On four-point nonlocal boundary value problems of nonlinear impulsive equations of fractional order

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Abstract: This paper is motivated from some recent papers treating the boundary value problems for impulsive fractional differential equations. We first give some notations, recall some concepts and preparation results. Second, we establish a general framework to find the solutions for impulsive fractional boundary value problems, which will provide an effective way to deal with such problems. Third, some sufficient conditions for the existence of the solutions are established by applying fixed point methods. Our results complements previous work in the area of four-point boundary value problems of fractional order.

Key–Words: Fractional differential equation; Impulse; Four-point boundary value problem; Fixed point theorem.

1 Introduction

Boundary value problems for nonlinear differential equations arise in a variety of areas of applied mathematics, physics and variational problems of control theory. A point of central importance in the study of nonlinear boundary value problems is to understand how the properties of nonlinearity in a problem influence the nature of the solutions to the boundary value problems. Multi-point nonlinear boundary value problems, which refer to a different family of boundary conditions in the study of disconjugacy theory [1] and take into account the boundary data at intermediate points of the interval under consideration, have been addressed by many authors, for example, see [2-11] and the references therein. The multi-point boundary conditions are important in various physical problems of applied science when the controllers at the end points of the interval (under consideration) dissipate or add energy according to the sensors located at intermediate points.

In recent years, differential equations of fractional order have been addressed by several researchers with the sphere of study ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. Fractional differential equations appear naturally in various fields of science and engineering such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex medium, viscoelasticity, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. In consequence, fractional differential equations have been of great interest. For details, see [12-19] and the references therein.

Impulsive differential equations, which provide a natural description of observed evolution processes, are regarded as important mathematical tools for the better understanding of several real world problems in applied sciences. In fact, the theory of impulsive differential equations is much richer than the corresponding theory of ordinary differential equations without impulse effects since a simple impulsive differential equation may exhibit several new phenomena such as rhythmical beating, merging of solutions and noncontinuability of solutions. For the general theory and applications of impulsive differential equations, we refer the reader to the references [20-24]. On the other hand, the impulsive boundary value problems for nonlinear fractional differential equations have not been addressed so extensively and many aspects of these problems are yet to be explored. For some recent contribution on impulsive differential equations of fractional order, see [25-32,36-38] and the references therein.

In [33], J. Wang, Y. Zhou, M. Feckan considered the boundary value problems for the following impulsive fractional differential equations

$$
\begin{gathered}
{}^{C}D^{q}u(t) = f(t, u(t)),\ t \in J',\ q \in (1,2),\\
\Delta u(t_k) = y_k,\ \Delta u'(t_k) = \bar{y}_k, k = 1, 2, \cdots, m,\\
u(0) = 0, \qquad u'(1) = 0.
\end{gathered}
$$

In [34], G. Wang, B. Ahmad, L.Zhang investigated the existence and uniqueness of solutions for a mixed boundary value problem of nonlinear impulsive differential equations of fractional order

$$^{C}D^{\alpha}u(t) = f(t, u(t)),\ 1 < \alpha \le 2,\ t \in J',$$

$$\Delta u(t_k) = I_k(u(t_k)),$$
$$\Delta u'(t_k) = I_k^*(u(t_k)), k = 1, 2, \cdots, p,$$
$$Tu'(0) = -au(0) - bu(T),$$
$$Tu'(T) = cu(0) + du(T), \quad a, b, c, d \in R.$$

In a recent paper [32], the authors concerned with the existence of solutions for the three-point impulsive boundary value problem involving nonlinear fractional differential equations:

$$^{C}D^{q}u(t) = f(t, u(t)),\ 0 < t < 1,\ t \ne t_k,$$

$$k = 1, 2, \cdots, p,$$
$$\Delta u|_{t=t_k} = I_k(u(t_k)),$$
$$\Delta u'|_{t=t_k} = \bar{I}_k(u(t_k)), k = 1, 2, \cdots, p,$$
$$u(0) + u'(0) = 0, \qquad u(1) + u'(\xi) = 0,$$

where $^{C}D^{q}$ is the Caputo fractional derivative, $q \in R,\ 1 < q \le 2$.

In [35], B. Ahmad and G. Wang investigated the existence and uniqueness of solutions for a four-point impulsive nonlocal boundary value problem of nonlinear differential equations of fractional order

$$^{C}D^{q}x(t) = f(t, x(t)),\ t \in J_1 = J \setminus \{t_1, t_2, \cdots, t_p\},$$

$$\Delta x(t_k) = I_k(x(t_k^-)),$$
$$\Delta x'(t_k) = F_k(x(t_k^-)), k = 1, 2, \cdots, p,$$
$$x'(0) + ax(\eta_1) = 0,$$
$$bx'(1) + x(\eta_2) = 0,\ 0 < \eta_1 \le \eta_2 < 1,$$

where $^{C}D^{q}$ is the Caputo fractional derivative, $1 < q \le 2$.

