y)$ given in (9) is not in $L^1(\mathbb{R}^+)$, therefore, $\{z \in \mathbb{C} : \Re z < -\alpha\} \subset \sigma(\mathcal{A})$.

Without loss of generality we can assume that the functions $e^{-\int_0^y[\mu_j(s+r)-\alpha_j]ds} (j = 4, 5)$ are uniformly bounded in $(y, r)$. Set

$$\begin{cases} N_j = \sup_{y,r\geq 0} e^{-\int_0^y[\mu_j(s+r)-\alpha_j]ds}, j = 4, 5, \\ N = max\{N_4, N_5\}, \end{cases} \quad (12)$$

when $\Re z > -\alpha$, we have the following estimates

$$\int_0^{+\infty}|P_j(y)|dy$$

$$
\begin{aligned}
&\leq |P_j(0)|\int_0^{+\infty} e^{-\int_0^y[\Re z+\mu_j(s)]ds}dy \\
&\quad +\int_0^{+\infty} dy\int_0^y |f_j(r)|e^{-\int_r^y[\Re z+\mu_j(s)]ds}dr \\
&\leq \frac{|P_j(0)|}{\Re z+\alpha}+\int_0^{+\infty}|f_j(r)|dr\int_r^{+\infty}e^{-(y-r)(\Re z+\alpha)}dy \\
&\leq \frac{|P_j(0)|+\|f_j\|_1}{\Re z+\alpha},
\end{aligned}
$$

so we have $P_j(y)\in L^1(\mathbb{R}^+), j=4,5.$

Note that these functions in (9) are the formal solution of the differential equations in (8). Substituting them into (8) and the boundary conditions (2) lead to algebraic equations with unknown variations $P_0, P_1, P_2, P_3, P_4(0), P_5(0)$:

$$
\begin{cases}
(z+h_0)P_0-\mu_1P_1-\mu_2P_2-P_4(0)[1-zG_4(z)] \\
-P_5(0)[1-zG_5(z)]=F_0, \\
zP_1-2\lambda_1P_0+h_1P_1-\mu_3P_3=f_1, \\
zP_2-\lambda_2P_0+h_2P_2-\mu_3P_3=f_2, \\
zP_3-2\lambda_1P_1-2\lambda_1P_2+h_3P_3=f_3, \\
\lambda_1P_3-P_4(0)=0, \\
\lambda_{c0}P_0+\lambda_{c1}P_1+\lambda_{c2}P_2+\lambda_{c3}P_3-P_5(0)=0.
\end{cases}
$$

Eliminating $P_4(0), P_5(0)$ from above equations yield

$$
\begin{cases}
[z+h_0+\lambda_{c0}zG_5(z)-\lambda_{c0}]P_0 \\
+[\lambda_{c1}zG_5(z)-\mu_1-\lambda_{c1}]P_1 \\
+[\lambda_{c2}zG_5(z)-\mu_2-\lambda_{c2}]P_2 \\
+[\lambda_1zG_4(z)+\lambda_{c3}zG_5(z) \\
-\lambda_1-\lambda_{c3}]P_3=F_0, \\
zP_1-2\lambda_1P_0+h_1P_1-\mu_3P_3=f_1, \\
zP_2-\lambda_2P_0+h_2P_2-\mu_3P_3=f_2, \\
zP_3-2\lambda_1P_1-2\lambda_1P_2+h_3P_3=f_3,
\end{cases} \tag{13}
$$

where

$$
G_j(z)=\int_0^{+\infty}e^{-\int_0^y[z+\mu_j(s)]ds}dy, \tag{14}
$$

$j=4,5$, and the inhomogeneous term $F_0$ is

$$
F_0=f_0+F_4(z)+F_5(z),
$$

where

$$
F_j(z)=\int_0^{\infty}\mu_j(y)dy\int_0^y f_j(r)e^{-\int_r^y[z+\mu_j(s)]ds}dr,
$$

$j=4,5$. A direct calculation gives the determinant of the coefficient matrix of (13)

$$
\begin{aligned}
D(z)=&[z+\lambda_{c0}zG_5(z)+h_0-\lambda_{c0}]d_{11}(z) \\
&+[\lambda_{c1}zG_5(z)-\lambda_{c1}-\mu_1]d_{12}(z) \\
&+[\lambda_{c2}zG_5(z)-\lambda_{c2}-\mu_2]d_{13}(z) \\
&+[\lambda_1zG_4(z)+\lambda_{c3}zG_5(z)-\lambda_1-\lambda_{c3}]d_{14}(z),
\end{aligned}
$$

where $d_{ij}(z)(i,j=1,2,3,4)$ are the algebraic cofactor of $D(z)$.

If $z_1\in\mathbb{C}$ such that $D(z_1)\neq 0$, solving the algebraic equations (13) we can get

$$
\begin{cases}
P_0^{(z_1)}=\frac{1}{D(z_1)}[d_{11}(z_1)F_0 \\
+d_{21}(z_1)f_1+d_{31}(z_1)f_2+d_{41}(z_1)f_3], \\
P_1^{(z_1)}=\frac{1}{D(z_1)}[d_{12}(z_1)F_0 \\
+d_{22}(z_1)f_1+d_{32}(z_1)f_2+d_{42}(z_1)f_3], \\
P_2^{(z_1)}=\frac{1}{D(z_1)}[d_{13}(z_1)F_0 \\
+d_{23}(z_1)f_1+d_{33}(z_1)f_2+d_{43}(z_1)f_3], \\
P_3^{(z_1)}=\frac{1}{D(z_1)}[d_{14}(z_1)F_0 \\
+d_{24}(z_1)f_1+d_{34}(z_1)f_2+d_{44}(z_1)f_3].
\end{cases} \tag{15}
$$

From boundary conditions (2) we can get

$$
\begin{cases}
P_4^{(z_1)}(0)=\frac{\lambda_1}{D(z_1)}[d_{14}(z_1)F_0 \\
+d_{24}(z_1)f_1+d_{34}(z_1)f_2+d_{44}(z_1)f_3], \\
P_5^{(z_1)}(0)=\sum\limits_{j=0}^{3}\frac{\lambda_{cj}}{D(z_1)}[d_{1,j+1}(z_1)F_0 \\
+d_{2,j+1}(z_1)f_1+d_{3,j+1}(z_1)f_2 \\
+d_{4,j+1}(z_1)f_3].
\end{cases} \tag{16}
$$

According to (9) we have

$$
\begin{cases}
P_4^{(z_1)}(y)=e^{-\int_0^y[z_1+\mu_4(s)]ds}\{P_4^{(z_1)}(0) \\
+\int_0^y f_4(r)e^{\int_0^r[z_1+\mu_4(s)ds]}dr\}, \\
P_5^{(z_1)}(y)=e^{-\int_0^y[z_1+\mu_5(s)]ds}\{P_5^{(z_1)}(0) \\
+\int_0^y f_5(r)e^{\int_0^r[z_1+\mu_5(s)]ds}dr\}.
\end{cases} \tag{17}
$$

Thus we obtain unique a solution of (8) in $X$ whose entries are determinant by (15) and (17). Therefore $z_1\in\rho(\mathcal{A})$.

