Blowup Analysis for a Nonlocal Reaction Diffusion Equation with Potential

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Abstract: In this paper we investigate a nonlocal reaction diffusion equation with potential, under Neumann boundary. We obtain the complete classification of the parameters for which the solution blows up in finite time or exists globally. Moreover, we study the blowup rate and the blowup set for the blowup solution.

Key–Words: Nonlocal diffusion, Blow up, Blowup rate, Blowup set

1 Introduction

It is well known the reaction diffusion system is the classical model in describing the spatial-temporal patten (see, e.g.[9]). However, the Laplacian operator in the reaction diffusion system is not sufficiently accurate in modelling the spatial diffusion of the individual in some cases, especially in many biological areas (see [8]). As stated in [8, 10], one way to overcome this disadvantage is to introduce the following nonlocal evolution equation

$$u_t(x,t) = \int_{R^N} J(x-y)(u(y,t)-u(x,t))\mathrm{d}y. \quad (1)$$

This equation and their variations, have been widely used to model diffusion process, for example, in biology, dislocations dynamics, etc. See, for example, [1, 5] and references therein. As stated in [5], if $u(x,t)$ is thought of as a density at the point x and time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then the convolution $(J*u)(x,t) := \int_{R^N} J(x-y)u(y,t)\mathrm{d}y$ is the rate at which individuals are arriving at x from all other places and $-u(x,t) = -\int_{R^N} J(x-y)u(x,t)\mathrm{d}y$ is the rate at which they are leaving location x to travel to all other sites.

In the past decades, some works have shown that equation (1) shares many properties with the classical heat equation

$$u_t - \Delta u = 0$$

such as bounded stationary solutions are constants, a maximum principle holds for both of them, etc. However, there is no regularizing effect in general. So it is an interesting topic to compare the properties of solutions of such nonlocal diffusion equation with corresponding local diffusion cases, see [11, 12] and references therein.

Motivated by above works, we devote our attention to the blowup analysis of the following nonlocal diffusion equation

$$\begin{aligned} u_t(x,t) &= \int_\Omega J(x-y)(u(y,t)-u(x,t))\mathrm{d}y \\ &\quad +a(x)u^p(x,t) - b(x)u^q(x,t), \\ &\qquad\qquad x\in\Omega,\ t>0, \\ u(x,0) &= u_0(x),\ x\in\Omega. \end{aligned} \quad (2)$$

Here $J: R^N \to R$ is nonnegative, bounded , symmetric radially and strictly decreasing function with $\int_{R^N} J(z) = 1$, p and q are both positive constants. Ω is a bounded connected and smooth domain. And the potential functions $a(x)$, $b(x)$ are both in $C^1(\overline{\Omega})$, satisfy $a(x) \geq A_1$, $b(x) \geq B_1$ for some positive constants A_1, B_1 , respectively. We take the initial datum, $u_0(x)$, nonnegative, nontrivial and bounded. As we will see through these pages, this equation will share many properties with corresponding local diffusion problems.

Note that in our problem (2) we are integrating in Ω. In this case, we are imposing the condition that the diffusion takes place only in Ω. No individual may enter or leave the domain. This is so called Neumann boundary conditions, see [4]. For more study about the nonlocal diffusion operator, we refer to [2, 3] and references therein.

A solution of (2) is a function $u(x,t) \in C^1([0,T); C(\overline{\Omega}))$ satisfying (2). As usual, we say the solution of problem (2) u blows up in finite time if there exists a $T < +\infty$ such that $\| u(x,t) \|_{L^\infty} < +\infty$ for all $t \in [0,T)$ and

$$\lim_{t\to T} \| u(x,t) \|_{L^\infty} = +\infty.$$

For the blowup properties of the solution to problem (2), we have following results. Firstly, we determine the complete classification of the parameters for which the solution blows up in finite time or exists globally.

Theorem 1 (i) *If* $p > \max\{q,1\}$, *then the equation (2) with large initial data have solutions blowing up in finite time, while the solutions of (2) with small initial data exist globally.*

(ii) *If* $p \le \max\{q,1\}$, *then all solutions of (2) are global. Moreover, if* $p < q$ *or* $p = q$ *and* $\max_{x\in\overline{\Omega}}(a(x) - b(x)) \le 0$, *all solutions are uniformly bounded, while if* $p = q$ *and* $\min_{x\in\overline{\Omega}}(a(x) - b(x)) > 0$, *there exist unbounded global solutions.*

Once we have obtained the values of the parameters for which blowup occurs, the next step is to concern the blowup rate. To this end, we suppose that

(H) $\mathrm{supp}J(x) \cap \mathrm{supp}u_0(x) \ne \emptyset$.

Theorem 2 *Let* $p > \max\{q,1\}$ *and* $u(x,t)$ *be a solution of (2) blowing up at time* T. *Then*

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}} \max_{x\in\overline{\Omega}} u(x,t) = ((p-1)\max_{x\in\overline{\Omega}} a(x))^{-\frac{1}{p-1}}. \tag{3}$$

Remark 3 *Note that the blowup rate here depends on the potential. This is different from the usual local diffusion case.*

Next we consider the spacial location of the blowup set. As usual, the blowup set of solution $u(x,t)$ is defined as follows:

$$B(u) = \{x \in \bar{\Omega}\,;\ \text{there exist}\ (x_n,\,t_n) \to (x,\,T)\ \text{such that}\ u(x_n,t_n) \to \infty\}.$$

where T is the maximal existence time of u. For a general domain Ω we can localize the blowup set near any pint in Ω just by taking an initial condition being very large near that point and not so large in the rest of the domain. This is the following result.

Theorem 4 *Let* $p > 2$. *For any* $x_0 \in \Omega$ *and* $\varepsilon > 0$, *there exists an initial data* u_0 *such that the corresponding solution* $u(x,t)$ *of (2) blows up at finite time* T *and its blowup set* $B(u)$ *is contained in* $B_\epsilon(x_0) = \{x \in \bar{\Omega}\,;\ \| x - x_0 \| < \epsilon\}$.

Considering the radial symmetric case, we have the following result.

Theorem 5 *Let* $p > \max\{q,2\}$ *and* $\Omega = B_R = \{|x| < R\}$. *If the potential functions* $a(x)$, $b(x)$ *are radial symmetric and satisfy* $a'(r) \le 0,\ \ b'(r) \ge 0$. *And the initial data* $u_0 \in C^1(\bar{B_R})$ *is a radial nonnegative function with a unique maximum at the origin, that is,* $u_0 = u_0(r) \ge 0$, $u_0'(r) < 0$ *for* $0 < r \le R$, $u'(0) = 0$ *and* $u_0''(0) < 0$, *then the blowup set* $B(u)$ *of the solution* u *of (2) consists only of the original point* $x = 0$.