Motivated by the above works, in this paper, we study the existence of solutions for the four-point nonlocal boundary value problems of nonlinear impulsive equations of fractional order

$$\begin{aligned} &^{C}D^{q}u(t) = f(t, u(t)),\ 0 < t < 1,\\ &t \ne t_k, \quad k = 1, 2, \cdots, p, \end{aligned} \tag{1}$$

$$\begin{aligned} &\Delta u|_{t=t_k} = I_k(u(t_k)),\\ &\Delta u'|_{t=t_k} = \bar{I}_k(u(t_k)), k = 1, 2, \cdots, p, \end{aligned} \tag{2}$$

$$\begin{aligned} &u(0) - \beta u'(\xi) = 0,\\ &u(1) + \gamma u'(\eta) = 0, \end{aligned} \tag{3}$$

where $^{C}D^{q}$ is the Caputo fractional derivative, $q \in R,\ 1 < q \le 2,\ f : [0, 1] \times R \to R$ is a continuous function, $I_k, \bar{I}_k :\ R \to R,\ \beta, \gamma > 0,\ 0 < \xi < \eta < 1,\ \xi \ne t_k,\ \eta \ne t_k,\ k = 1, 2, \cdots, p$ and $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-),\ \Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-),\ u(t_k^+)$ and $u(t_k^-)$ represent the right hand limit and the left hand limit of the function $u(t)$ at $t = t_k$, and the sequences $\{t_k\}$ satisfy that $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1,\ p \in N$.

To the best of our knowledge, there is no paper that consider the four-point impulsive boundary value problem involving nonlinear differential equations of fractional order (1)- (3). The main difficulty of this problem is that the corresponding integral equation is very complex because of the impulse effects. In this paper, we study the existence of solutions for four-point impulsive boundary value problem (1)- (3). By use of Banach's fixed point theorem and Schauder's fixed point theorem, some existence results are obtained.

2 Preliminaries and Lemmas

The material in this section is basic in some sense. For the reader's convenience, we present some necessary definitions from fractional calculus theory and preliminary results.

Definition 1 *[32] The Riemann-Liouville fractional integral of order q for function y is defined as*

$$I^{q}y(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} y(s) ds,\ \ q > 0.$$

Definition 2 *[32] The Caputo's derivative for function y is defined as*

$$^{C}D^{q}y(t) = \frac{1}{\Gamma(n - q)} \int_0^t \frac{y^{(n)}(s)ds}{(t - s)^{q+1-n}},\ \ n = [q]+1,$$

where $[q]$ denotes the integer part of real number q.

Lemma 3 *[32] Let $q > 0$, then the fractional differential equation*

$$^{C}D^{q}u(t) = 0$$

has solutions

$$u(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},\ \ c_i \in R,$$
$$i = 1, 2, \cdots, n, n = [q] + 1.$$

Lemma 4 [32] *Let $q > 0$, then*

$$I^{q\ C}D^q u(t) = u(t) + c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}$$

for some $c_i \in R,\ i = 1, 2, \cdots, n, n = [q] + 1$.

For the sake of convenience, we introduce the following notations.

Let $\Delta = 1+\beta+\gamma,\ J = [0, 1],\ J_0 = [0, t_1],\ J_1 = (t_1, t_2],\ \cdots, J_{p-1} = (t_{p-1}, t_p],\ J_p = (t_p, 1],\ J' = J \setminus \{t_1, t_2, \cdots, t_p\}$, and $PC(J) = \{u : [0, 1] \to R|\ u \in C(J'),\ u(t_k^+)$ and $u(t_k^-)$ exist, and $u(t_k^-) = u(t_k),\ 1 \le k \le p\}$. Obviously, $PC(J)$ is a Banach space with the norm $\|u\| = \sup_{0\le t\le 1} |u(t)|$.

Lemma 5 *Let $y \in C[0, 1]$ and $\xi, \eta \in (t_l, t_{l+1}),\ l$ is a nonnegative integer $0 \le l \le p,\ 1 < q \le 2$. A function $u \in PC(J)$ is a solution of the boundary value problem*

$$\begin{aligned} &^C D^q u(t) = y(t),\ 0 < t < 1,\\ &t \ne t_k,\ k = 1, 2, \cdots, p, \end{aligned} \tag{4}$$

$$\begin{aligned} &\Delta u|_{t=t_k} = I_k(u(t_k)),\\ &\Delta u'|_{t=t_k} = \bar{I}_k(u(t_k)), \quad k = 1, 2, \cdots, p, \end{aligned} \tag{5}$$

$$\begin{aligned} u(0) - \beta u'(\xi) = 0,\\ u(1) + \gamma u'(\eta) = 0, \end{aligned} \tag{6}$$

if and only if u is a solution of the integral equation

$$u(t) = \begin{cases} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} y(s)ds + M_1(\beta + t) \\ +M_2[\beta - (\beta+\gamma)t] \\ +\frac{\beta+\beta\gamma-\beta t}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}y(s)ds, \\ \qquad\qquad t \in J_0, \\ \frac{1}{\Gamma(q)}\int_{t_k}^t (t-s)^{q-1} y(s)ds \\ +\frac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)ds \\ +\frac{1}{\Gamma(q-1)}\sum_{i=1}^{k}(t-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds \\ +\sum_{i=1}^{k}(t-t_i)\bar{I}_i(u(t_i)) + \sum_{i=1}^{k} I_i(u(t_i)) \\ +M_1(\beta+t) + M_2[\beta-(\beta+\gamma)t] \\ +\frac{\beta+\beta\gamma-\beta t}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}y(s)ds, \\ \qquad\qquad t \in J_k, \end{cases} \tag{7}$$