For $z_0\in C$ with $\Re z_0>-\alpha$, the functions $G_4(z_0)$ and $G_5(z_0)$ defined by (14) have meaning. If $D(z_0)=0$, the homogeneous algebraic equations

$$
\begin{cases}
(z+h_0)P_0-\mu_1P_1-[1-zG_4(z)]P_4(0) \\
-[1-zG_5(z)]P_5(0)-\mu_2P_2=0, \\
zP_1-2\lambda_1P_0+h_1P_1-\mu_3P_3=0, \\
zP_2-\lambda_2P_0+h_2P_2-\mu_3P_3=0, \\
zP_3-2\lambda_1P_1-2\lambda_1P_2+h_3P_3=0, \\
\lambda_1P_3-P_4(0)=0, \\
\lambda_{c0}P_0+\lambda_{c1}P_1+\lambda_{c2}P_2 \\
+\lambda_{c3}P_3-P_5(0)=0,
\end{cases} \tag{18}
$$

have a nonzero solution of the form

$$\begin{cases} P_0^{(z_0)} = (z_0+h_1)\{2\lambda_1\mu_3[\lambda_2(z_0+h_1) \\ +2\lambda_1(z_0+h_2)] - (z_0+h_2)\}, \\ P_1^{(z_0)} = -2\lambda_1\{(z_0+h_2) - [\lambda_2(z_0+h_1) \\ +2\lambda_1(z_0+h_2)][(2\lambda_1-\lambda_2)\mu_3]\}, \\ P_2^{(z_0)} = \lambda_2(z_0+h_1)[2\lambda_1(\lambda_2-2\lambda_1)\mu_3 \\ -\lambda_2(z_0+h_1)(z_0+h_3)], \\ P_3^{(z_0)} = -2\lambda_1\lambda_2(z_0+h_1)[\lambda_2(z_0+h_1) \\ +2\lambda_1(z_0+h_2)], \\ P_4^{(z_0)}(0) = \lambda_1 P_3^{(z_0)}, \\ P_5^{(z_0)}(0) = \lambda_{c0}P_0^{(z_0)} + \lambda_{c1}P_1^{(z_0)} \\ +\lambda_{c2}P_2^{(z_0)} + \lambda_{c3}P_3^{(z_0)}. \end{cases} \tag{19}$$

**Remark:** Let $a = -2\lambda_1\lambda_2$, $b = 2\lambda_1(z_0+h_2)$, $c = -2\lambda_1\mu_3$, $d = \lambda_2(z_0+h_1)$,$e = -\lambda_2\mu_3$, $f = -2\lambda_1$, $g = z_0+h_3$, then

$$A = \begin{pmatrix} a & 0 & b & c \\ a & d & 0 & e \\ 0 & f & f & g \end{pmatrix} \to$$

$$\begin{pmatrix} r & 0 & 0 & d[cf(b+d)-b] \\ 0 & r & 0 & a[b+f(b+d)(e-c)] \\ 0 & 0 & r & ad[dg-f(e-c)] \end{pmatrix}$$

where $r = adf(b+d)$.

Using (13) with $f_4(r) = 0$ and $f_5(r) = 0$, we can show that the functions

$$\begin{cases} P_0^{(z_0)} = (z_0+h_1)\{2\lambda_1\mu_3[\lambda_2(z_0+h_1) \\ +2\lambda_1(z_0+h_2)] - (z_0+h_2)\}, \\ P_1^{(z_0)} = -2\lambda_1\{(z_0+h_2) - [\lambda_2(z_0+h_1) \\ +2\lambda_1(z_0+h_2)][(2\lambda_1-\lambda_2)\mu_3]\}, \\ P_2^{(z_0)} = \lambda_2(z_0+h_1)[2\lambda_1(\lambda_2-2\lambda_1)\mu_3 \\ -\lambda_2(z_0+h_1)(z_0+h_3)], \\ P_3^{(z_0)} = -2\lambda_1\lambda_2(z_0+h_1)[\lambda_2(z_0+h_1) \\ +2\lambda_1(z_0+h_2)], \\ P_4^{(z_0)}(y) = P_4^{(z_0)}(0)e^{-\int_0^y [z_0+\mu_4(s)]ds}, \\ P_5^{(z_0)}(y) = P_5^{(z_0)}(0)e^{-\int_0^y [z_0+\mu_5(s)]ds}, \end{cases} \tag{20}$$

satisfy the homogeneous equations (8) with $f_0 = f_1 = f_2 = f_3 = f_4(y) = f_5(y) = 0$ and boundary conditions (2) and $P_j^{(z_0)}(y) \in L^1(R^+)(j = 4,5)$ for $\Re z_0 > -\alpha$. Set

$$P^{(z_0)} = (P_0^{(z_0)}, P_1^{(z_0)}, P_2^{(z_0)}, P_3^{(z_0)}, P_4^{(z_0)}(y), P_5^{(z_0)}(y)),$$

we have $P^{(z_0)} \in D(\mathcal{A})$ and $\mathcal{A}P^{(z_0)} = z_0 P^{(z_0)}$. So $z_0$ is an eigenvalue of $\mathcal{A}$.

Summarizing the discussion above, we have proved the following results.

**Theorem 2** *Let $X$ and $\mathcal{A}$ be defined as before, and $\alpha$ be defined by (10)(11). Then the following assertions are true:*
(1) $\{z \in \mathbb{C} : \Re z < -\alpha\} \subset \sigma(\mathcal{A})$;
(2) $\{z \in \mathbb{C} : \Re z > -\alpha, D(z) \neq 0\} \subset \rho(\mathcal{A})$;
(3) $\{z \in \mathbb{C} : \Re z > -\alpha, D(z) = 0\}$ *consists of all eigenvalues of* $\mathcal{A}$;
(4) $\sigma(\mathcal{A})$ *distributes symmetrically with respect to the real axis.*

Let $z_1 \in \mathbb{C}$ with $\Re z_1 > -\alpha$ and $D(z_1) \neq 0$, we have $z_1 \in \rho(\mathcal{A})$. So the solution of (8) is given by $P = R(z_1, \mathcal{A})F$. According to the previous calculation we have norm estimate

$$\|P\|_X = \sum_{i=0}^{3} |P_i^{(z_1)}| + \sum_{j=4}^{5} \int_0^{+\infty} |P_j(y)|dy$$
$$\leq \sum_{i=0}^{3} |P_i^{(z_1)}| + \sum_{j=4}^{5} \frac{|P_j^{(z_1)}(0)|}{\Re z_1 + \alpha} + \sum_{j=4}^{5} \frac{\|f_j\|_1}{\Re z_1 + \alpha}.$$