The remainder of this paper is organized as follows. In Section 2, we give the existence and uniqueness of the solutions as well as the comparison principle. In section 3, we prove the blowup and global existence condition. And then we prove the blowup rate and blowup set results in Section 4 and section 5, respectively. In Section 6, we will give some numerical experiments to demonstrate our results. And the last section is devoted to our conclusion.

2 Existence, Uniqueness and Comparison Principle

We begin our study of problem (2) with a result of existence and uniqueness of continuous solution and comparison principle.

Firstly, existence and uniqueness of solution is a consequence of Banach's fixed point theorem. We look for $u \in C^1([0,T);C(\overline{\Omega}))$ satisfying (2). Fix $t_0 > 0$, and consider the Banach space $X_{t_0} = C^1([0,T);C(\overline{\Omega}))$ with the norm

$$\| \omega \|_{X_{t_0}} = \max_{0\le t\le t_0} \| \omega(\cdot,t) \|_{L^\infty(\overline{\Omega})} + \max_{0\le t\le t_0} \| \omega_t(\cdot,t) \|_{L^\infty(\overline{\Omega})}.$$

We define the following operator $T : X_{t_0} \to X_{t_0}$

$$\begin{aligned} & T_{\omega_0}(\omega)(x,t) \\ = \ & \omega_0(x) + \int_0^t \int_\Omega J(x-y)(\omega(y,s) - \omega(x,s))\mathrm{d}y ds \\ & + \int_0^t \Big(a(x) \mid \omega \mid^{p-1} \omega(x,s) \\ & - b(x) \mid \omega \mid^{q-1} \omega(x,s) \Big) ds. \end{aligned}$$

We could prove the solution to (2) is a fixed point of operator T in a convenient ball of X_{t_0}. Thus, we have the following result.

Theorem 6 *For every* $u_0 \in C(\overline{\Omega})$ *there exists a unique solution* u *of (2) such that* $u \in C^1([0,T);C(\overline{\Omega}))$ *and* T *(finite or infinite) is the maximal existence time of solution .*

In order to prove Theorem 6, we need the following lemma.

Lemma 7 *The operator T_{ω_0} is well defined, mapping X_{t_0} into itself. Moreover, let $\omega_0,\, z_0 \in C(\overline{\Omega})$ and $\omega,\, z \in X_{t_0}$. Then there exists a positive constant $C = C(p,\, q,\, \| \omega \|_{X_{t_0}},\, \| z \|_{X_{t_0}},\, \| J \|_\infty,\, \Omega)$ such that*

$$\begin{aligned}&\| T_{\omega_0}(\omega) - T_{z_0}(z) \|_{X_{t_0}} \\ &\le \| \omega_0 - z_0 \|_{L^\infty(\overline{\Omega})} + Ct \| \omega - z \|_{X_{t_0}}\,. \end{aligned} \tag{4}$$

Thus, T_{ω_0} is a strict contraction if t_0 is small enough and $u_0 \equiv v_0$.

Proof. This proof consists of several steps.

Step 1. We show that T_{ω_0} maps X_{t_0} into X_{t_0}. Notice that $\int_\Omega J(x-y)dx \le \int_{R^N} J(x)dx = 1$. For any $(x,t) \in \overline{\Omega} \times [0, t_0]$ we have,

$$\begin{aligned}&|T_{\omega_0}(\omega(x,t)) - \omega_0| \\ &\le \Big| \int_0^t \int_\Omega J(x-y)(\omega(y,s) - \omega(x,s))\mathrm{d}y ds \Big| \\ &\quad + \Big| \int_0^t \Big(a(x) \mid \omega \mid^{p-1} \omega(x,s) \\ &\quad - b(x) \mid \omega \mid^{q-1} \omega(x,s) \Big) ds \Big| \\ &\le \Big| \int_0^t \int_\Omega J(x-y)(\omega(y,s) - \omega(x,s))\mathrm{d}y ds \Big| \\ &\quad + \Big| \int_0^t \Big(a(x) \mid \omega \mid^{p-1} \omega(x,s) \Big) ds \Big| \\ &\quad + \Big| \int_0^t \Big(b(x) \mid \omega \mid^{q-1} \omega(x,s) \Big) ds \Big| \\ &\le C_1 t (2 \| \omega \|_{X_{t_0}} + \| \omega \|^p_{X_{t_0}} + \| \omega \|^q_{X_{t_0}}).\end{aligned}$$

This implies T_{ω_0} is continues at $t = 0$.

Similarly, for any $(x, t_1),\, (x, t_2) \in \overline{\Omega} \times (0, t_0]$, we have

$$\begin{aligned}&|T_{\omega_0}(\omega(x,t_1)) - T_{\omega_0}(\omega(x,t_2))| \\ &= \Big| \int_{t_1}^{t_2} \Big(\int_\Omega J(x-y)(\omega(y,s) - \omega(x,s))\mathrm{d}y \\ &\quad + a(x) \mid \omega \mid^{p-1} \omega(x,s) \\ &\quad - b(x) \mid \omega \mid^{q-1} \omega(x,s) \Big) ds \Big| \\ &\le C_2 (t_2 - t_1)(2 \| \omega \|_{X_{t_0}} + \| \omega \|^p_{X_{t_0}} \\ &\quad + \| \omega \|^q_{X_{t_0}}).\end{aligned}$$

Thus, T_{ω_0} is continues for any $t \in (0,\, t_0]$.

On the other hand, the convolution $\int_\Omega J(x-y)(u(y,t) - u(x,t))\mathrm{d}y$ is uniformly continuous. Thus, T_{ω_0} is continuous as a function of x. Therefore, we conclude that $T_{\omega_0}(\omega) \in C([0,T]; C(\overline{\Omega}))$ for any $\omega_0 \in C(\overline{\Omega})$ and $\omega \in C([0,T]; C(\overline{\Omega}))$. Moreover, we could easily prove that $\int_\Omega J(x-y)(\omega(y,t) - \omega(x,t))\mathrm{d}y + a(x) \mid \omega \mid^{p-1} \omega(x,t) - b(x) \mid \omega \mid^{q-1} \omega(x,t)$ is continues for any $t \in (0,\, t_0]$ in a similar way. That is, $T_{\omega_0} \in C^1([0,T); C(\overline{\Omega}))$ for any $\omega_0 \in C(\overline{\Omega})$ and $\omega \in C^1([0,T); C(\overline{\Omega}))$. Therefore, T_{ω_0} maps X_{t_0} into itself.