where

$$\begin{aligned} M_1 &= -\tfrac{1}{\Delta\Gamma(q)}\textstyle\int_{t_p}^1 (1-s)^{q-1}y(s)ds \\ &\quad -\tfrac{1}{\Delta\Gamma(q)}\textstyle\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)ds \\ &\quad -\tfrac{1}{\Delta\Gamma(q-1)}\textstyle\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds \\ &\quad -\tfrac{1}{\Delta}\textstyle\sum_{i=1}^{p}(1-t_i)\bar{I}_i(u(t_i)) - \tfrac{1}{\Delta}\sum_{i=1}^{p} I_i(u(t_i)) \\ &\quad -\tfrac{\gamma}{\Delta\Gamma(q-1)}\textstyle\int_{t_l}^{\eta}(\eta-s)^{q-2}y(s)ds, \end{aligned}$$

$$\begin{aligned} M_2 &= \tfrac{1}{\Delta\Gamma(q-1)}\textstyle\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds \\ &\quad +\tfrac{1}{\Delta}\textstyle\sum_{i=1}^{l}\bar{I}_i(u(t_i)). \end{aligned}$$

Proof: Suppose that u is a solution of (4)- (6). By Lemma 4, we have

$$\begin{aligned} u(t) &= I^q y(t) - C_1 - C_2 t \\ &= \tfrac{1}{\Gamma(q)}\textstyle\int_0^t (t-s)^{q-1}y(s)ds - C_1 - C_2 t, \\ &\qquad\qquad t \in J_0, \end{aligned} \tag{8}$$

for some $C_1, C_2 \in R$. Then we have

$$u'(t) = \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}y(s)ds - C_2,\ t \in J_0.$$

If $t \in J_1$, then we have

$$u(t) = \frac{1}{\Gamma(q)}\int_{t_1}^t (t-s)^{q-1}y(s)ds - d_1 - d_2(t-t_1),$$

$$u'(t) = \frac{1}{\Gamma(q-1)}\int_{t_1}^t (t-s)^{q-2}y(s)ds - d_2,$$

for some $d_1, d_2 \in R$. Thus

$$u(t_1^-) = \frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}y(s)ds - C_1 - C_2 t_1,$$

$$u(t_1^+) = -d_1,$$

$$u'(t_1^-) = \frac{1}{\Gamma(q-1)}\int_0^{t_1}(t_1-s)^{q-2}y(s)ds - C_2,$$

$$u'(t_1^+) = -d_2,$$

In view of $\Delta u|_{t=t_1} = u(t_1^+) - u(t_1^-) = I_1(u(t_1))$ and $\Delta u'|_{t=t_1} = u'(t_1^+) - u'(t_1^-) = \bar{I}_1(u(t_1))$, we have

$$-d_1 = \frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}y(s)ds + I_1(u(t_1)) - C_1 - C_2 t_1,$$

$$-d_2 = \frac{1}{\Gamma(q-1)}\int_0^{t_1}(t_1-s)^{q-2}y(s)ds + \bar{I}_1(u(t_1)) - C_2.$$

Hence, we obtain

$$\begin{aligned}u(t) &= \tfrac{1}{\Gamma(q)}\int_{t_1}^{t}(t-s)^{q-1}y(s)ds\\ &+\tfrac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}y(s)ds\\ &+\tfrac{t-t_1}{\Gamma(q-1)}\int_0^{t_1}(t_1-s)^{q-2}y(s)ds+\\ &+(t-t_1)\bar{I}_1(u(t_1))+I_1(u(t_1))-C_1-C_2t,\\ &\quad t\in J_1.\end{aligned}$$

In a similar way, we can obtain

$$\begin{aligned}u(t) &= \tfrac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}y(s)ds\\ &+\tfrac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)ds\\ &+\tfrac{1}{\Gamma(q-1)}\sum_{i=1}^{k}(t-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds\\ &+\sum_{i=1}^{k}(t-t_i)\bar{I}_i(u(t_i))+\sum_{i=1}^{k}I_i(u(t_i))-C_1-C_2t,\\ &\quad t\in J_k,\ k=1,2,\cdots,p,\end{aligned}\tag{9}$$

$$\begin{aligned}u'(t) &= \tfrac{1}{\Gamma(q-1)}\int_{t_k}^{t}(t-s)^{q-2}y(s)ds\\ &+\tfrac{1}{\Gamma(q-1)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds\\ &+\sum_{i=1}^{k}\bar{I}_i(u(t_i))-C_2,\\ &\quad t\in J_k,\ k=1,2,\cdots,p.\end{aligned}\tag{10}$$

By (8), (9), (10), we have

$$\begin{aligned}u(0) &= -C_1\\ u(1) &= \tfrac{1}{\Gamma(q)}\int_{t_p}^{1}(1-s)^{q-1}y(s)ds\\ &+\tfrac{1}{\Gamma(q)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)ds\\ &+\tfrac{1}{\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds\\ &+\sum_{i=1}^{p}(1-t_i)\bar{I}_i(u(t_i))\\ &+\sum_{i=1}^{p}I_i(u(t_i))-C_1-C_2,\end{aligned}$$

$$\begin{aligned}u'(\xi) &= \tfrac{1}{\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}y(s)ds\\ &+\tfrac{1}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds\\ &+\sum_{i=1}^{l}\bar{I}_i(u(t_i))-C_2,\end{aligned}$$

$$\begin{aligned}u'(\eta) &= \tfrac{1}{\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}y(s)ds\\ &+\tfrac{1}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds\\ &+\sum_{i=1}^{l}\bar{I}_i(u(t_i))-C_2.\end{aligned}$$