On the other hand,

$$\begin{aligned} & \sum_{i=0}^{3} |P_i^{(z_1)}| + \sum_{j=4}^{5} \frac{|P_j^{(z_1)}(0)|}{\Re z_1 + \alpha} \\ \leq & \ (1 + \frac{\lambda_{c0}}{\Re z_1 + \alpha})|P_0^{(z_1)}| + (1 + \frac{\lambda_{c1}}{\Re z_1 + \alpha})|P_1^{(z_1)}| \\ + & \ (1 + \frac{\lambda_{c2}}{\Re z_1 + \alpha})|P_2^{(z_1)}| + (1 + \frac{\lambda_1 + \lambda_{c3}}{\Re z_1 + \alpha})|P_3^{(z_1)}| \\ \leq & \ N_0 \sum_{i=0}^{3} |P_i^{(z_1)}|, \end{aligned}$$

where

$$N_0 = \max\{1 + \frac{\lambda_{c0}}{\Re z_1 + \alpha}, 1 + \frac{\lambda_{c1}}{\Re z_1 + \alpha},$$
$$1 + \frac{\lambda_{c2}}{\Re z_1 + \alpha}, 1 + \frac{\lambda_1 + \lambda_{c3}}{\Re z_1 + \alpha}\}.$$

According to (15) it holds

$$\sum_{i=0}^{3} |P_i^{(z_1)}|$$
$$\leq \frac{\max\limits_{1 \leq j \leq 4} \sum\limits_{i=1}^{4} |d_{ij}(z_1)|(|F_0| + |f_1| + |f_2| + |f_3|)}{|D(z_1)|}.$$

Since

$$|F_j(z)|$$
$$\leq \int_0^{+\infty} \mu_j(y)dy \int_0^y |f_j(r)|e^{-\int_r^y [\Re z + \mu_j(s)]ds} dr$$

$$\leq \int_0^{+\infty} |f_j(r)|dr$$
$$\int_r^{+\infty} [\Re z + \mu_j(y)] e^{-\int_r^y [\Re z + \mu_j(s)]ds} dy$$
$$+ |\Re z| \int_0^{+\infty} |f_j(r)|dr \int_r^{+\infty} e^{-\int_r^y [\Re z + \mu_j(s)]ds} dy$$
$$\leq (1 + \frac{|\Re z|}{\Re z + \alpha}) \|f_j\|_1, j = 4, 5,$$

while

$$|F_0| + |f_1| + |f_2| + |f_3|$$
$$\leq |f_0| + |f_1| + |f_2| + |f_3| + |F_4(z_1)| + |F_5(z_1)|$$
$$\leq |f_0| + |f_1| + |f_2| + |f_3|$$
$$+ (1 + \frac{|\Re z_1|}{\Re z_1 + \alpha}) \|f_4\|_1 + (1 + \frac{|\Re z_1|}{\Re z_1 + \alpha}) \|f_5\|_1$$
$$\leq (1 + \frac{|\Re z_1|}{\Re z_1 + \alpha}) \|F\|_X,$$

so we have

$$\sum_{i=0}^{3} |P_i^{(z_1)}|$$
$$\leq \frac{\max_{1 \leq j \leq 4} \sum_{i=1}^{4} |d_{ij}(z_1)|}{|D(z_1)|} (1 + \frac{|\Re z_1|}{\Re z_1 + \alpha}) \|F\|_X.$$

Since $d_{ij}(z)(i, j = 1, 2, 3, 4)$ are the algebraic cofactor of $D(z)$, they all are at most 3-order polynomial of $z$, then we can get that there is a positive constant $M$ such that

$$\max_{1 \leq j \leq 4} \sum_{i=1}^{4} |d_{ij}(z_1)| \leq M(|z_1|^3 + \frac{1}{\Re z_1 + \alpha})$$

for any $\Re z_1 > -\alpha$, therefore we have

$$\|P\|_X \leq \sum_{i=0}^{3} |P_i^{(z_1)}| + \sum_{j=4}^{5} \frac{|P_j^{(z_1)}(0)|}{\Re z_1 + \alpha}$$
$$+ \sum_{j=4}^{5} \frac{\|f_j\|_1}{\Re z_1 + \alpha}$$
$$\leq N_0 \sum_{i=0}^{3} |P_i^{(z_1)}| + \frac{\|f_4\|_1 + \|f_5\|_1}{\Re z_1 + \alpha}$$
$$\leq \frac{N_0 M}{|D(z_1)|} (|z_1|^3 + \frac{1}{\Re z_1 + \alpha})$$
$$(1 + \frac{|\Re z_1|}{\Re z_1 + \alpha}) \|F\|_X + \frac{1}{\Re z_1 + \alpha} \|F\|_X$$
$$\leq H(z_1) \|F\|_X,$$

where

$$H(z_1) = \frac{N_0 M}{|D(z_1)|} (|z_1|^3 + \frac{1}{\Re z_1 + \alpha})$$
$$(1 + \frac{|\Re z_1|}{Re z_1 + \alpha}) + \frac{1}{\Re z_1 + \alpha}.$$

Since $D(z)$ is analysis in the half plane $\Re z > -\alpha$, we have

$$\lim_{|\Im z| \to +\infty} \frac{|z|^4 + h_0 h_1 h_2 h_3}{D(z)} = 1,$$

and the limit is uniformly in the region $\Re z + \alpha \geq \delta > 0$. So the term $\frac{1}{|D(z_1)|}(|z_1|^3 + \frac{1}{\Re z_1 + \alpha})$ is bounded as $|\Im z_1| \to +\infty$ with $\Re z_1 + \alpha \geq \delta > 0$. Thus we can define the positive number

$$M(\Re z_1) = \frac{|D(z_1)|}{|z_1|^4 + h_0 h_1 h_2 h_3} (\Re z_1 + \alpha) H(z_1).$$

Obviously, when $\Re z_1 + \alpha \geq \delta > 0, M(\Re z_1)$ is uniformly bounded. In addition, $\mathcal{A}$ is a dissipative operator in $X$([18]), we also have $\|R(z_1, \mathcal{A})\| \leq (\Re z_1)^{-1}$ as $\Re z_1 > 0$. So far we have proved the following result.

**Theorem 3** *Let $D(z)$ be defined as before, then for any $\Re z > -\alpha, D(z) \neq 0$, there exists a nonnegative function $M(\Re z)$ such that*

$$\|R(z, \mathcal{A})\| \leq \frac{(|z|^4 + h_0 h_1 h_2 h_3) M(\Re z)}{|D(z)|(\Re z + \alpha)}.$$

*In particular, when $\Re z > 0$, it holds that*

$$\|R(z, \mathcal{A})\| \leq (\Re z)^{-1}.$$

As a consequence of Theorem 3, we have the following corollary thank to the semigroup theory([19]).

