Slowly Changing Function Oriented Growth Analysis of Differential Monomials and Differential Polynomials

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Abstract: In this paper, we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using L^*-order and L^*-type and differential monomials, differential polynomials generated by one of the factors.

Key–Words: Meromorphic function, transcendental entire function, composition, growth, L^*-order, L^*-type, differential monomials, differential polynomial, slowly changing function.

1 Introduction Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [5] and [17]. In the sequel we use the following notation :

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \forall k = 1, 2, 3, \cdots$$

and $\log^{[0]} x = x$.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Singh and Barker[13] defined it in the following way:

Definition 1 *[13] A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon > 0$,*

$$\frac{1}{k^{\varepsilon}} \le \frac{L(kr)}{L(r)} \le k^{\varepsilon}, \text{ for } r \ge r(\varepsilon)$$

and uniformly for $k \ge 1$.

If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0 .$$

Somasundaram and Thamizharasi [14] introduced the notions of L-order and L-order for entire functions. The more generalized concept for L-order and L-type for entire and meromorphic functions are L^*-order and L^*-type respectively. Their definitions are as follows:

Definition 2 *[14] The L^*-order $\rho_f^{L^*}$ and the L^* -lower order$\lambda_f^{L^*}$ of an entire function f are defined as*

$$\rho_f^{L^*} = \lim\sup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[re^{L(r)}]}$$

and

$$\lambda_f^{L^*} = \lim\inf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log\left[re^{L(r)}\right]},$$

When f is meromorphic, one can easily verify that

$$\rho_f^{L^*} = \lim\sup_{r\to\infty} \frac{\log T(r, f)}{\log\left[re^{L(r)}\right]}$$

and

$$\lambda_f^{L^*} = \lim\inf_{r\to\infty} \frac{\log T(r, f)}{\log\left[re^{L(r)}\right]}.$$

Definition 3 *[14] The L^*-type $\sigma_f^{L^*}$ of an entire function f is defined as follows:*

$$\sigma_f^{L^*} = \lim\sup_{r\to\infty} \frac{\log M(r, f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} , \ 0 < \rho_f^{L^*} < \infty .$$

For meromorphic f,

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}\ ,\ 0 < \rho_f^{L^*} < \infty\ .$$

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. Also let $n_{0j}, n_{1j}, ...n_{kj} (k \geq 1)$ be non-negative integers such that for each j, $\sum_{i=0}^{k} n_{ij} \geq 1$. We call

$$M_j[f] = A_j\,(f)^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} ... \left(f^{(k)}\right)^{n_{kj}}$$

where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f. The numbers $\gamma_{M_j} = \sum_{i=0}^{k} n_{ij}$ and $\Gamma_{M_j} = \sum_{i=0}^{k} (i+1) n_{ij}$ are called respectively the degree and weight of $M_j[f]$ ([4],[12]). The expression $P[f] = \sum_{i=0}^{s} M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P = \max_{1\leq j\leq s} \gamma_{M_j}$ and $\Gamma_P = \max_{1\leq j\leq s} \Gamma_{Mj}$ are called respectively the degree and weight of $P[f]$([4],[12]). Also we call the numbers $\underline{\gamma_P} = \min_{1\leq j\leq s} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma_p} = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f i.e., for which $n_{0j} = 0$ for $j = 1, 2, ...s$. We consider only those $P[f]$, $P_0[f]$ singularities of whose individual terms do not cancel each other. We also denote by $M[f]$ a differential monomial generated by a transcendental meromorphic function f.

In the sequel the following definitions are also well known :

Definition 4 *Let 'a' be a complex number, finite or infinite. The Nevanlinna deficiency and the Valiron deficiency of 'a' with respect to a meromorphic function f are defined as*

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

and

$$\Delta(a;f) = 1 - \liminf_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \limsup_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}.$$

Definition 5 *The quantity $\Theta(a; f)$ of a meromorphic function f is defined as follows*

$$\Theta(a;f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

Definition 6 *[16] For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $n(r, a; f \mid= 1)$, the number of simple zeros of $f - a$ in $|z| \leq r$. $N(r, a; f \mid= 1)$ is defined in terms of $n(r, a; f \mid= 1)$ in the usual way. We put*

$$\delta_1(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f\mid=1)}{T(r,f)}\ ,$$

the deficiency of 'a' corresponding to the simple a-points of f i.e., simple zeros of $f - a$.