In view of the boundary condition $u(0)-\beta u'(\xi)=0$, we have

$$\begin{aligned}C_1 &= -\tfrac{\beta}{\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}y(s)ds\\ &-\tfrac{\beta}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds\\ &-\beta\sum_{i=1}^{l}\bar{I}_i(u(t_i))+C_2\beta.\end{aligned}\tag{11}$$

By the boundary condition $u(1)+\gamma u'(\eta)=0$, we have

$$\begin{aligned}&\tfrac{1}{\Gamma(q)}\int_{t_p}^{1}(1-s)^{q-1}y(s)ds\\ &+\tfrac{1}{\Gamma(q)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)ds\\ &+\tfrac{1}{\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds\\ &+\sum_{i=1}^{p}(1-t_i)\bar{I}_i(u(t_i))+\sum_{i=1}^{p}I_i(u(t_i))-C_1-C_2\\ &+\tfrac{\gamma}{\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}y(s)ds\\ &+\tfrac{\gamma}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds\\ &+\gamma\sum_{i=1}^{l}\bar{I}_i(u(t_i))-C_2\gamma=0.\end{aligned}\tag{12}$$

From (11), (12), we obtain

$$\begin{aligned}C_1 &= \tfrac{\beta}{\Delta\Gamma(q)}\int_{t_p}^{1}(1-s)^{q-1}y(s)ds\\ &+\tfrac{\beta}{\Delta\Gamma(q)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)ds\\ &+\tfrac{\beta}{\Delta\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds\\ &+\tfrac{\beta}{\Delta}\sum_{i=1}^{p}(1-t_i)\bar{I}_i(u(t_i))+\tfrac{\beta}{\Delta}\sum_{i=1}^{p}I_i(u(t_i))\\ &+\tfrac{\beta\gamma}{\Delta\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}y(s)ds\\ &-\tfrac{\beta}{\Delta\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds\\ &-\tfrac{\beta}{\Delta}\sum_{i=1}^{l}\bar{I}_i(u(t_i))\\ &-\tfrac{\beta+\beta\gamma}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}y(s)ds.\end{aligned}\tag{13}$$

$$\begin{aligned}C_2 &= \tfrac{1}{\Delta\Gamma(q)}\int_{t_p}^{1}(1-s)^{q-1}y(s)ds\\ &+\tfrac{1}{\Delta\Gamma(q)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)ds\\ &+\tfrac{1}{\Delta\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds\\ &+\tfrac{1}{\Delta}\sum_{i=1}^{p}(1-t_i)\bar{I}_i(u(t_i))+\tfrac{1}{\Delta}\sum_{i=1}^{p}I_i(u(t_i))\\ &+\tfrac{\gamma}{\Delta\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}y(s)ds\\ &+\tfrac{\beta+\gamma}{\Delta\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)ds\\ &+\tfrac{\beta+\gamma}{\Delta}\sum_{i=1}^{l}\bar{I}_i(u(t_i))+\tfrac{\beta}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}y(s)ds.\end{aligned}\tag{14}$$

By (8), (9), (13), (14), we get (7).

Conversely, we assume that u is a solution of the integral equation (7). In view of the relations ${}^{C}D^{p}I^{p}y(t) = y(t)$ for $p > 0$, we get

$$ {}^{C}D^{q}u(t) = y(t),\ \ 0 < t < 1, $$
$$ t \neq t_k,\ k = 1, 2, \cdots, p, 1 < q \leq 2. $$

Moreover, it can easily be verified that

$$ \Delta u|_{t=t_k} = I_k(u(t_k)),\ \Delta u'|_{t=t_k} = \bar{I}_k(u(t_k)), $$
$$ k = 1, 2, \cdots, p, $$
$$ u(0) - \beta u'(\xi) = 0, \qquad u(1) + \gamma u'(\eta) = 0. $$

The proof is completed.

Lemma 6 *Let* $\xi, \eta \in (t_l, t_{l+1}),\ l$ *is a nonnegative integer,* $0 \leq l \leq p$.

Define an operator $T:\ PC(J) \to PC(J)$ *by*

$$ \begin{aligned} (Tu)(t) &= \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, u(s)) ds \\ &+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, u(s)) ds \\ &+ \frac{1}{\Gamma(q-1)} \sum_{0<t_k<t} (t - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{q-2} f(s, u(s)) ds \\ &+ \sum_{0<t_k<t} (t - t_k) \bar{I}_k(u(t_k)) + \sum_{0<t_k<t} I_k(u(t_k)) \\ &+ M_1(\beta + t) + M_2[\beta - (\beta + \gamma)t] \\ &+ \frac{\beta + \beta\gamma - \beta t}{\Delta\Gamma(q-1)} \int_{t_l}^{\xi} (\xi - s)^{q-2} f(s, u(s)) ds, \end{aligned} \tag{15} $$