**Corollary 4** *$\mathcal{A}$ generates a $C_0$ semigroup on $X$ of contraction and the system (7) is well-posed in $X$.*

**Theorem 5** *If the functions $\mu_j(y)(j = 4, 5)$ satisfy conditions*

$$\sup_{r \geq 0} \int_r^{+\infty} e^{-\int_r^y \mu_j(s)ds} dy < +\infty, \quad (21)$$

*$j = 4, 5$, then $\{z : \Re z = 0, \Im z \neq 0\} \subset \rho(\mathcal{A})$.*

**Proof:** For any $z = ib, b \in \mathbb{R}, b \neq 0$, the matrix $\Delta$ of coefficients of (18) is

$$\Delta(ib) = \left( \begin{array}{cccc} ib + h_0 & -\mu_1 & -\mu_2 & 0 \\ -2\lambda_1 & ib + h_1 & 0 & -\mu_3 \\ -\lambda_2 & 0 & ib + h_2 & -\mu_3 \\ 0 & -2\lambda_1 & -2\lambda_1 & ib + h_2 \\ 0 & 0 & 0 & -\lambda_1 \\ -\lambda_{c0} & -\lambda_{c1} & -\lambda_{c2} & -\lambda_{c3} \end{array} \right.$$

$$\begin{matrix} ibG_4(ib)-1 & ibG_5(ib)-1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix} \Bigg) .$$

Since $h_0 = 2\lambda_1 + \lambda_{c0} + \lambda_2, h_1 = 2\lambda_1 + \mu_1 + \lambda_{c1}$, $h_2 = 2\lambda_1 + \mu_2 + \lambda_{c2}, h_3 = \lambda_1 + 2\mu_3 + \lambda_{c3}$, $|ibG_4(ib) - 1| < 1, |ibG_5(ib) - 1| < 1$, then $\Delta(ib)$ is a strictly diagonal-dominant matrix about column, which implies $det\triangle(ib) \neq 0$. Therefore, $ib$ is not the eigenvalue of $\mathcal{A}$. In this case, the equation (18) has uniquely a solution $P_0^{(ib)}, P_1^{(ib)}, P_2^{(ib)}, P_3^{(ib)}$. Under conditions (21), we can verify that the functions $P_4^{(ib)}(y), P_5^{(ib)}(y))$ defined by (17) with $z_1 = ib(b \neq 0)$ are in $L^1(R^+)$, so the resolvent equation $(ibI - \mathcal{A})P = F$ has a uniquely solution in $D(\mathcal{A})$. Therefore

$$\{z : \Re z = 0, \Im z \neq 0\} \subset \rho(\mathcal{A}).$$

□

**Theorem 6** *Let $\alpha$ be defined as (10)(11). If $\alpha > 0$, then $\sigma(\mathcal{A})$ has the following properties:*
*(1) For any $\delta > 0$, there are at most finitely many eigenvalues of $\mathcal{A}$ in the region $\{z \in C : \Re z + \alpha \geq \delta\}$.*
*(2) There exists a constant $\varepsilon > 0$, such that the region $\{z \in C : \Re z > -\varepsilon\}$ has only one eigenvalue $\gamma_0 = 0$.*

**Proof:** In the half plane $S_0 = \{z : \Re z > -\alpha\}$ we have proved that there are eigenvalues of $\mathcal{A}$ only, and $z$ is a eigenvalue of $\mathcal{A}$ if and only if $D(z) = 0$. When $\alpha > 0$, there is only one zero of $D(z)$ on the imaginary axis according to Theorem 5. We consider the zero of $D(z)$ in the region $G = \{z : 0 > \Re z > -\alpha + \delta\}$. Observing that the functions

$$h_j(z) = \int_0^{+\infty} \mu_j(y) e^{-\int_0^y [z+\mu_j(s)]ds} dy (j = 4, 5)$$

are analysis in the region. The Riemann Lemma asserts that

$$\lim_{|\Im z| \to +\infty} h_j(z) = 0, j = 4, 5.$$

Therefore

$$\lim_{|\Im z| \to +\infty} \frac{D(z)}{z^4} = 1$$

is uniformly in the region $G$. So $D(z)$ has at most finite zeros in $G$. So $\mathcal{A}$ has at most finite eigenvalue in $G$.

Let $D_0(z) = \frac{D(z)}{z}$, then $z \neq 0$ is a zero of $D(z)$ if and only if it is that of $D_0(z)$. Since $\overline{D_0(z)} = D_0(\overline{z})$, its zero are symmetrically with respect to the real axis. Note that $\alpha > 0$ implies $D_0(ib) \neq 0, b \in R$. Let the zeros of $D_0(z)$ in the region $G$ be $z_k (k = 1, 2, \ldots, m)$, we can set

$$\varepsilon = \min_{1 \leq k \leq m} |\Re z_k|.$$

There is no zero of $D_0(z)$ as $\Re z > -\varepsilon$. Hence there is only one eigenvalue $\gamma_0 = 0$ of $\mathcal{A}$ in $S_0$. □

**Definition 7** *([20]) Let $\lambda_0$ be a eigenvalue of $\mathcal{A}$, we call $\lambda_0$ a dominant eigenvalue if $\lambda_0$ is a simple eigenvalue that is greater than the real part of any other point of the spectrum, and if the eigenfunction associated with $\lambda_0$ is positive; we call $\lambda_0$ strictly dominant if it is dominant and if there exists an $\varepsilon > 0$ such that the real part of any other point of spectrum is less than $\lambda_0 - \varepsilon$.*

From Theorem 1 and Theorem 6 we have the following result.