Step 2. We prove the estimate (4). For any $(x,t) \in \overline{\Omega} \times [0,\, t_0]$, we have

$$\begin{aligned}&|T_{\omega_0}(\omega(x,t)) - T_{z_0}(z(x,t))| \\ &\le \|\omega_0 - z_0\|_{L^\infty \overline{\Omega}} \\ &\quad + \Big| \int_0^t \Big(a(x) \mid \omega \mid^{p-1} \omega - a(x) \mid z \mid^{p-1} z \Big) ds \Big| \\ &\quad + \Big| \int_0^t \Big(b(x) \mid z \mid^{q-1} \omega - b(x) \mid \omega \mid^{q-1} \omega \Big) ds \Big| \\ &\quad + \Big| \int_0^t \int_\Omega J(x-y) \Big(\omega(y,s) - z(y,s) \\ &\quad - (\omega(x,s) - z(x,s)) \Big) \mathrm{d}y ds \\ &\le \|\omega_0 - z_0\|_{L^\infty(\overline{\Omega})} \\ &\quad + (C_3 p \eta^{p-1} + C_4 q \zeta^{q-1}) \Big| \int_0^t (\omega(x,s) - z(x,s)) ds \\ &\quad + \Big| 2 \int_0^t \|\omega(\cdot,s) - z(\cdot,s)\|^\infty_L(\overline{\Omega}) \int_\Omega J(x-y) \mathrm{d}y ds \\ &\le \|\omega_0 - z_0\|_{L^\infty(\overline{\Omega})} \\ &\quad + (C_3 p \eta^{p-1} + C_4 q \zeta^{q-1} + 2K|\Omega|) t \|\omega - z\|_{X_{t_0}},\end{aligned}$$

where $\eta \le \max\{\|\omega\|_{X_{t_0}},\, \|z\|_{X_{t_0}}\}$, $\zeta \le \max\{\|\omega\|_{X_{t_0}},\, \|z\|_{X_{t_0}}\}$. The arbitrariness of $(x,t) \in \overline{\Omega} \times [0,\, t_0]$ gives the desired estimate (4).

Step 3. Finally, we prove that the operator T_{ω_0} is a strict contraction if t_0 is small enough. To this end, choosing t_0 such that $Ct_0 < 1$ and $\omega_0 = z_0$, we have

$$\|T_{\omega_0}(\omega(x,t)) - T_{z_0}(z(x,t))\|_{X_{t_0}} \le Ct_0 \|\omega - z\|_{X_{t_0}}.$$

Hence it is enough to choose t_0 such that $Ct_0 < 1/2$ to obtain a strict contraction in the ball $B(u_0, 2\|u_0\|_{L^\infty(\overline{\Omega})})$. The proof of Lemma 7 is complete. □

Now we could easily prove Theorem 6 , concerning the existence and uniqueness of solutions.

Proof of Theorem 6 By Lemma 7, we have that the operator T_{ω_0} is a strict contraction in the time interval $[0,\, t_0]$, the Banach's fixed point theorem implies that problem (2) have a unique solution in the space $X_{t_0} = C([0,T]; C(\overline{\Omega}))$. If $\|u\|_{X_{t_0}} < +\infty$, taking $u(\cdot,\, t_0) \in C(\overline{\Omega})$ as the initial data in the problem (2) and arguing as before, we can extend the solution up to some interval $[0,\, t_1)$ for some $t_1 > t_0$. Therefore, we conclude that if the maximal existence time of the solution, T, is finite time then the solution blows up in $L^\infty(\overline{\Omega})$ norm.

Next, we will study the comparison principle. As usual we first give the definition of supersolution and subsolution.

Definition 8 *A function $\overline{u} \in C^1([0,T);C(\overline{\Omega}))$ is a supersolution of (2) if it satisfies*

$$\begin{aligned}\overline{u}_t(x,t) &\geq \int_\Omega J(x-y)(\overline{u}(y,t)-\overline{u}(x,t))\mathrm{d}y \\ &\quad +a(x)\overline{u}^p(x,t)-b(x)\overline{u}^q(x,t),\\ \overline{u}(x,0) &\geq u_0(x).\end{aligned}$$

Subsolutions are defined similarly by reversing the inequalities.

To obtain the comparison principle for problem (2), we first give a maximum principle.

Lemma 9 *Suppose that $w(x,t) \in C^1([0,T);C(\overline{\Omega}))$ satisfies*

$$\begin{aligned}w_t(x,t) &\geq \int_\Omega J(x,y)(w(y,t)-w(x,t))\mathrm{d}y \\ &+ \ c_1 w(x,t), \qquad x\in\Omega,\, t>0 \qquad (5)\end{aligned}$$

with $w(x,0) \geq 0$ for $x \in \overline{\Omega}$ and $suppJ(x)\bigcap suppw(x,t) \neq \emptyset$, c_1 is a bounded function, then $w(x,t)>0$, $x\in\Omega$, $t>0$.

Proof. We first show $w(x,t)\geq 0$ for $x\in\Omega$, $t>0$. Assume that $w(x,t)$ is negative somewhere. Let $\theta(x,t)=e^{-\lambda t}w(x,t)\,(\lambda>0,\,\lambda\geq 2\sup|c_1|)$. If we take (x_0,t_0) a point where θ attains its negative minimum, there holds $t_0>0$ and

$$\begin{aligned}\theta_t(x_0,t_0) &= -\lambda e^{-\lambda t_0}w(x_0,t_0)+e^{-\lambda t_0}w_t(x_0,t_0)\\ &\geq e^{-\lambda t_0}\int_\Omega J(x_0-y)(w(y,t_0)\\ &\quad -w(x_0,t_0))\mathrm{d}y+(-\lambda+c_1)w(x_0,t_0)\\ &> 0,\end{aligned}$$

which is a contradiction. Thus $\theta(x,t)\geq 0$ for $x\in\Omega$, $t>0$. And so does $w(x,t)$.