Yang [15] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta_1(a; f) > 0$ and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) \leq 4$.

Definition 7 *[9] For $a \in \mathbb{C} \cup \{\infty\}$, let $n_p(r, a; f)$ denotes the number of zeros of $f - a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly p times; and $N_p(r, a; f)$ is defined in terms of $n_p(r, a; f)$ in the usual way. We define*

$$\delta_p(a;f) = 1 - \limsup_{r\to\infty} \frac{N_p(r,a;f)}{T(r,f)}.$$

Definition 8 *[3] $P[f]$ is said to be admissible if*

(i) $P[f]$ is homogeneous, or

(ii) $P[f]$ is non homogeneous and $m(r, f) = S(r, f)$.

Lakshminarasimhan [6] introduced the idea of the functions of L-bounded index. Later Lahiri and Bhattacharjee [8] worked on the entire functions of L-bounded index and of non uniform L-bounded index. In the paper we investigate the comparative growth of composite entire and meromorphic functions and differential monomials, differential polynomials generated by one of their factors using L^*-order and L^*-type.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 9 *[1] If f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

Lemma 10 *[11] Let f and g be any two entire functions. Then for all $r > 0$,*

$$T(r, f \circ g) \geq \frac{1}{3} \log M \left\{ \frac{1}{8} M\left(\frac{r}{4}, g\right) + o(1), f \right\}.$$

Lemma 11 *[7] Let g be an entire function with $\lambda_g < \infty$ and $a_i (i = 1, 2, 3, \cdots, n;\ n \leq \infty)$ are entire functions satisfying $T(r, a_i) = o\{T(r, g)\}$. If $\sum_{i=1}^{n} \delta(a_i, g) = 1$ then $\lim_{r\to\infty} \frac{T(r,g)}{\log M(r,g)} = \frac{1}{\pi}$.*

Lemma 12 *[3] Let $P_0[f]$ be admissible.If f is of finite order or of non zero lower order and $\sum_{a\neq\infty} \Theta(a; f) = 2$ then*

$$\lim_{r\to\infty} \frac{T(r, P_0[f])}{T(r, f)} = \Gamma_{P_0[f]}.$$

Lemma 13 *[3] Let f be either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$. Then for homogeneous $P_0[f]$,*

$$\lim_{r\to\infty} \frac{T(r, P_0[f])}{T(r, f)} = \gamma_{P_0[f]}.$$

Lemma 14 *Let f be a meromorphic function of finite order or of non zero lower order. If $\sum_{a\neq\infty} \Theta(a; f) = 2$, then the L^*-order (L^*-lower order) of admissible $P_0[f]$ is same as that of f. Also the L^*-type of $P_0[f]$ is $\Gamma_{P_0[f]}$ times that of f when f is of finite positive L^*-order.*

Proof. By Lemma 12, $\lim_{r\to\infty} \frac{\log T(r,P_0[f])}{\log T(r,f)}$ exists and is equal to 1.

$$\begin{aligned}
\rho^{L^*}_{P_0[f]} &= \limsup_{r\to\infty} \frac{\log T(r, P_0[f])}{\log \left[re^{L(r)}\right]} \\
&= \limsup_{r\to\infty} \frac{\log T(r, f)}{\log \left[re^{L(r)}\right]} . \lim_{r\to\infty} \frac{\log T(r, P_0[f])}{\log T(r, f)} \\
&= \rho^{L^*}_f .1 \\
&= \rho^{L^*}_f .
\end{aligned}$$

In a similar manner, $\lambda^{L^*}_{P_0[f]} = \lambda^{L^*}_f$.