**Theorem 8** *If $\alpha > 0$, then $\gamma_0 = 0$ is a strictly dominant eigenvalue of $\mathcal{A}$.*

# 4 Analysis of stability

Firstly, from [18] we known that $\gamma_0 = 0$ is a simple eigenvalue of $\mathcal{A}$. On the other hand, from discussion in the above section, we can get that the corresponding positive eigenvector is

$$\hat{P}_0 = \frac{1}{Z}(P_0^{(0)}, P_1^{(0)}, P_2^{(0)}, P_3^{(0)}, P_4^{(0)}(y), P_5^{(0)}(y)),$$

where

$$\begin{cases} P_0^{(0)} = h_1[2\lambda_1\mu_3(\lambda_2 h_1 + 2\lambda_1 h_2) - h_2], \\ P_1^{(0)} = 2\lambda_1[(\lambda_2 h_1 + 2\lambda_1 h_2) \\ (2\lambda_1 - \lambda_2)\mu_3 - h_2], \\ P_2^{(0)} = \lambda_2 h_1[2\lambda_1(\lambda_2 - 2\lambda_1)\mu_3 - \lambda_2 h_1 h_3], \\ P_3^{(0)} = 2\lambda_1\lambda_2 h_1(\lambda_2 h_1 + 2\lambda_1 h_2), \\ P_4^{(0)}(y) = P_4^{(0)}(0) e^{-\int_0^y \mu_4(s)ds}, \\ P_5^{(0)}(y) = P_5^{(0)}(0) e^{-\int_0^y \mu_5(s)ds}, \end{cases} \tag{22}$$

and

$$\begin{aligned} Z = {} & h_1[2\lambda_1\mu_3(\lambda_2 h_1 + 2\lambda_1 h_2) - h_2] \\ & +2\lambda_1[(\lambda_2 h_1 + 2\lambda_1 h_2)(2\lambda_1 - \lambda_2)\mu_3 - h_2] \\ & +\lambda_2 h_1[2\lambda_1(\lambda_2 - 2\lambda_1)\mu_3 - \lambda_2 h_1 h_3] \\ & +2\lambda_1\lambda_2 h_1(\lambda_2 h_1 + 2\lambda_1 h_2) \\ & +2\lambda_1^2\lambda_2 h_1(\lambda_2 h_1 + 2\lambda_1 h_2)G_4(0) \\ & +\{\lambda_{c0} h_1[2\lambda_1\mu_3(\lambda_2 h_1 + 2\lambda_1 h_2) - h_2] \\ & +2\lambda_{c1}\lambda_1[(\lambda_2 h_1 + 2\lambda_1 h_2)(2\lambda_1 - \lambda_2)\mu_3 - h_2] \end{aligned}$$

$$+\lambda_{c2}\lambda_2 h_1[2\lambda_1(\lambda_2-2\lambda_1)\mu_3-\lambda_2 h_1 h_3] +2\lambda_{c3}\lambda_1\lambda_2 h_1(\lambda_2 h_1+2\lambda_1 h_2)\}G_5(0). \quad (23)$$

**Remark:** The positive eigenvector $\hat{P}_0$ of $\mathcal{A}$ satisfies $\|\hat{P}_0\|_X = 1$. If we take $Q_0 = (1,1,1,1,1,1)$, then $Q_0 \in D(\mathcal{A}^*)$ and $\mathcal{A}^*Q_0 = 0$, i.e. $Q_0$ is an eigenvector of $\mathcal{A}^*$, the corresponding eigenvalue is $\gamma_0 = 0$. Furthermore, we have

$$<\hat{P}_0, Q_0> = \frac{1}{Z}\{\sum_{i=0}^{3} P_i^{(0)} + \sum_{j=4}^{5}\int_0^{+\infty} P_j^{(0)}(y)dy\}$$

$$= \|\hat{P}_0\|_X = 1.$$

In what follows we shall study the linear stability and the exponential stability of system (7).

**Definition 9** *([21]) We say the solution of system (7) is linearly stable, if there is a positive constant $\varepsilon > 0$, such that the nonzero spectrum of the operator $\mathcal{A}$ satisfies*

$$\max\{\Re z : z \in \sigma(\mathcal{A}), z \neq 0\} \leq -\varepsilon$$

*and $z = 0$ is a simple eigenvalue of $\mathcal{A}$.*

From Theorem 1 and Theorem 6, we have the following result.

**Theorem 10** *If $\alpha > 0$, then the solution of system (7) is linearly stable.*

Let $\mathcal{A}$ be a generator of the $C_0$ semigroup $T(t)$, $H_0$ be an initial value. The linear stability implies that

$$T(t)H_0 = (H_0, Q)\widehat{P} + T_1(t)H_0,$$

$$\lim_{t\to+\infty} T_1(t)H_0 = 0,$$

where $Q = (1,1,1,1,1,1)$, and $\widehat{P} = \xi\widetilde{P}$ is a positive solution such that $(\widehat{P}, Q) = 1$([18]). In general, linear stability does not imply that there exist $\omega > 0, M > 0$, such that

$$||T_1(t)H_0|| \leq Me^{-\omega t}, t \geq 0, \quad (24)$$

i.e., the exponential stability.

In the following part we shall study the exponential stability of system (7). Let $\lambda \in \rho(\mathcal{A})$, the following definitions can be referenced in [19]:

$$s(\mathcal{A}) = \sup\{\Re\lambda \mid \lambda \in \sigma(\mathcal{A})\},$$

$$\omega_1(T) = \lim_{t\to+\infty}\frac{\ln||T(t)R(\lambda,\mathcal{A})||}{t},$$

$$s_0(\mathcal{A}) = \sup\left\{w > s(\mathcal{A}) \mid \sup_{\Re\lambda=w}||R(\lambda,\mathcal{A})|| < +\infty\right\},$$

$$\omega_0(T) = \lim_{t\to+\infty}\frac{\ln||T(t)||}{t}.$$

From [22, pp343, Theorem 5.1.9] and [23, pp119, Corollary 4.2.7], we have the following Lemma.

**Lemma 11** *Let $T(t)$ be a $C_0$ semigroup on Banach space $X$, and $\mathcal{A}$ be the generator, then*

$$s(\mathcal{A}) \leq \omega_1(T) < s_0(\mathcal{A}) \leq \omega_0(T).$$

**Theorem 12** *Let $c_j = inf\{\mu_j(y), y \in \mathbb{R}^+\}$, if $c = min\{c_j, j = 4,5\} > 0$, then the solution of system (7) is exponentially stable.*

**Proof:** Since $\gamma_0 = 0$ is an isolated eigenvalue of $\mathcal{A}$, let $E_0(\mathcal{A})$ be the corresponding spectral mapping, then we have the following decomposition of $X$:

$$X = X_0 + Y, X_0 = E_0(\mathcal{A})X, Y = (I - E_0(\mathcal{A}))X,$$

where $X_0$ is an eigen-subspace corresponding to $\gamma_0 = 0$. It is well-known that $Y$ also is a Banach space. Set $T_1(t) = T(t)|_Y, \mathcal{A}_1 = \mathcal{A}|_Y$, then $\mathcal{A}_1$ is a generator of semigroup $T_1(t)$, and $\sigma(\mathcal{A}_1) = \sigma(\mathcal{A})\backslash\{0\}$.

(1) $s_0(\mathcal{A}_1) \leq 0$.