Now, we suppose $\theta(x_1,t_1)=0$ for some (x_1,t_1), that is, θ attains its minimum at (x_1,t_1) from the first step. Then $w(x_1,t_1)=0$. As $suppJ(x)\bigcap suppw(x,t)\neq\emptyset$, we have

$$\begin{aligned}\theta_t(x_1,t_1) &\geq e^{-\lambda t_1}\int_\Omega J(x_1-y)(w(y,t_1))\mathrm{d}y\\ &> 0.\end{aligned}$$

This is a contradiction. The conclusion follows. □

Remark 10 *From the proof of Lemma 9, $w(x,t)\geq 0$ for $x\in\Omega$, $t>0$ is also validity when the condition $suppJ(x)\bigcap suppw(x,t)\neq\emptyset$ fails.*

Lemma 11 *If $p\geq 1$, $q\geq 1$ and $\overline{u}$, $\underline{u}$ be super and subsolutions to (2), respectively. Then $\overline{u}(x,t)\geq \underline{u}(x,t)$ for every $(x,t)\in\overline{\Omega}\times[0,T)$.*

Proof. Let $w(x,t)=\overline{u}-\underline{u}$, it is easy to verify that $w(x,t)$ satisfies (5) when $p\geq 1, q\geq 1$. We could obtain our conclusion from Lemma 9. □

Remark 12 *When $p<1$ or $q<1$ the conclusion is also validity if $\overline{u}$ and $\underline{u}$ are bounded away from 0.*

3 Blowup and Global Existence

In this section, we will analyze the blowup condition and give the proof of Theorem 1. For convenience of writing, we introduce the following notation. Let $A_2=\max_{x\in\overline{\Omega}}a(x)$, $B_2=\max_{x\in\overline{\Omega}}b(x)$. We will use this notation in the rest of this paper.

Proof of Theorem 1 (i). We first show that if the initial data $u_0(x)$ is large enough, solutions of (2) blow up in finite time.

In the case of $p>q>1$. Integrating equation $(2)_1$ in Ω and applying Fubini's theorem, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x,t)\mathrm{d}x &= \int_\Omega a(x)u^p(x,t)\mathrm{d}x\\ &\quad -\int_\Omega b(x)u^q(x,t)\mathrm{d}x \quad (6)\end{aligned}$$

Using Hölder's inequality and noting the bound of the potential, we could get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x,t)\mathrm{d}x\\ &\geq A_1\int_\Omega u^p(x,t)\mathrm{d}x-B_2|\Omega|^{\frac{p-q}{p}}\left(\int_\Omega u^p(x,t)\mathrm{d}x\right)^{\frac{q}{p}}\\ &=\left(\int_\Omega u^p(x,t)\mathrm{d}x\right)^{\frac{q}{p}}\Big[A_1\left(\int_\Omega u^p(x,t)\mathrm{d}x\right)^{\frac{p-q}{p}}\\ &\quad -B_2|\Omega|^{\frac{p-q}{p}}\Big],\end{aligned}$$

where $|\Omega|$ is assumed to be the measure of Ω. Given positive constant $m>(\frac{B_2}{A_1})^{\frac{1}{p-q}}$ and $u_0\geq m$, we have by the comparison principle that the solution $u(x,t)$ of problem (2) satisfies $u(x,t)\geq m$. Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x,t)\mathrm{d}x &\geq \left(\int_\Omega u^p(x,t)\mathrm{d}x\right)^{\frac{q}{p}}\cdot\\ &\quad \Big(A_1 m^{p-q}|\Omega|^{\frac{p-q}{p}}-B_2|\Omega|^{\frac{p-q}{p}}\Big).\end{aligned}$$

Then we use Jensen's inequality to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x,t)\mathrm{d}x > C\left(\int_\Omega u(x,t)\mathrm{d}x\right)^q, \qquad (7)$$

where C is a positive constant independent of the solution u. From this inequality, we could easily obtain that $u(x,t)$ blow up in finite time.

In the case of $p > 1 \geq q$, it follows from $u^q \leq u+1$ and Jensen's inequality that

$$\begin{aligned}
&\int_\Omega a(x)u^p(x,t)\mathrm{d}x - b(x)\int_\Omega u^q(x,t)\mathrm{d}x \\
&\geq \int_\Omega A_1 u^p(x,t)\mathrm{d}x - B_2\int_\Omega u^q(x,t)\mathrm{d}x \\
&\geq A_1\int_\Omega u^p(x,t)\mathrm{d}x - B_2\int_\Omega u(x,t)\mathrm{d}x - B_2|\Omega| \\
&\geq A_1|\Omega|^{1-p}\left(\int_\Omega u(x,t)\mathrm{d}x\right)^p - B_2\int_\Omega u(x,t)\mathrm{d}x \\
&\quad -B_2|\Omega|.
\end{aligned}$$

Substituting this inequality into (6), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x,t)\mathrm{d}x \\
&\geq A_1|\Omega|^{1-p}\left(\int_\Omega u(x,t)\mathrm{d}x\right)^p - B_2\int_\Omega u(x,t)\mathrm{d}x \\
&\quad -B_2|\Omega|.
\end{aligned}$$

Therefore, if we take the initial data u_0 large enough such that $A_1|\Omega|^{1-p}\left(\int_\Omega u_0(x)\mathrm{d}x\right)^p - B_2\int_\Omega u_0(x)\mathrm{d}x - B_2|\Omega| > 0$, then $\int_\Omega u(x,t)\mathrm{d}x$ blows up in finite time. So does $u(x,t)$.

Next we show when the initial data $u_0(x)$ is small, solutions of (2) exist globally.

Consider constant M. Let $0 < M \leq (\min_{x\in\overline{\Omega}} \frac{b(x)}{a(x)})^{\frac{1}{p-q}}$. Then $M_t \geq a(x)M^p - b(x)M^q$. Henceforth, if $u_0(x) \leq M$, M is a supersolution of equation (2). From the comparison principle, we know solutions of (2) are global in this case.

(ii). We only need to look for a global supersolution of equations (2). Indeed, it is easy to construct spacial homogeneous global supersolution of (2). To see this, we let $\overline{u} = Ce^{\alpha t}$, where C and α are positive constants to be determined.

For any given initial data u_0, we note that $\overline{u}(t_0) \geq \|u_0\|_\infty$ for t_0 sufficiently large and $\overline{u}$ is bounded away from 0. Thus by the comparison principle and Remark 12, to make $\overline{u}$ be a supersolution of (2) we only need to show the existence of C and α satisfying

$$a(x)C^p e^{p\alpha t} \leq b(x)C^q e^{q\alpha t} + \alpha C e^{\alpha t}. \qquad (8)$$

If $p \leq 1 < q$, for any given α, we can take $C = (A_2^{-1}B_1)^{\frac{1}{p-q}}$ such that (8) holds.