Again by Lemma 12,

$$\begin{aligned}
\sigma^{L^*}_{P_0[f]} &= \limsup_{r\to\infty} \frac{T(r, P_0[f])}{\left[re^{L(r)}\right]^{\rho^{L^*}_{P_0[f]}}} \\
&= \lim_{r\to\infty} \frac{T(r, P_0[f])}{T(r, f)} . \limsup_{r\to\infty} \frac{T(r, f)}{\left[re^{L(r)}\right]^{\rho^{L^*}_f}} \\
&= \Gamma_{P_0[f]} . \sigma^{L^*}_f .
\end{aligned}$$

This proves the lemma. □

Lemma 15 *Let f be a meromorphic function of finite order or of non zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$. Then the L^*-order (L^*-lower order) of homogeneous $P_0[f]$ and f are same. Also the L^*-type of $P_0[f]$ is $\gamma_{P_0[f]}$ times that of f when f is of finite positive L^*-order.*

We omit the proof of the lemma because it can be carried out in the line of Lemma 14 and with the help of Lemma 13.

Lemma 16 *[10] Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$. Then*

$$\lim_{r\to\infty} \frac{T(r, M[f])}{T(r, f)} = \Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; f),$$

where

$$\Theta(\infty; f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r, f)}{T(r, f)}.$$

Lemma 17 *If f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$, then L^*-order (L^*-lower order) of $M[f]$ are same as those of f. Also the L^*-type of $M[f]$ is $\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; f)$ times that of f when f is of finite positive L^*-order.*

We omit the proof of the lemma because it can be carried out in the line of Lemma 14 and with the help of Lemma 16.

3 Theorems.

In this section we present the main results of the paper.

It is needless to mention that in the paper, the admissibility and homogenity of $P_0[f]$ will be needed as per the requirements of the theorems.

Theorem 18 *Let f be a meromorphic function and g be an entire function of finite order or of non zero lower order such that (i) $0<\rho_f^{L^*}<\infty$, (ii) $\sigma_g^{L^*}<\infty$, (iii) $0<\sigma_f^{L^*}<\infty$ and (iv) $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ or $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$. Then*
(a) if $L(M(r,g))=o\{T(r,P_0[g])\}$ then

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[g])+L(M(r,g))}\leq\frac{\rho_f^{L^*}}{\gamma_{P_0[g]}},$$

and (b) if $T(r,P_0[g])=o\{L(M(r,g))\}$ then

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[g])+L(M(r,g))}\leq\rho_f^{L^*}.$$

Proof. Since $T(r,g)\leq\log^+M(r,g)$ in view of Lemma 9 we obtain for all sufficiently large values of r that

$$T(r,f\circ g)\leq\{1+o(1)\}T(M(r,g),f)$$

i.e.,

$$\log T(r,f\circ g)\leq\log\{1+o(1)\}+\log T(M(r,g),f)$$

i.e.,

$$\log T(r,f\circ g)\leq o(1)+\left(\rho_f^{L^*}+\varepsilon\right)\{\log M(r,g)+L(M(r,g))\}. \quad (1)$$

Using the definition of L^*-type we obtain from (1) for all sufficiently large values of r that

$$\log T(r,f\circ g)\leq o(1)+\left(\rho_f^{L^*}+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}+\left(\rho_f^{L^*}+\varepsilon\right)L(M(r,g)). \quad (2)$$

Again from the definition of L^*-type and in view of Lemma 13 and Lemma 15 we get for a sequence of values of r tending to infinity that

$$T(r,P_0[g])\geq\left(\sigma_{P_0[g]}^{L^*}-\varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{P_0[g]}^{L^*}}$$

$$i.e.,\ T(r,P_0[g])\geq\left\{\gamma_{P_0[g]}.\sigma_g^{L^*}-\varepsilon\right\}\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}$$

$$i.e.,\ \left\{re^{L(r)}\right\}^{\rho_g^{L^*}}\leq\frac{T(r,P_0[g])}{\left(\gamma_{P_0[g]}.\sigma_g^{L^*}-\varepsilon\right)}. \quad (3)$$

Now from (2) and (3) it follows for a sequence of values of r tending to infinity that

$$\log T(r,f\circ g)\leq o(1)+\left(\rho_f^{L^*}+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)\frac{T(r,P_0[g])}{\left(\gamma_{P_0[g]}.\sigma_g^{L^*}-\varepsilon\right)}+\left(\rho_f^{L^*}+\varepsilon\right)L(M(r,g))$$

i.e.,

$$\frac{\log T(r,f\circ g)}{T(r,P_0[g])+L(M(r,g))}\leq\frac{o(1)}{T(r,P_0[g])+L(M(r,g))}+\frac{\frac{\left(\rho_f^{L^*}+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)}{\left(\gamma_{P_0[g]}.\sigma_g^{L^*}-\varepsilon\right)}}{1+\frac{L(M(r,g))}{T(r,P_0[g])}}+\frac{\left(\rho_f^{L^*}+\varepsilon\right)}{1+\frac{T(r,P_0[g])}{L(M(r,g))}}. \quad (4)$$