Since $\gamma_0 = 0$ is a resolvent point of $\mathcal{A}_1$, it is sufficient to prove:

$$\sup_{|\beta|>\delta>0}||R(i\beta,\mathcal{A}_1)|| < +\infty.$$

In fact, for $z = i\beta$, similar to Theorem 2, for any $f = (f_0, f_1, f_2, f_3, f_4(y), f_5(y)) \in Y$, $\|f\| = 1$, the resolvent equation $(i\beta I - \mathcal{A}_1)P = f$ has exactly one solution $P = (P_0, P_1, P_2, P_3, P_4(y), P_5(y)) \in D(\mathcal{A}_1)$, and

$$\begin{cases} P_0 = \frac{I_0(\beta)}{\beta^4+I(\beta)}, P_1 = \frac{I_1(\beta)}{\beta^4+I(\beta)}, \\ P_2 = \frac{I_2(\beta)}{\beta^4+I(\beta)}, P_3 = \frac{I_3(\beta)}{\beta^4+I(\beta)}, \\ P_j(y) = e^{-\int_0^y[i\beta+\mu_j(u)]du}\{P_j(0) \\ +\int_0^y f_j(r)e^{\int_0^r[i\beta+\mu_j(u)]du}dr\}, j = 4,5, \end{cases} \quad (25)$$

where $I(\beta), I_0(\beta), I_1(\beta), I_2(\beta)$ and $I_3(\beta)$ all are 3-order polynomial of $\beta$, then there exists a $\tau > 0$, such that

$$\sup_{z=i\beta,|\beta|>\delta>0}|P_i| \leq \tau, i = 0,1,2,3.$$

Observing

$$|F_j(i\beta)| \leq \int_0^{+\infty}|\mu_j(y)e^{-\int_0^y[i\beta+\mu_j(u)]du}|dy = 1,$$

and

$$\begin{aligned}&|G_j(i\beta)| = |\int_0^{+\infty}\mu_j(y)e^{-\int_0^y[i\beta+\mu_j(u)]du}dy \\ &\int_0^y f_j(r)e^{\int_0^r[i\beta+\mu_j(u)]du}dr| \\ &\leq \int_0^{+\infty}|f_j(r)|e^{\int_0^r\mu_j(u)du}dr\int_r^{\infty}\mu_j(y)e^{-\int_0^y\mu_j(u)du}dy \\ &= \int_0^{+\infty}|f_j(r)|dr = ||f_j||_1, j = 4,5,\end{aligned}$$

we have

$$\int_0^{+\infty}|P_j(y)|dy$$
$$\leq \int_0^{+\infty}|P_j(0)|e^{-\int_0^y \mu_j(u)du}dy$$
$$+\int_0^{+\infty}dy\int_0^y|f_j(r)|e^{-\int_r^y \mu_j(u)du}dr$$
$$\leq |P_j(0)|\int_0^{+\infty}e^{-yc}dy$$
$$+\int_0^{+\infty}dy\int_0^y|f_j(r)|e^{-(y-r)c}dr$$
$$\leq |P_j(0)|\int_0^{+\infty}e^{-yc}dy$$
$$+\int_0^{+\infty}|f_j(r)|dr\int_r^{+\infty}e^{-(y-r)c}dy$$
$$=\frac{|P_j(0)|}{c}+\frac{||f_j||_1}{c}, j=4,5,$$

therefore

$$||R(i\beta,\mathcal{A}_1)f||=||P||_X$$
$$=|P_0|+|P_1|+|P_2|+|P_3|$$
$$+\sum_{j=4}^{5}\int_0^{+\infty}|P_j(y)|dy$$
$$\leq(4\tau+\frac{2}{c}+\frac{\lambda_1\tau}{c}+\frac{\tau}{c}\sum_{i=0}^{3}\lambda_{ci})\|f\|.$$

So there exists an $M_0>0$, such that

$$\lim_{|\beta|\to+\infty}||R(i\beta,\mathcal{A}_1)||\leq M_0.$$

This means

$$\sup_{|\beta|>\delta>0}||R(i\beta,\mathcal{A}_1)||<+\infty,$$

i.e., $s_0(\mathcal{A}_1)\leq 0$.

(2) The solution of system (7) is exponentially stable.

According to Lemma 11, we have $\omega_1(T_1)<s_0(\mathcal{A}_1)\leq 0$. Thus we can choose $\varepsilon>0$, such that $\omega_1(T_1)+\varepsilon<0$. Using the definition of $\omega_1(T_1)$, there exists an $M>0$, such that

$$||T_1(t)R(z,\mathcal{A}_1)||\leq Me^{[\omega_1(T_1)+\varepsilon]t},\quad t\geq 0,$$

i.e.,

$$||T_1(t)g||\leq Me^{[\omega_1(T_1)+\varepsilon]t}||(zI-\mathcal{A}_1)g|| \qquad (26)$$

for $g\in D(\mathcal{A}_1)$. Hence the solution $P(t)$ of system (7) can be written

$$P(t)=T(t)\tilde{P}=(\tilde{P},Q)\widehat{P}+T_1(t)\tilde{P}$$
$$=\widehat{P}+T_1(t)\tilde{P}\in D(\mathcal{A}), t>0.$$

This means $T_1(t)\tilde{P}\in D(\mathcal{A})$. Set $g=T_1(s)\tilde{P}, s>0$, then $g\in D(\mathcal{A}_1)$. From (26) we get

$$||T_1(t+s)\tilde{P}||=||T_1(t)g||$$
$$\leq Me^{-|\omega_1(T_1)+\varepsilon|t}||(zI-\mathcal{A}_1)g||, t>0.$$

Therefore

$$||T(t+s)\tilde{P}-\widehat{P}||=||T_1(t)\tilde{P}||$$
$$\leq Me^{-|\omega_1(T_1)+\varepsilon|t}||(zI-\mathcal{A}_1)g||, t>0,$$

i.e., the system (7) is exponentially stable. □

## 5 Finite expansion of solution

From now on we suppose that $\alpha>0$. According to results of Theorem 6, for any small $\delta>0$, the region $\{z:0\geq\Re z\geq-\alpha+\delta\}$ has only finite many eigenvalues of $\mathcal{A}$. Without loss of generality we assume that $\{z:\Re z=-\alpha+\delta\}\in\rho(\mathcal{A})$.