If $q \leq 1$ and thus $p \leq 1$, we can choose C and α satisfying $\alpha = A_2 C^{p-1}$, which make (8) validity.

Next, we show all global solutions are uniformly bounded when $p < q$ or $p = q$ and $\max_{x\in\overline{\Omega}}(a(x) - b(x)) \leq 0$. In fact, (2) has constant supersolution $\overline{u} = A$ whenever $p < q$ or $p = q$ and $\max_{x\in\overline{\Omega}}(a(x) - b(x)) \leq 0$. To see this, we choose A large enough such that

$$b(x)A^q \geq a(x)A^p, \quad A \geq \|u_0\|_\infty,$$

which imply that $\overline{u}$ is a supersolution of (2).

At last we show there exist global unbounded solutions when $p = q$ and $\min_{x\in\overline{\Omega}}(a(x) - b(x)) > 0$.

Define function $f(t)$ as follows.

If $p = q < 1$, $f(t) = ((A_1 - B_2)(1-p)t + f^{1-p}(0))^{\frac{1}{1-p}}$.

If $p = q = 1$, $f(t) = e^{(A_1 - B_2)t}$.

It is easy to see that if $f(0) \leq \max_\Omega u_0(x)$, $f(t)$ is a subsolution of equations (2). It is obvious that when $p < 1$, $f(t)$ is unbounded. □

4 Blowup Rate Estimate

In this section, we study the blowup rate and prove Theorem 2.

Proof of Theorem 2. Let $U(t) = u((x(t),t) = \max_{x\in\overline{\Omega}} u(x,t)$. It is easy to see that $U(t)$ is Lipschitz continuous and thus it is differential everywhere (see [6]). From the first equality of (2) we have

$$\begin{aligned}
&U'(t) \\
&\leq \int_\Omega J(x(t),y)(u(y,t) - u(x(t),t))dy \\
&\quad +a(x(t))u^p(x(t),t) - b(x(t))u^q(x(t),t) \\
&\leq A_2 u^p(x(t),t) \qquad (9)
\end{aligned}$$

at any point of differentiability of $U(t)$. Here we used $\nabla u((x(t),t) = 0$.

Noticing that $p > 1$ and integrating (9) from t to T, we obtain

$$\max_{x\in\overline{\Omega}} u(x,t) \geq (A_2(p-1))^{-\frac{1}{p-1}}(T-t)^{-\frac{1}{p-1}}. \quad (10)$$

Next we will establish the upper estimate. For any $(x,t) \in \overline{\Omega}\times[0,T)$, we have

$$\begin{aligned}
u_t(x,t) &\geq -u(x,t) + a(x)u^p(x,t) - b(x)u^q(x,t) \\
&= u^p(x,t)\Big(a(x) - u^{-(p-1)}(x,t) \\
&\qquad -b(x)u^{-(p-q)}(x,t)\Big).
\end{aligned}$$

In particular,

$$U'(t) \geq U^p(t)\left(A_2 - U(t)^{-(p-1)} - B_2 U(t)^{-(p-q)}\right).$$

From the lower estimate (10) we get

$$\begin{aligned}&U'(t)\\&\geq U^p(t)\Big(A_2 - A_2(p-1)(T-t)\\&\quad -B_2(p-1)^{\frac{p-q}{p-1}}(T-t)^{\frac{p-q}{p-1}}\Big).\end{aligned}$$

Integrating in (t,T), we get

$$\begin{aligned}&\max_{x\in\overline{\Omega}} u(x,t)\\&\leq \Big(A_2(p-1)(T-t) - \frac{A_2(p-1)^2}{2}(T-t)^2\\&\quad -B_2\frac{(p-1)^{\frac{3p-q-2}{p-1}}}{2p-q-1}(T-t)^{\frac{2p-q-1}{p-1}}\Big)^{-\frac{1}{p-1}},\end{aligned}$$

as $u(x,t)$ could not attain its maximum on the boundary of Ω from Lemma 7 . Combining with (10), the conclusion of Theorem 2 is proved if one take the limit as $t \to T$. □

5 Blowup Set

Next we will concern the blowup set for the solution to problem (2). We will first localize the blowup set near any point in $\overline{\Omega}$ just by taking an initial condition being very large near that point and not so large in the rest of the domain.

Proof of Theorem 4. Given $x_0 \in \overline{\Omega}$ and $\varepsilon > 0$, we could construct an initial condition u_0 such that

$$B(u) \subset B_\varepsilon(x_0) = \{x \in \overline{\Omega} : \|x - x_0\| < \varepsilon\}. \quad (11)$$

In fact, we will consider u_0 concentrated near x_0 and small away from x_0.

Let φ be a nonnegative smooth function such that $\operatorname{supp}(\varphi) \subset B_{\frac{\varepsilon}{2}}(x_0)$ and $\varphi(x) > 0$ for $x \in B_{\frac{\varepsilon}{2}}(x_0)$.

Next, let

$$u_0(x) = M\varphi(x) + \delta.$$

We want to choose M large and δ small such that (11) holds.

First we can assume that T is as small as we need by taking M large enough. In fact, we have

$$T \leq \frac{C(\Omega, p, A_1, B_2, \varphi)}{M^{q-1}} \quad \text{or} \quad T \leq \frac{C(\Omega, p, A_1, B_2, \varphi)}{M^{p-1}}$$

from the proof of Theorem 1.

Now, from the proof of blowup rate, we have

$$\begin{aligned}&\max_{x\in\overline{\Omega}} u(x,t)\\&\leq \Big(A_2(p-1)(T-t) - \frac{A_2(p-1)^2}{2}(T-t)^2\\&\quad -B_2\frac{(p-1)^{\frac{3p-q-2}{p-1}}}{2p-q-1}(T-t)^{\frac{2p-q-1}{p-1}}\Big)^{-\frac{1}{p-1}}\\&\leq C(T-t)^{-\frac{1}{p-1}}.\end{aligned}$$

Henceforth, for any $\overline{x} \in \Omega$,

$$\begin{aligned}u_t(\overline{x},t) &= \int_\Omega J(x,y)(u(y,t) - u(\overline{x},t))dy\\&\quad +a(x)u^p(\overline{x},t) - b(x)u^q(\overline{x},t)\\&\leq \int_\Omega J(\overline{x},y)u(y,t)dy + A_2u^p(\overline{x},t)\\&\leq C(T-t)^{-\frac{1}{p-1}} + A_2u^p(\overline{x},t),\end{aligned}$$

which show that $u(\overline{x},t)$ is a subsolution to

$$w_t = C(T-t)^{-\frac{1}{p-1}} + A_2w^p(t). \quad (12)$$

And then, if $u(\overline{x},0) \leq w(0)$, we have

$$u(\overline{x},t) \leq w(t). \quad (13)$$

Next, we only need to prove that a solution w to (12) with initial value $w(0) = \delta$ remains bounded up to $t = T$, provided that δ and T are small enough.