If $L(M(r,g))=o\{T(r,P_0[g])\}$ then from (4) we obtain that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[g])+L(M(r,g))}\leq\frac{\left(\rho_f^{L^*}+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)}{\gamma_{P_0[g]}.\left(\sigma_g^{L^*}-\varepsilon\right)}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[g])+L(M(r,g))}\leq\frac{\rho_f^{L^*}}{\gamma_{P_0[g]}}.$$

Thus the first part of Theorem 18 follows.

Again if $T(r,P_0[g])=o\{L(M(r,g))\}$ then from (4) it follows that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[g])+L(M(r,g))}\leq\left(\rho_f^{L^*}+\varepsilon\right).$$

As $\varepsilon(>0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[g])+L(M(r,g))}\leq\rho_f^{L^*}.$$

Thus the second part of Theorem 18 follows. □

Remark 19 *With the help of Lemma 15, the conclusion of Theorem 18 can also be drawn under the hypothesis*

$$\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$$

or

$$\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$$

instead of $\sum_{a \neq \infty} \Theta(a; g) = 2$.

In the line of Theorem 18 and with the help of Lemma 17 we may state the following theorem without proof :

Theorem 20 *Let f be a meromorphic function and g be a transcendental entire function of finite order or of non zero lower order such that*
(i) $0 < \rho_f^{L^*} < \infty$,
(ii) $\sigma_g^{L^*} < \infty$,
(iii) $0 < \sigma_f^{L^*} < \infty$ *and*
(iv) $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$.
Thus
(a) *if* $L(M(r, g)) = o\{T(r, M[g])\}$ *then*

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, M[g]) + L(M(r, g))} \leq \frac{\rho_f^{L^*}}{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)},$$

and
(b) *if* $T(r, M[g]) = o\{L(M(r, g))\}$ *then*

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, M[g]) + L(M(r, g))} \leq \rho_f^{L^*}.$$

Theorem 21 *Let f be meromorphic with finite order or non zero lower order and g be entire with*
(i) $0 < \rho_f^{L^*} < \infty$,
(ii) $\rho_f^{L^*} = \rho_g^{L^*}$,
(iii) $\sigma_g^{L^*} < \infty$,
(iv) $0 < \sigma_f^{L^*} < \infty$ *and*
(v) $\sum_{a \neq \infty} \Theta(a; f) = 2$.
Then
(a) *if* $L(M(r, g)) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\}$ *as* $r \to \infty$ *and for some positive* $\alpha < \rho_f^{L^*}$ *then*

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, P_0[f]) + L(M(r, g))} \leq \frac{\rho_f^{L^*} \cdot \sigma_g^{L^*}}{\Gamma_{P_0[f]} \cdot \sigma_f^{L^*}},$$

and
(b) *if* $T(r, P_0[f]) = o\{L(M(r, g))\}$ *then*

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, P_0[f]) + L(M(r, g))} \leq \rho_f^{L^*}.$$

Proof. In view of condition (ii) we obtain from (2) for all sufficiently large values of r that

$$\log T(r, f \circ g) \leq o(1) + \left(\rho_f^{L^*} + \varepsilon\right)\left(\sigma_g^{L^*} + \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_f^{L^*}} + \left(\rho_f^{L^*} + \varepsilon\right) L(M(r, g)). \quad (5)$$

Again from the definition of L^*-type and in view of Lemma 12 and Lemma 14 we get for a sequence of values of r tending to infinity that

$$T(r, P_0[f]) \geq \left(\sigma_{L(f)}^{L^*} - \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{P_0[f]}^{L^*}}$$

$$i.e.,\ T(r, P_0[f]) \geq \left\{\Gamma_{P_0[f]} . \sigma_f^{L^*} - \varepsilon\right\}\left\{re^{L(r)}\right\}^{\rho_f^{L^*}}$$

$$i.e.,\ \left\{re^{L(r)}\right\}^{\rho_f^{L^*}} \leq \frac{T(r, P_0[f])}{\left(\Gamma_{P_0[f]} . \sigma_f^{L^*} - \varepsilon\right)}. \quad (6)$$