where

$$ \begin{aligned} M_1 = &- \tfrac{1}{\Delta\Gamma(q)} \int_{t_p}^{1} (1-s)^{q-1} f(s, u(s)) ds \\ &- \tfrac{1}{\Delta\Gamma(q)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} f(s, u(s)) ds \\ &- \tfrac{1}{\Delta\Gamma(q-1)} \sum_{i=1}^{p} (1 - t_i) \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} f(s, u(s)) ds \\ &- \tfrac{1}{\Delta} \sum_{i=1}^{p} (1 - t_i) \bar{I}_i(u(t_i)) - \tfrac{1}{\Delta} \sum_{i=1}^{p} I_i(u(t_i)) \\ &- \tfrac{\gamma}{\Delta\Gamma(q-1)} \int_{t_l}^{\eta} (\eta - s)^{q-2} f(s, u(s)) ds, \end{aligned} $$

and

$$ \begin{aligned} M_2 \ = \ & \tfrac{1}{\Delta\Gamma(q-1)} \sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} f(s, u(s)) ds \\ & + \tfrac{1}{\Delta} \sum_{i=1}^{l} \bar{I}_i(u(t_i)). \end{aligned} $$

Clearly, the fixed points of the operator T *are solutions of problem (1)-(3).*

3 Main result

Theorem 7 *Assume that:*
(C_1) *There exists a constant* $L_1 > 0$ *such that*

$$ |f(t, x) - f(t, y)| \leq L_1 |x - y|, $$

for each $t \in J$ *and all* $x, y \in R$.
(C_2) *There exist constants* $L_2, L_3 > 0$ *such that*

$$ |I_k(x) - I_k(y)| \leq L_2 |x - y|, $$
$$ |\bar{I}_k(x) - \bar{I}_k(y)| \leq L_3 |x - y|, $$

for each $t \in J$ *and all* $x, y \in R,\ k = 1, 2, \cdots, p$.

If the condition

$$ \begin{aligned} &\frac{[(1+\beta+\gamma) + (2+2\beta+\gamma)(p+1)]L_1}{\Delta\Gamma(q+1)} \\ &+ \frac{[2(1+\beta+\gamma)p + \beta + 2\beta\gamma + \gamma]L_1}{\Delta\Gamma(q)} \\ &+ \frac{(2+2\beta+\gamma)pL_2}{\Delta} + \frac{2(1+\beta+\gamma)pL_3}{\Delta} < 1 \end{aligned} $$

is satisfied, then problem (1)-(3) has a unique solution.

Proof: Let $x, y \in PC(J)$. Then for each $t \in J$, we have

$$ \begin{aligned} &|(Tx)(t) - (Ty)(t)| \\ &\leq \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} |f(s, x(s)) - f(s, y(s))| ds \\ &+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \\ &|f(s, x(s)) - f(s, y(s))| ds \\ &+ \frac{1}{\Gamma(q-1)} \sum_{0<t_k<t} (t - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{q-2} \\ &|f(s, x(s)) - f(s, y(s))| ds \\ &+ \sum_{0<t_k<t} (t - t_k) |\bar{I}_k(x(t_k)) - \bar{I}_k(y(t_k))| \\ &+ \sum_{0<t_k<t} |I_k(x(t_k)) - I_k(y(t_k))| \\ &+ \frac{\beta + t}{\Delta\Gamma(q)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} \\ &|f(s, x(s)) - f(s, y(s))| ds \\ &+ \frac{\beta + t}{\Delta\Gamma(q-1)} \sum_{i=1}^{p} (1 - t_i) \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} \\ &|f(s, x(s)) - f(s, y(s))| ds \\ &+ \frac{\beta + t}{\Delta} \sum_{i=1}^{p} (1 - t_i) |\bar{I}_i(x(t_i)) - \bar{I}_i(y(t_i))| \\ &+ \frac{\beta + t}{\Delta} \sum_{i=1}^{p} |I_i(x(t_i)) - I_i(y(t_i))| \end{aligned} $$

$$
\begin{aligned}
&+\frac{\gamma(\beta+t)}{\Delta\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}\\
&|f(s,x(s))-f(s,y(s))|ds\\
&+\frac{(\beta+\gamma)t-\beta}{\Delta\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\\
&|f(s,x(s))-f(s,y(s))|ds\\
&+\frac{(\beta+\gamma)t-\beta}{\Delta}\sum_{i=1}^{l}|\bar{I}_i(x(t_i))-\bar{I}_i(y(t_i))|\\
&+\frac{\beta+\beta\gamma-\beta t}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}\\
&|f(s,x(s))-f(s,y(s))|ds\\
&\le\frac{L_1\|x-y\|}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}ds\\
&+\frac{(\Delta+\beta+1)L_1\|x-y\|}{\Delta\Gamma(q)}\\
&\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}ds\\
&+\frac{(\Delta+\beta+1+\gamma)L_1\|x-y\|}{\Delta\Gamma(q-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}ds\\
&+\frac{\gamma(\beta+1)L_1\|x-y\|}{\Delta\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}ds\\
&+\frac{(\beta+\beta\gamma)L_1\|x-y\|}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}ds\\
&+\frac{\beta+1+\Delta}{\Delta}\sum_{i=1}^{p}L_2\|x-y\|\\
&+\frac{\beta+1+\gamma+\Delta}{\Delta}\sum_{i=1}^{p}L_3\|x-y\|\\
&\le\frac{[(1+\beta+\gamma)+(2+2\beta+\gamma)(p+1)]L_1}{\Delta\Gamma(q+1)}\\
&+\frac{[2(1+\beta+\gamma)p+\beta+2\beta\gamma+\gamma]L_1}{\Delta\Gamma(q)}\\
&+\frac{(2+2\beta+\gamma)pL_2}{\Delta}+\frac{2(1+\beta+\gamma)pL_3}{\Delta}\|x-y\|.
\end{aligned}
$$