Let

$$z(s) = (T-t)^{1/(p-1)}w(t), \quad s = -ln(T-t).$$

Then $z(s)$ satisfies

$$z'(s) = Ce^{-s} - \frac{1}{p-1}z(s) + A_2z^p(s),$$

$$z(-lnT) = T^{1/(p-1)}\delta.$$

On the other hand, $p > 2$ shows that for T and δ small (T is small if M is large), we have

$$CT - \frac{1}{p-1}\delta T^{\frac{1}{p-1}} + A_2\delta^pT^{\frac{p}{p-1}} < 0.$$

So $z'(s) < 0$ for all $s > -lnT$. From this and Lemma 4.2 of [7], we know

$$z(s) \to 0, \quad s \to \infty.$$

Combining the equation verified by z we obtain that for given positive constant $\gamma\,(< \frac{1}{p(p-1)})$, there exists $s_0 > 0$ such that

$$z'(s) \leq Ce^{-s} - \left(\frac{1}{p-1} - \gamma\right)z(s)$$

for $s > s_0$.

Let $v(s)$ be a solution of

$$v'(s) = Ce^{-s} - \left(\frac{1}{p-1} - \gamma\right) v(s)$$

with $v(s_0) \geq z_(s_0)$. Integrating this equation we get

$$v(s) = C_1 e^{-s} + C_2 e^{-(\frac{1}{p-1}-\gamma)s}.$$

By a comparison argument we could get that for every $s > s_0$,

$$z(s) \leq v(s) = C_1 e^{-s} + C_2 e^{-(\frac{1}{p-1}-\gamma)s}. \qquad (14)$$

Now we go back to $z'(s) = Ce^{-s} - \frac{1}{p-1} z(s) + z^p(s)$. We have

$$z'(s) + \frac{1}{p-1} z(s) = Ce^{-s} + z^p(s),$$

then

$$\left(e^{\frac{1}{p-1}s} z(s)\right)' = e^{\frac{1}{p-1}s}(Ce^{-s} + z^p).$$

Integrating form s_0 to s, one could get

$$\begin{aligned} & z(s) \\ = \;& e^{-\frac{1}{p-1}s}\left(C_1 + \int_{s_0}^{s} e^{\frac{1}{p-1}\sigma}(Ce^{-\sigma} + z^p) d\sigma\right) \\ = \;& e^{-\frac{1}{p-1}s} \cdot \\ & \left(C_1 + \int_{s_0}^{s} e^{-\frac{p-2}{p-1}\sigma}(C + e^{\sigma} z^p) d\sigma\right). \qquad (15) \end{aligned}$$

Using (14) and $\gamma < \frac{1}{p(p-1)}$, we have

$$e^s z^p \leq C_1^p e^{-(p-1)s} + C_2^p e^{-(\frac{p}{p-1} - p\gamma - 1)s} \to 0$$

as $s \to +\infty$.

And thus from (15), we get

$$\begin{aligned} z(s) &\leq e^{-\frac{1}{p-1}s}\left(C_1 + C_3 \int_{s_0}^{s} e^{-\frac{p-2}{p-1}\sigma} d\sigma\right) \\ &\leq C_1 e^{-\frac{1}{p-1}s} + C_4 e^{-s}. \end{aligned}$$

As $p > 2$, we have

$$z(s) \leq Ce^{-\frac{1}{p-1}} s.$$

This implies that $w(t) \leq C$, for $0 \leq t < T$. From the boundedness of w and (13) we get $u(\overline{x}, t) \leq w(t) \leq C$ for every $\overline{x} \in \overline{\Omega} \setminus B_\epsilon(x_0)$, as we wished.

Next, we will consider the radial symmetric case, that is, the proof of Theorem 5. For the convenience of writing, we only deal with the one dimensional case, $\Omega = (-l, l)$. The radial case is analogous, we leave the details to the reader.

First, we prove a lemma that show if the initial data has a unique maximum at the origin and $a'(x) \leq 0$, $b'(x) \geq 0$, then the solution has a unique maximum at this point for every $t \in (0, T)$.

Lemma 13 *Under the hypothesis of Theorem 5 we have that the solution $u(x,t)$ is symmetric and such that $u_x < 0$ in $(0, L] \times (0, T)$.*

Proof. Symmetry follows from uniqueness since $h(x,t) = u(-x,t)$ is also a solution to (2).

Denote $w(x,t) = u_x$. Then w satisfies the following equation

$$\begin{aligned} & w_t(x,t) \\ & = \int_{-L}^{L} J'(x-y)(u(y,t) - u(x,t)dy) \\ & -w(x,t) \int_{-L}^{L} J(x-y)dy + pa(x)u^{p-1}(x,t)w(x,t) \\ & -qb(x)u^{q-1}w(x,t) + a'(x)u^p(x,t) - b'(x)u^q(x,t). \end{aligned}$$

If we assume that there exists a point $(x_0, t_0) \in (0, L] \times (0, T)$ at which $w(x_0, t_0)$, we get

$$\begin{aligned} & w_t(x_0, t) \\ & = \int_{-L}^{L} J'(x_0 - y)u(y,t)dy + a'(x_0)u^p(x_0, t) \\ & -b'(x_0)u^q(x_0, t). \end{aligned}$$

Here we used that J' is odd and the symmetry of u.

From this equation it is easy to obtain a contradiction as $a'(x) \leq 0$ and $b'(x) \geq 0$.

Now we are ready to prove Theorem 5.

Proof of Theorem 5. Let us perform the following change of variables

$$\begin{aligned} & z(x,s) = (T-t)^{1/(p-1)} u(x,t), \\ & s = -\ln(T-t). \end{aligned} \qquad (16)$$

Our remainder proof consist of two steps.

Step 1. We first prove the only blowup point that verifies the blowup estimate (3) is $x = 0$. And this shows that for $x \neq 0$, $z(x,s)$ does not converge to $C_p = (A_2(p-1))^{-\frac{1}{p-1}}$ as $s \to +\infty$.

We conclude by contradiction. Assume that $(T - t)^{1/(p-1)} u(x_0, t) \to C_p$ for a $x_0 > 0$.