Let $T(t)$ be a $C_0$ semigroup on Banach space $X$, and $\mathcal{A}$ be the generator, then for $\omega>0$ it holds that

$$T(t)P=\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}e^{zt}R(z,\mathcal{A})Pdz, P\in X.$$

For sufficient large $M$, let

$$S_M(t)P=\frac{1}{2\pi i}\int_{-\alpha+\delta-iM}^{-\alpha+\delta+iM}e^{zt}R(z,\mathcal{A})Pdz,$$

then we have

$$\frac{1}{2\pi i}\int_{\alpha-iM}^{\alpha+iM}e^{zt}R(z,\mathcal{A})Pdz$$
$$=\sum_{z_i\in\sigma_p(\mathcal{A}),\Re z_i\geq-\alpha+\delta}\frac{1}{2\pi i}\int_{|z-z_i|=\varepsilon}e^{zt}R(z,\mathcal{A})Pdz$$
$$+\frac{1}{2\pi i}\int_{-\alpha+\delta}^{\omega}e^{(iM+s)t}R(iM+s,\mathcal{A})Pds$$
$$-\frac{1}{2\pi i}\int_{-\alpha+\delta}^{\omega}e^{(-iM+s)t}R(-iM+s,\mathcal{A})Pds+S_M(t)P.$$

Using the estimates in Theorem 3, we can get

$$\lim_{M\to+\infty}\frac{1}{2\pi i}\int_{-\alpha+\delta}^{\omega}e^{(iM+s)t}R(iM+s,\mathcal{A})Pds=0,$$

$$\lim_{M\to+\infty}\frac{1}{2\pi i}\int_{-\alpha+\delta}^{\omega} e^{(-iM+s)t}R(-iM+s,\mathcal{A})Pds=0.$$

Therefore we have

$$T(t)P=\sum_{z_i\in\sigma_p(\mathcal{A}),\Re z_i\geq-\alpha+\delta} T(t)E(z_i,\mathcal{A})P+S(t)P,$$

where

$$S(t)P=\lim_{M\to+\infty}S_M(t)P$$
$$=\frac{1}{2\pi i}\int_{-\alpha+\delta-i\infty}^{-\alpha+\delta+i\infty} e^{zt}R(z,\mathcal{A})Pdz.$$

Obviously this presentation implies that there exists a constant $C>0$ such that

$$\|S(t)P\|\leq Ce^{(-\alpha+\delta)t}\|P\|, P\in X.$$

Therefore we have the following result.

**Theorem 13** *Suppose that, for $\delta>0$ small enough, the eigenvalues of $\mathcal{A}$ in the region $\{z:\Re z>-\alpha+\delta\}$ are given by $0, z_1, \overline{z_1}, z_2, \overline{z_2},\ldots, z_m, \overline{z_m}$ with $\Re z_j\geq\Re z_{j+1}$. Then we have the finite expansion of the semigroup $T(t)$ :*

$$T(t)P=<P,Q>\hat{P}_0$$
$$+\sum_{j=1}^{m}T(t)[E(z_j,\mathcal{A})+E(\overline{z_j},\mathcal{A})]P+S(t)P,$$

*where $Q=(1,1,1,1,1,1)$, and*

$$\hat{P}_0=\frac{1}{Z}(P_0^{(0)},P_1^{(0)},P_2^{(0)},P_3^{(0)},P_4^{(0)}(y),P_5^{(0)}(y)),$$

*whose entries determinant by* (22)(23).

From Section 4, we see that $\hat{P}_0$ is the stead-state solution with $\|\hat{P}_0\|=1$. Du to $\Re z_j<0$, from Theorem 13 we see that for any $P\in X$,

$$\lim_{t\to+\infty}T(t)P=<P,Q>\hat{P}_0.$$

In particular, we have the following estimate for its convergence.

**Corollary 14** *Suppose that $\alpha>0$ and $-\omega_1<\Re z_1<0$, then for any initial value $P(0)$ we have*

$$\|P(t)-<P(0),Q>\hat{P}_0\|\leq 2e^{-\omega_1 t}\|P(0)\|, t\geq 0,$$

*where $P(t)=T(t)P(0)$.*

**Proof:** Since the Riesz spectral project corresponding to $\gamma_0=0$ is given by

$$E(\gamma_0,\mathcal{A})F=\frac{1}{2\pi i}\int_{|z|=\varepsilon} e^{zt}R(z,\mathcal{A})Fdz$$
$$=<F,Q>\hat{P}_0$$

for any $F\in X$. This leads to

$$\|E(\gamma_0,\mathcal{A})\|=\|Q\|\|\hat{P}_0\|=1.$$

Since $T(t)$ is a semigroup in the subspace $(I-E(\gamma_0,\mathcal{A}))X$, we have

$$\|P(t)-<P(0),Q>\hat{P}_0\|$$
$$=\|T(t)(I-E(\gamma_0,\mathcal{A}))P(0)\|$$
$$\leq 2e^{-\omega_1 t}\|P(0)\|.$$

The desired result follows. □

**Remark:** In Corollary 14, usually we have $\omega_1\neq\Re z_1$. If $z_1$ is an eigenvalue of $\mathcal{A}$ without the second order root vector, then we can take $\omega_1=\Re z_1$.

# 6 Some indices of the system

Quasi-exponential decaying of the system means that one can see the stead state of system in a relatively shot period. For the system under consideration, the dynamic solution of system is given by

$$P(t)=T(t)P(0)$$
$$=(P_0(t),P_1(t),P_2(t),P_3(t),P_4(y,t),P_5(y,t)),$$

with initial value $P(0)=(1,0,0,0,0,0)$ and the stead state of system is $<P(0),Q>\hat{P}_0=\hat{P}_0$, where

$$\hat{P}_0=\frac{1}{Z}(P_0^{(0)},P_1^{(0)},P_2^{(0)},P_3^{(0)},P_4^{(0)}(y),P_5^{(0)}(y))$$

and $\hat{P}_0$ defined by (22)(23).

For a system $S$, whose dynamic solution is $P(t)$ with initial data $\|P_0\|=1$ and the stead state is $\hat{P}_0$, if there is a time $\tau_0$ such that when $t>\tau_0$ it holds that $\|P(t)-P_0\|\leq 0.25$, then we say that we can see the stead state of the system at $\tau_0$.

According to Corollary 14, we have estimate

$$\|P(t)-\hat{P}_0\|\leq 2e^{-\omega_1 t}.$$

Obviously, for $\tau_0=\frac{3ln2}{\omega_1}$, when $t>\tau_0$ we have

$$\|P(t)-\hat{P}_0\|\leq 0.25.$$

Therefore we can see the stead state at $\tau_0=\frac{3ln2}{\omega_1}$.