Now from (5) and (6) it follows for a sequence of values of r tending to infinity that

$$\log T(r, f \circ g) \leq o(1) + \left(\rho_f^{L^*} + \varepsilon\right)\left(\sigma_g^{L^*} + \varepsilon\right)\frac{T(r, P_0[f])}{\left(\Gamma_{P_0[f]} . \sigma_f^{L^*} - \varepsilon\right)} + \left(\rho_f^{L^*} + \varepsilon\right) L(M(r, g))$$

$$ie.,\ \frac{\log T(r, f \circ g)}{T(r, P_0[f]) + L(M(r, g))} \leq \frac{o(1)}{T(r, P_0[f]) + L(M(r, g))} + \frac{\frac{\left(\rho_f^{L^*} + \varepsilon\right)\left(\sigma_g^{L^*} + \varepsilon\right)}{\left(\Gamma_{P_0[f]} . \sigma_f^{L^*} - \varepsilon\right)}}{1 + \frac{L(M(r,g))}{T(r, P_0[f])}} + \frac{\left(\rho_f^{L^*} + \varepsilon\right)}{1 + \frac{T(r, P_0[f])}{L(M(r,g))}}. \quad (7)$$

If $L(M(r, g)) = o\{T(r, L(f))\}$ then from (7) we get that

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, P_0[f]) + L(M(r, g))} \leq \frac{\left(\rho_f^{L^*} + \varepsilon\right)\left(\sigma_g^{L^*} + \varepsilon\right)}{\Gamma_{P_0[f]} . \left(\sigma_f^{L^*} - \varepsilon\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows from above that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[f])+L(M(r,g))}\leq\frac{\rho_f^{L^*}\cdot\sigma_g^{L^*}}{\Gamma_{P_0[f]}\cdot\sigma_f^{L^*}}\,.$$

Thus the first part of Theorem 21 follows.

Again if $T(r,L(f))=o\{L(M(r,g))\}$ then from (7) it follows that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[f])+L(M(r,g))}\leq\left(\rho_f^{L^*}+\varepsilon\right)\,.$$

As $\varepsilon\,(>0)$ is arbitrary we obtain from above that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[f])+L(M(r,g))}\leq\rho_f^{L^*}\,.$$

Thus the second part of Theorem 21 follows. □

Now in the line of Theorem 21 one may state the following two theorems without proof :

Theorem 22 *Let f be a meromorphic function with finite order or non zero lower order and g be entire such that*

(i) $0<\rho_f^{L^*}<\infty$,

(ii) $\rho_f^{L^*}=\rho_g^{L^*}$,

(iii) $\sigma_g^{L^*}<\infty$,

(iv) $0<\sigma_f^{L^*}<\infty$ *and*

(v) $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ *or* $\delta(\infty;f)=\sum\limits_{a\neq\infty}\delta(a;f)=1$.

Then

(a) *if* $L(M(r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ *as* $r\to\infty$ *and for some positive* $\alpha<\rho_f^{L^*}$ *then*

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[f])+L(M(r,g))}\leq\frac{\rho_f^{L^*}\cdot\sigma_g^{L^*}}{\gamma_{P_0[f]}\cdot\sigma_f^{L^*}}\,,$$

and

(b) *if* $T(r,P_0[f])=o\{L(M(r,g))\}$ *then*

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[f])+L(M(r,g))}\leq\rho_f^{L^*}\,.$$

Theorem 23 *Let f be a transcendental meromorphic function with finite order or non zero lower order and g be entire such that*

(i) $0<\rho_f^{L^*}<\infty$,

(ii) $\rho_f^{L^*}=\rho_g^{L^*}$,

(iii) $\sigma_g^{L^*}<\infty$,

(iv) $0<\sigma_f^{L^*}<\infty$ *and*

(v) $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$.