Let $v(t) = u(0,t) - u(x_0, t)$. Then

$$\begin{aligned} v'(t) &= \int_{-l}^{l} J(-y)(u(y,t) - u(0,t))dy \\ & \quad - \int_{-l}^{l} J(x_0 - y)(u(y,t) - u(x_0,t))dy \\ & \quad + a(0)u^p(0,t) - a(x_0)u^p(x_0,t) \\ & \quad - b(0)u^q(0,t) + b(x_0)u^q(x_0,t) \\ &\geq \int_{-l}^{l} J(-y)(u(y,t) - u(0,t))dy \\ & \quad - \int_{-l}^{l} J(x_0 - y)(u(y,t) - u(x_0,t))dy \\ & \quad + pa(0)\xi^{p-1}(t)v(t) - qb(x_0)\eta^{q-1}(t)v(t), \end{aligned}$$

where $\xi(t)$ and $\eta(t)$ are between $u(0,t)$ and $u(x_0,t)$. Hence

$$\begin{aligned}
v'(t) &\geq \int_{-l}^{l}(J(-y)-J(x_0-y))u(y,t)dy \\
&\quad +\int_{-l}^{l}(J(y-x_0)-J(y))u(0,t)dy \\
&\quad -v(t)+pa(0)\xi^{p-1}(t)v(t) \\
&\quad -qb(x_0)\eta^{q-1}(t)v(t) \\
&= \int_{-l}^{l}(J(y-x_0)-J(y))(u(0,t)-u(y,t))dy \\
&\quad -v(t)+pa(0)\xi^{p-1}(t)v(t) \\
&\quad -qb(x_0)\eta^{q-1}(t)v(t) \\
&\geq (-C_1+pa(0)\xi^{p-1}(t)-qb(x_0)\eta^{q-1}(t))v(t),
\end{aligned}$$

for some positive constant.

Integrating the above inequality, we obtain

$$\begin{aligned}
&\ln(v)(t)-\ln(v)(t_0) \\
&\geq \int_{t_0}^{t}(-C_1+A_2p\xi^{p-1}(s)-B_2q\eta^{q-1}(s))ds.
\end{aligned}$$

Remember that $(T-t)^{1/(p-1)}u(x_0,t)\to C_p$, $(T-t)^{1/(p-1)}u(0,t)\to C_p$, we have

$$\lim_{t\to T}\xi(t)(T-t)^{\frac{1}{p-1}}=\lim_{t\to T}\eta(t)(T-t)^{\frac{1}{p-1}}=C_p.$$

And this implies that

$$\begin{aligned}
&\int_{t_0}^{t}(-C_1+A_2p\xi^{p-1}(s)-B_2q\eta^{q-1}(s))ds \\
&\geq A_2p\int_{t_0}^{t}\frac{C_p^{p-1}-\delta_1}{T-s}ds \\
&\quad -B_2q\int_{t_0}^{t}(C_p^{q-1}+\delta_2)(T-s)^{-\frac{q-1}{p-1}}ds-C_2.
\end{aligned}$$

$p>q$ implies that such that $(T-s)^{-\frac{q-1}{p-1}}\leq\delta_3(T-s)^{-1}$ as $s\to T$ for given $\delta_3>0$. Hence

$$\begin{aligned}
&\int_{t_0}^{t}(-C_1+A_2p\xi^{p-1}(s)-B_2q\eta^{q-1}(s))ds \\
&\geq A_2p\int_{t_0}^{t}\frac{C_p^{p-1}-\delta}{T-s}ds-C_2 \\
&=-A_2p(C_p^{p-1}-\delta)ln(T-t)-C_2
\end{aligned}$$

for some $\delta>0$.

Hence

$$v(t)\geq C(T-t)^{-pA_2(C_p^{p-1}-\delta)}=C(T-t)^{A_2p\delta-\frac{p}{p-1}}.$$

Using this fact, we have

$$\begin{aligned}
0 &= \lim_{t\to T}(T-t)^{1/(p-1)}v(t) \\
&\geq C\lim_{t\to T}(T-t)^{\frac{1}{p-1}-\frac{p}{p-1}+A_2p\delta} \\
&= +\infty.
\end{aligned}$$

This contradiction proves our claim.

Step 2. We will show the only possible blowup point is $x=0$.

Remembering the transform (16), $z(x,s)$ satisfies

$$\begin{aligned}
z_s &= e^{-s}\int_{-l}^{l}J(x-y)(z(y,s)-z(x,s))dy \\
&\quad -\frac{1}{p-1}z+a(x)z^p-b(x)e^{\frac{q-p}{p-1}s}z^q.
\end{aligned}$$

Note that the blowup rate of u implies that $z(x,s)\leq C$ for every $(x,s)\in[-l,l]\times(-lnT,\infty)$. Therefore,

$$z_s(x,s)\leq Ce^{-s}-\frac{1}{p-1}z(x,s)+A_2z^p(x,s). \quad (17)$$

From this we know that if there exists s_0 such that $A_2z^p(x,s_0)-\frac{1}{p-1}z(x,s_0)<-Ce^{-s_0}$ then $z(x,s)\to 0$ as $s\to\infty$ (see Lemma 4.2 in [7]).

Moreover, if there exists s_0 such that $A_2z^p(x,s_0)-\frac{1}{p-1}z(x,s_0)>Ce^{-s_0}$ then $z(x,s)$ blows up in finite time $\overline{s}$. This follows from Lemma 4.3 of [7]) using that

$$z_s(x,s)\geq -Ce^{-s}-\frac{1}{p-1}z(x,s)+A_2z^p(x,s).$$

Thus if $z(x,s)$ does not converge to zero and does not blow up in finite time, then $z(x,s)$ satisfies

$$Ce^{-s}\geq A_2z^p(x,s)-\frac{1}{p-1}z(x,s)\geq -Ce^{-s}.$$

Henceforth,

$$z^p(x,s)-\frac{1}{p-1}z(x,s)\to 0 \quad (s\to+\infty).$$

As $z(x,s)$ is continuous , bounded and does not go to zero, we conclude that $z(x,s)\to C_p$.

Now we could conclude that $z(x,s)$ verifies $z(x,s)\to 0(s\to+\infty)$, or $z(x,s)\to C_p(s\to+\infty)$, or $z(x,s)$ blows up in finite time.