From

$$
\begin{aligned}
&\frac{[(1+\beta+\gamma)+(2+2\beta+\gamma)(p+1)]L_1}{\Delta\Gamma(q+1)}\\
&+\frac{[2(1+\beta+\gamma)p+\beta+2\beta\gamma+\gamma]L_1}{\Delta\Gamma(q)}\\
&+\frac{(2+2\beta+\gamma)pL_2}{\Delta}+\frac{2(1+\beta+\gamma)pL_3}{\Delta}<1,
\end{aligned}
$$

we have

$$\|Tx-Ty\|<\|x-y\|,$$

so T is a contraction. As a consequence of Banach's fixed point theorem, we deduce that T has a fixed point which is a solution of problem (1)-(3).

Theorem 8 *Assume that:*

(C_3) *The function* $f:\ [0,1]\times R\to R$ *is continuous, and there exists a constant* $N_1>0$ *such that* $|f(t,u)|\le N_1$,*for each* $t\in J$ *and all* $u\in R$.

(C_4) *The functions* $I_k,\ \bar{I}_k:\ R\to R$ *are continuous, and there exist constants* $N_2,\ N_3>0$ *such that*

$$|I_k(u)|\le N_2,\ \ |\bar{I}_k(u)|\le N_3$$

for all $u\in R,\ k=1,2,\cdots,p$. *Then problem (1)-(3) has at least one solution.*

Proof: We shall prove this theorem in four steps.

Step 1: T is continuous.

Let $\{u_n\}$ be a sequence such that $u_n\to u$ in $PC(J)$.

$$
\begin{aligned}
&|(Tu_n)(t)-(Tu)(t)|\\
&\le\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}|f(s,u_n(s))-f(s,u(s))|ds\\
&+\frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\\
&\qquad|f(s,u_n(s))-f(s,u(s))|ds\\
&+\frac{1}{\Gamma(q-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{q-2}\\
&\qquad|f(s,u_n(s))-f(s,u(s))|ds\\
&+\sum_{0<t_k<t}(t-t_k)|\bar{I}_k(u_n(t_k))-\bar{I}_k(u(t_k))|\\
&+\sum_{0<t_k<t}|I_k(u_n(t_k))-I_k(u(t_k))|\\
&+\frac{\beta+t}{\Delta\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\\
&|f(s,u_n(s))-f(s,u(s))|ds\\
&+\frac{\beta+t}{\Delta\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\\
&|f(s,u_n(s))-f(s,u(s))|ds\\
&+\frac{\beta+t}{\Delta}\sum_{i=1}^{p}(1-t_i)|\bar{I}_i(u_n(t_i))-\bar{I}_i(u(t_i))|\\
&+\frac{\beta+t}{\Delta}\sum_{i=1}^{p}|I_i(u_n(t_i))-I_i(u(t_i))|\\
&+\frac{\gamma(\beta+t)}{\Delta\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}\\
&|f(s,u_n(s))-f(s,u(s))|ds\\
&+\frac{(\beta+\gamma)t-\beta}{\Delta\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\\
&|f(s,u_n(s))-f(s,u(s))|ds\\
&+\frac{(\beta+\gamma)t-\beta}{\Delta}\sum_{i=1}^{l}|\bar{I}_i(u_n(t_i))-\bar{I}_i(u(t_i))|\\
&+\frac{\beta+\beta\gamma-\beta t}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}\\
&|f(s,u_n(s))-f(s,u(s))|ds\\
&\le\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}|f(s,u_n(s))-f(s,u(s))|ds
\end{aligned}
$$

$$+\frac{\Delta+\beta+1}{\Delta\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}$$
$$|f(s,u_n(s))-f(s,u(s))|ds$$
$$+\frac{\Delta+\beta+1+\gamma}{\Delta\Gamma(q-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}$$
$$|f(s,u_n(s))-f(s,u(s))|ds$$
$$+\frac{\gamma(\beta+1)}{\Delta\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}|f(s,u_n(s))-f(s,u(s))|ds$$
$$+\frac{\beta+\beta\gamma}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}|f(s,u_n(s))-f(s,u(s))|ds$$
$$+\frac{\beta+1+\Delta}{\Delta}\sum_{i=1}^{p}|I_i(u_n(t_i))-I_i(u(t_i))|$$
$$+\frac{\beta+1+\gamma+\Delta}{\Delta}\sum_{i=1}^{p}|\bar{I}_i(u_n(t_i))-\bar{I}_i(u(t_i))|.$$

Since $f,\ I,\ \bar{I}$ are continuous functions, then we have

$$\|Tu_n-Tu\|\to 0 \quad \text{as } n\to\infty.$$

Step 2: T maps bounded sets into bounded sets.