The instantaneous availability of the system is a probability of the system in work, which is defined by

$$V(t) = P_0(t) + P_1(t) + P_2(t) + P_3(t).$$

Since

$$\sum_{i=0}^{3} |P_i(t) - P_i^{(0)}| \leq \|P(t) - \hat{P}_0\| \leq 2e^{-\omega_1 t},$$

where

$$\begin{aligned} &P_0^{(0)} + P_1^{(0)} + P_2^{(0)} + P_3^{(0)} \\ &= \frac{h_1[2\lambda_1\mu_3(\lambda_2 h_1 + 2\lambda_1 h_2) - h_2]}{Z} \\ &+ \frac{2\lambda_1[(\lambda_2 h_1 + 2\lambda_1 h_2)(2\lambda_1 - \lambda_2)\mu_3 - h_2]}{Z} \\ &+ \frac{\lambda_2 h_1[2\lambda_1(\lambda_2 - 2\lambda_1)\mu_3 - \lambda_2 h_1 h_3]}{Z} \\ &+ \frac{2\lambda_1\lambda_2 h_1(\lambda_2 h_1 + 2\lambda_1 h_2)}{Z}, \end{aligned}$$

then for $t > \frac{3ln2}{\omega_1}$, we have

$$\begin{aligned} V(t) &= \frac{h_1[2\lambda_1\mu_3(\lambda_2 h_1 + 2\lambda_1 h_2) - h_2]}{Z} \\ &+ \frac{2\lambda_1[(\lambda_2 h_1 + 2\lambda_1 h_2)(2\lambda_1 - \lambda_2)\mu_3 - h_2]}{Z} \\ &+ \frac{\lambda_2 h_1[2\lambda_1(\lambda_2 - 2\lambda_1)\mu_3 - \lambda_2 h_1 h_3]}{Z} \\ &+ \frac{2\lambda_1\lambda_2 h_1(\lambda_2 h_1 + 2\lambda_1 h_2)}{Z} + O(t), \end{aligned}$$

$|O(t)| \leq 0.25$. Obviously, the probability of the system failure is

$$\int_0^{+\infty} P_4(y,t)dy + \int_0^{+\infty} P_5(y,t)dy.$$

It has an estimate

$$\begin{aligned} 1 - V(t) &= 1 - \frac{h_1[2\lambda_1\mu_3(\lambda_2 h_1 + 2\lambda_1 h_2) - h_2]}{Z} \\ &+ \frac{2\lambda_1[(\lambda_2 h_1 + 2\lambda_1 h_2)(2\lambda_1 - \lambda_2)\mu_3 - h_2]}{Z} \\ &+ \frac{\lambda_2 h_1[2\lambda_1(\lambda_2 - 2\lambda_1)\mu_3 - \lambda_2 h_1 h_3]}{Z} \\ &+ \frac{2\lambda_1\lambda_2 h_1(\lambda_2 h_1 + 2\lambda_1 h_2)}{Z} \pm 0.25. \end{aligned}$$

Note that the $Z$ as (23) is a decrease function with respect to the repair rates $\mu_4(y), \mu_5(y)$. When the repair rates are strength, the availability of the system increases and hence the reliability of system is enhanced.

**Acknowledgements:** The research was supported by the Natural Science Foundation of China (10871226, 61179041) and in the case of the first author.

*References:*


[1] S. K. Srinivasan and R. Subramanian, Reliability analysis of a three unit warm standby redundant system with repair, *Ann. Oper. Res.,* 143, 227-235, 2006.

[2] J. C. Ke and K. H. Wang, Vacation policies for machine repair problem with two type spares, *Applied Mathematical Modelling,* 31, 880-894, 2007.

[3] J. M. Zhao, A. H. C. Chan, C. Roeerts and K. B. Madelin, Reliability evaluation and optimisation of imperfect inspections for a component with multi-defects, *Reliability Engineering and System Safety,* 92, 65-73, 2007.

[4] W. W. Hu, H. B. Xu, J. Y. Yu and G. T. Zhu, Exponential stability of a reparable multi-state device, *Journal of System Science and Complexity,* 20, 437-443, 2007.

[5] M. J. Kallen and J. M. van Noortwijk, Optimal maintenance decisions under imperfect inspection, *Reliability Engineering and System Safety,* 90, 177-185, 2005.

[6] W. L. Wang and G. Q. Xu, Stability analysis of a complex standby system with constant waiting and different repairman criteria incorporating environmental failure, *Applied Mathematical Modelling,* 33(2), 724-743, 2009.

[7] W. L. Wang and G. Q. Xu, The well-posedness of an $M/G/1$ queue with second optional service and server breakdown, *Computers and Mathematics with Applications,* 57(5), 729-739, 2009.

[8] L. N. Guo, H. B. Xu, C. Gao and G. T. Zhu, Stability analysis of a new kind series system, *IMA Journal of Applied Mathematics,* 75, 439-460, 2010.

[9] Y. L. Chen and G. Q. Xu, Analysis of the solution to a labelled cell populations system, 4th International Conference on *Bioinformatics and Biomedical Engineering(iCBBE),* IEEE, Chengdu, 1-7, 2010.

[10] H. Y. Wang, G. Q. Xu and Z. J. Han, Modeling of health status on given public and its analysis of well-posedness, *Journal of Systems Engineering,* 23(4), 385-391, 2007.

[11] W. Z. Yuan and G. Q. Xu, Spectral analysis of a two unit deteriorating standby system with repair, *WSEAS Transactions on Mathematics,* 10(4), 125-138, 2011.

[12] B. S. Dhillon and O. C. Anude, Common-cause failure analysis of a parallel system with warm standby. *Microelectron. Reliab.* 33(9), 1321-1342, 1993.

[13] B. S. Abbas and Kuo Way, Stochastic effectiveness models for human-machine system. *IEEE Trasactions on Systems. Man. and Cybernetics,* 20(4), 826-834, 1990.

[14] N. Yang and B. S. Dhillon, Availability analysis of a repairable standby human-machine system, *Microelectron. Reliab.* 35(11), 1401-1413, 1995.

[15] B. S. Dhillon and N. Yang, Stochastic analysis of standeby system with common-cause failures and human errors. *Microelectron. Reliab.* 32(12), 1699-1712, 1992.

[16] L. R. Goel, G. C. Sharma and P. Gupta, Stochastic analysis of a human-machine system with critical human error. *Microelectron. Reliab.* 25(4), 669-674, 1995.

[17] K. W. Lee, J. J. Higgins and F. A. Tillman, Stochastic modelling of human-performance reliability. *IEEE Trasactions on Reliability,* 37(5), 269-271, 1988.

[18] W. W. Hu. Asymptotic stability of a parallel repairable system with warm standby under common cause failure. *Acta Anal. Funct. Appl.* 8(1), 5-20, 2006.

[19] R. Nagel, One parameter semigroups of positive operators. *Lecture Notes in Math.* Vol.1184. *Springer-Verlag,* New York. 1986.

[20] H. G. Kapper, C. G. Lekkerkerker and J. Hejtmanek, Spectral methods in transport theory, *OT5. Birhauser Verlag,* Basel. 1982.

[21] Linghai Zhang, Existence, uniqueness and exponential stability of traveling wave solutions of some integral differential equations arising from neuronal networks. *J. Diff. Equas.* 197, 162-196, 2004.

[22] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrande, Vector-valued Laplace transforms and Cauchy problems. *Birkhauser Verlag,* Basel. 2001.

[23] J. Van Neerven, The asymptotic behaviour of semigroups of linear operators. *Birkhauser Verlag,* Basel. 1996.