From step 1 we know for $x\neq 0$, $z(x,s)$ is bounded and does not converge to C_p, so $z(x,s)\to 0$ as $s\to+\infty$. Combined with inequality (17), we could get

$$z_s(x,s)\leq Ce^{-s}-\left(\frac{1}{p-1}-\theta\right)z(x,s)$$

for any $\theta > 0$.

By a comparison argument as in the proof of Theorem 4, it follows that

$$z(x,s) \le C_1 e^{-s} + C_2 e^{-(\frac{1}{p-1}-\theta)s}. \tag{18}$$

Going back to the equation verified by $z(x,t)$ we obtain

$$\begin{aligned}&\left(e^{\frac{1}{p-1}s} z(x,s)\right)_s \\ &\le e^{\frac{1}{p-1}s}\Big(e^{-s}\int_{-l}^{l} J(x-y)(z(y,s)-z(x,s))dy \\ &\quad + A_2 z^p(x,s) - B_1 e^{\frac{q-p}{p-1}s} z^q(x,s)\Big).\end{aligned}$$

Integrating we get

$$\begin{aligned}&z(x,s) \\ &\le e^{-\frac{1}{p-1}s}\Big(C_1 + \int_{s_0}^{s} e^{-\frac{p-2}{p-1}\sigma}\big(\int_{-l}^{l} J(x-y)(z(y,s) \\ &\quad -z(x,s))dy + A_2 e^{\sigma} z^p(x,s) \\ &\quad -B_1 e^{\frac{q-1}{p-1}\sigma} z(x,s)d\sigma\big)\Big).\end{aligned}$$

On the other hand, (18) implies that $e^s z^p(x,s) \to 0$ as $s \to \infty$. Henceforth,

$$z(x,s) \le e^{-\frac{1}{p-1}s}\left(C_1 + C_2\int_{s_0}^{s} e^{-\frac{p-2}{p-1}\sigma} d\sigma\right).$$

Using that $p > 2$, one could have

$$z(x,s) \le C_3 e^{-\frac{1}{p-1}s}.$$

Remembering the transform (16), we have

$$u(x,t) = e^{\frac{1}{p-1}s} z(x,s) \le c_3.$$

And so our proof is complete. $\square$

6 Numerical Experiments

At the end of this paper, we will use several numerical examples to demonstrate our results about the location of blowup points. For this purpose, we discretize the problem in the spacial variable to obtain an ODE system. For simplicity, we only consider one classical case in which $a(x) = 1$, $b(x) = 1$. Taking $\Omega = [-4,4]$ and $-4 = x_{-N} < \cdots < x_N = 4$, $N = 100$, we consider the following system

$$\begin{aligned} u_i'(t) &= \sum_{j=-N}^{N} J(x_i - x_j)(u_j(t) - u_i(t)) \\ &\quad + (u_i)^p(t) - k(u_i)^q(t), \\ u_i(0) &= u_0(x_i). \end{aligned}$$

Next we choose $p = 3$, $q = 1$, $k = 1$ and

$$J(z) = \begin{cases} 1, & , \quad |z| \le 1/10, \\ 0, & , \quad |z| > 1/10. \end{cases}$$

In Fig.1 we choose a non-symmetric initial condition very large near the point $x_0 = 1$, $u_0(x) = 1/4 + 100(1 - |x-1|)_+$. We observe that the blowup set is localized in a neighborhood of $x_0 = 1$.

Next we choose a symmetric initial condition with a unique maximum at the origin, $u_0(x) = 16 - x_0^2$. We observe that the solution blows up only at the origin, Fig.2.

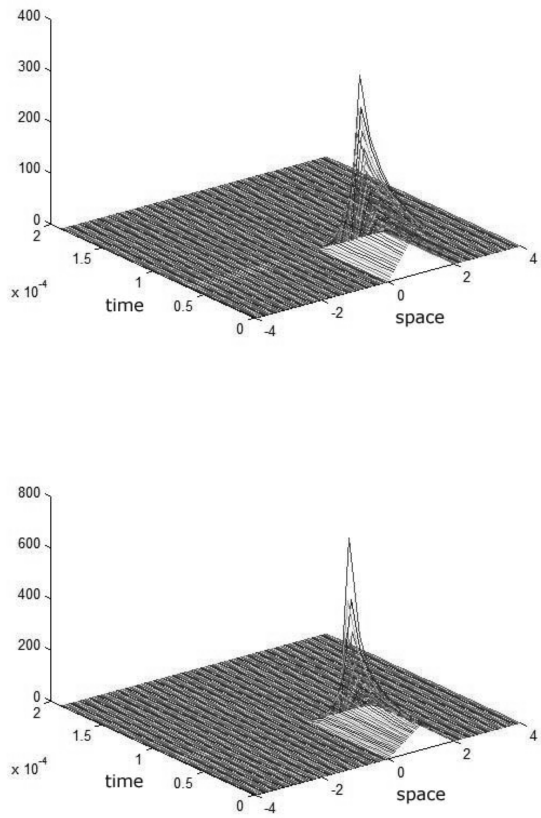
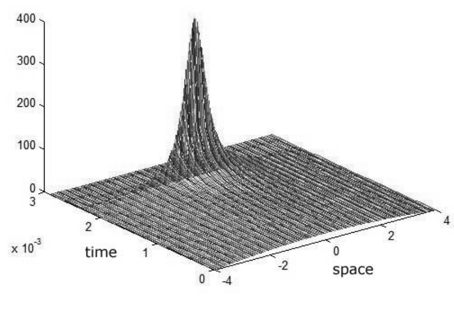


Fig.1 Evolution in time, non-symmetric datum.



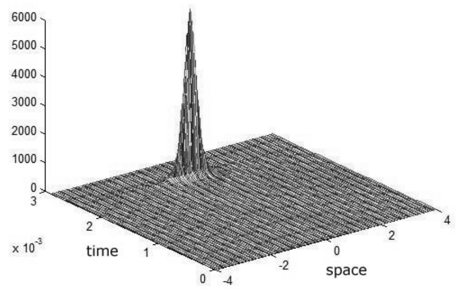


Fig.2 Evolution in time, symmetric datum.

7 Conclusion

This paper study the blowup properties of a nonlocal diffusion equation with reaction and absorption term. We established the complete classification of the global existence or finite time blowup of the solution. Moreover, we obtained the precise blowup rate for the blowup rate. Meanwhile, the localization of the blowup set for usual domain Ω or the radial symmetric domain, respectively. The results illustrated in the previous sections show that this equation shares many shares many blowup properties with corresponding local diffusion equation, such as the blowup classification. However, they have some difference, such as the blowup rate and blowup set.

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