We shall show that for any $\rho>0$, there exists a positive constant L such that for each $u\in\Omega_\rho=\{u\in PC(J)|\|u\|\le\rho\}$, we have $\|Tu\|\le L$. By (C_3) and (C_4), we have for each $t\in J$,

$$|(Tu)(t)|$$
$$\le\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}|f(s,u(s))|ds$$
$$+\frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}|f(s,u(s))|ds$$
$$+\frac{1}{\Gamma(q-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{q-2}$$
$$|f(s,u(s))|ds$$
$$+\sum_{0<t_k<t}(t-t_k)|\bar{I}_k(u(t_k))|+\sum_{0<t_k<t}|I_k(u(t_k))|$$
$$+\frac{\beta+t}{\Delta\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}|f(s,u(s))|ds$$
$$+\frac{\beta+t}{\Delta\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}$$
$$|f(s,u(s))|ds$$
$$+\frac{\beta+t}{\Delta}\sum_{i=1}^{p}(1-t_i)|\bar{I}_i(u(t_i))|$$
$$+\frac{\beta+t}{\Delta}\sum_{i=1}^{p}|I_i(u(t_i))|$$
$$+\frac{\gamma(\beta+t)}{\Delta\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}|f(s,u(s))|ds$$
$$+\frac{(\beta+\gamma)t-\beta}{\Delta\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}|f(s,u(s))|ds$$
$$+\frac{(\beta+\gamma)t-\beta}{\Delta}\sum_{i=1}^{l}|\bar{I}_i(u(t_i))|$$
$$+\frac{\beta+\beta\gamma-\beta t}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}|f(s,u(s))|ds$$
$$\le\frac{N_1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}ds$$
$$+\frac{(\Delta+\beta+1)N_1}{\Delta\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}ds$$
$$+\frac{(\Delta+\beta+1+\gamma)N_1}{\Delta\Gamma(q-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}ds$$
$$+\frac{\gamma(\beta+1)N_1}{\Delta\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}ds$$
$$+\frac{(\beta+\beta\gamma)N_1}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}ds$$
$$+\frac{(\beta+1+\Delta)pN_2}{\Delta}+\frac{(\beta+1+\gamma+\Delta)pN_3}{\Delta}$$
$$\le\frac{[(1+\beta+\gamma)+(2+2\beta+\gamma)(p+1)]N_1}{\Delta\Gamma(q+1)}$$
$$+\frac{[2(1+\beta+\gamma)p+\beta+2\beta\gamma+\gamma]N_1}{\Delta\Gamma(q)}$$
$$+\frac{(2+2\beta+\gamma)pN_2}{\Delta}+\frac{2(1+\beta+\gamma)pN_3}{\Delta}.$$

Thus

$$\|Tu\|\le\frac{[(1+\beta+\gamma)+(2+2\beta+\gamma)(p+1)]N_1}{\Delta\Gamma(q+1)}$$
$$+\frac{[2(1+\beta+\gamma)p+\beta+2\beta\gamma+\gamma]N_1}{\Delta\Gamma(q)}$$
$$+\frac{(2+2\beta+\gamma)pN_2}{\Delta}+\frac{2(1+\beta+\gamma)pN_3}{\Delta}:=L.$$

Step 3: T maps bounded sets into equicontinuous sets.

Let Ω_ρ be a bounded set of $PC(J)$ as in step 2, and let $u\in\Omega_\rho$. For each $t\in J_k,\ 0\le k\le p$, we have

$$|(Tu)'(t)|$$
$$\le\frac{1}{\Gamma(q-1)}\int_{t_k}^{t}(t-s)^{q-2}|f(s,u(s))|ds$$
$$+\frac{1}{\Gamma(q-1)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-2}|f(s,u(s))|ds$$
$$+\sum_{0<t_k<t}|\bar{I}_k(u(t_k))|$$
$$+\frac{1}{\Delta\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}|f(s,u(s))|ds$$
$$+\frac{1}{\Delta\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}$$
$$|f(s,u(s))|ds$$
$$+\frac{1}{\Delta}\sum_{i=1}^{p}(1-t_i)|\bar{I}_i(u(t_i))|+\frac{1}{\Delta}\sum_{i=1}^{p}|I_i(u(t_i))|$$
$$+\frac{\gamma}{\Delta\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}|f(s,u(s))|ds$$

$$
\begin{aligned}
&+\frac{(\beta+\gamma)}{\Delta\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}|f(s,u(s))|ds\\
&+\frac{(\beta+\gamma)}{\Delta}\sum_{i=1}^{l}|\bar{I}_i(u(t_i))|\\
&+\frac{\beta}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}|f(s,u(s))|ds\\
&\le \frac{N_1}{\Gamma(q-1)}\int_{t_k}^{t}(t-s)^{q-2}ds\\
&+\frac{(\Delta+\beta+1+\gamma)N_1}{\Delta\Gamma(q-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}ds\\
&+\frac{N_1}{\Delta\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}ds\\
&+\frac{\gamma N_1}{\Delta\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}ds\\
&+\frac{\beta N_1}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}ds\\
&+\frac{pN_2}{\Delta}+\frac{(\beta+1+\gamma+\Delta)pN_3}{\Delta}\\
&\le\left(\frac{p+1}{\Delta\Gamma(q+1)}+\frac{\Delta+(\Delta+1+\beta+\gamma)p+\beta+\gamma}{\Delta\Gamma(q)}\right)N_1\\
&+\frac{pN_2}{\Delta}+\frac{(\beta+1+\gamma+\Delta)pN_3}{\Delta}:=M.
\end{aligned}
$$

Hence, letting $t'', t' \in J_k,\ t' < t'',\ 0 \le k \le p$, we have

$$|(Tu)(t'')-(Tu)(t')| \le \int_{t'}^{t''}|(Tu)'(s)|ds \le M(t''-t').$$

So, $T(\Omega_\rho)$ is equiv-continuous on all J_k, $(k = 0,1,2,\cdots,p)$. We can conclude that $T: PC(J) \to PC(J)$ is completely continuous.

Step 4: A priori bounds.

We shall show that the set

$$\Omega = \{u \in PC(J) | u = \lambda Tu \ \text{ for some } \ 0 < \lambda < 1\}$$

is bounded. Let $u \in \Omega$, then $u = \lambda Tu$ for some $0 < \lambda < 1$. Thus, for each $t \in J$, we have

$$
\begin{aligned}
u(t) &= \frac{\lambda}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}f(s,u(s))ds\\
&+\frac{\lambda}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}f(s,u(s))ds\\
&+\frac{\lambda}{\Gamma(q-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{q-2}\\
&f(s,u(s))ds\\
&+\lambda\sum_{0<t_k<t}(t-t_k)\bar{I}_k(u(t_k))+\lambda\sum_{0<t_k<t}I_k(u(t_k))\\
&-\frac{\lambda(\beta+t)}{\Delta\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}f(s,u(s))ds\\
&-\frac{\lambda(\beta+t)}{\Delta\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\\
&f(s,u(s))ds\\
&-\frac{\lambda(\beta+t)}{\Delta}\sum_{i=1}^{p}(1-t_i)\bar{I}_i(u(t_i))\\
&-\frac{\lambda(\beta+t)}{\Delta}\sum_{i=1}^{p}I_i(u(t_i))\\
&-\frac{\lambda\gamma(\beta+t)}{\Delta\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}\\
&f(s,u(s))ds\\
&+\frac{\lambda[\beta-(\beta+\gamma)t]}{\Delta\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\\
&f(s,u(s))ds\\
&+\frac{\lambda[\beta-(\beta+\gamma)t]}{\Delta}\sum_{i=1}^{l}\bar{I}_i(u(t_i))\\
&+\frac{\lambda(\beta+\beta\gamma-\beta t)}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}\\
&f(s,u(s))ds\\
&\le\frac{N_1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}ds\\
&+\frac{(\Delta+\beta+1)N_1}{\Delta\Gamma(q)}\\
&\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}ds\\
&+\frac{(\Delta+\beta+1+\gamma)N_1}{\Delta\Gamma(q-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}ds\\
&+\frac{\gamma(\beta+1)N_1}{\Delta\Gamma(q-1)}\int_{t_l}^{\eta}(\eta-s)^{q-2}ds\\
&+\frac{(\beta+\beta\gamma)N_1}{\Delta\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}ds\\
&+\frac{(\beta+1+\Delta)pN_2}{\Delta}\\
&+\frac{(\beta+1+\gamma+\Delta)pN_3}{\Delta}\\
&\le\frac{[(1+\beta+\gamma)+(2+2\beta+\gamma)(p+1)]N_1}{\Delta\Gamma(q+1)}\\
&+\frac{[2(1+\beta+\gamma)p+\beta+2\beta\gamma+\gamma]N_1}{\Delta\Gamma(q)}\\
&+\frac{(2+2\beta+\gamma)pN_2}{\Delta}+\frac{2(1+\beta+\gamma)pN_3}{\Delta}.
\end{aligned}
$$

Thus, for every $t \in J$, we have

$$
\begin{aligned}
\|u\| &\le \frac{[(1+\beta+\gamma)+(2+2\beta+\gamma)(p+1)]N_1}{\Delta\Gamma(q+1)}\\
&+\frac{[2(1+\beta+\gamma)p+\beta+2\beta\gamma+\gamma]N_1}{\Delta\Gamma(q)}\\
&+\frac{(2+2\beta+\gamma)pN_2}{\Delta}+\frac{2(1+\beta+\gamma)pN_3}{\Delta}.
\end{aligned}
$$

This shows that Ω is bounded. By use of Schauder's fixed point theorem, we deduce that T has a fixed point which is a solution of problem (1)-(3). This completes the proof.

4 Conclusion

This paper is motivated from some recent papers treating the boundary value problems for impulsive fractional differential equations. We first give some notations, recall some concepts and preparation results. Second, we establish a general framework to find the solutions for impulsive fractional boundary value problems, which will provide an effective way to deal with such problems. Third, some sufficient conditions for the existence of the solutions are established by applying fixed point methods. Our results complements previous work in the area of four-point boundary value problems of fractional order. To the best of our knowledge, there is no paper that consider the four-point impulsive boundary value problem involving nonlinear differential equations of fractional order (1)- (3). The main difficulty of this problem is that the corresponding integral equation is very complex because of the impulse effects. In this paper, by use of Banach's fixed point theorem and Schauder's fixed point theorem, some existence results are obtained.

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