Then

(a) *if* $L(M(r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ *as* $r\to\infty$ *and for some positive* $\alpha<\rho_f^{L^*}$ *then*

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,M[f])+L(M(r,g))}\leq\frac{\rho_f^{L^*}\cdot\sigma_g^{L^*}}{\Gamma_M-(\Gamma_M-\gamma_M)\Theta(\infty;f)\cdot\sigma_f^{L^*}}\,,$$

and

(b) *if* $T(r,M[f])=o\{L(M(r,g))\}$ *then*

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,M[f])+L(M(r,g))}\leq\rho_f^{L^*}\,.$$

Theorem 24 *Let f be an entire function of finite order or of non zero lower order and g be an entire function with $0<\lambda_f^{L^*}\leq\rho_f^{L^*}<\infty$, $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$ and $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum\limits_{a\neq\infty}\delta(a;f)=1$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[2]}T(r,f\circ g)}{\log T(r,P_0[f])+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)}\geq\frac{\rho_g^{L^*}}{\rho_f^{L^*}}\,.$$

Proof. In view of Lemma 10, we have for all sufficiently large values of r,

$$T(r,f\circ g)\geq \frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o(1),f\right\}$$

$$i.e.,\ \log T(r,f\circ g)\geq o(1)+\log^{[2]}M\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o(1),f\right\}\,.\quad (8)$$

$$i.e.,\ \log T(r,f\circ g)\geq o(1)+\left(\lambda_f^{L^*}-\varepsilon\right)\left[\log\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o(1)\right\}+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)\right]$$

$$i.e.,\ \log T(r,f\circ g)\geq o(1)+\left(\lambda_f^{L^*}-\varepsilon\right)\cdot\left[\log\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)\left(1+\frac{o(1)}{\frac{1}{8}M\left(\frac{r}{4},g\right)}\right)\right\}+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)\right]$$

$$i.e.,\ \log T(r,f\circ g)\geq\left(\lambda_f^{L^*}-\varepsilon\right)\log M\left(\frac{r}{4},g\right).$$

$$\left\{\frac{\log M\left(\frac{r}{4},g\right)+\log\left(1+\frac{o(1)}{\frac{1}{8}M\left(\frac{r}{4},g\right)}\right)+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)}{\log M\left(\frac{r}{4},g\right)}\right\}$$

$$i.e.,\ \log^{[2]}T(r,f\circ g)\geq$$
$$\log^{[2]}M\left(\frac{r}{4},g\right)+\left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{L^*}+\varepsilon}\right)L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)$$
$$-\log\left[\exp\left\{\left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{L^*}+\varepsilon}\right)L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)\right\}\right]+$$
$$\log\left\{\frac{\left(\lambda_f^{L^*}-\varepsilon\right)\left[\log M\left(\frac{r}{4},g\right)+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)\right]+o(1)}{\log M\left(\frac{r}{4},g\right)}\right\}$$

$$i.e.,\ \log^{[2]}T(r,f\circ g)\geq$$
$$\log^{[2]}M\left(\frac{r}{4},g\right)+\left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{L^*}+\varepsilon}\right)L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)+$$
$$\log\left\{\frac{\left(\lambda_f^{L^*}-\varepsilon\right)\left[\log M\left(\frac{r}{4},g\right)+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)\right]+o(1)}{\exp\left\{\left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{L^*}+\varepsilon}\right)L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)\right\}\log M\left(\frac{r}{4},g\right)}\right\}$$

i.e.,

$$\log^{[2]}T(r,f\circ g)\geq\log^{[2]}M\left(\frac{r}{4},g\right)+\left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{L^*}+\varepsilon}\right)L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right).\quad(9)$$

Now from (9) it follows for a sequence of values of r tending to infinity that

$$\log^{[2]}T(r,f\circ g)\geq\left(\rho_g^{L^*}-\varepsilon\right)\log\left\{\frac{r}{4}e^{L\left(\frac{r}{4}\right)}\right\}+\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{L^*}+\varepsilon}\right)L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right).\quad(10)$$

In view of Lemma 15, we get for all sufficiently large values of r that

$$\log T(r,P_0[f])\leq\left(\rho_{P_0[f]}^{L^*}+\varepsilon\right)\log\left\{re^{L(r)}\right\}$$
$$i.e.,\ \log T(r,P_0[f])\leq\left(\rho_f^{L^*}+\varepsilon\right)\log\left\{re^{L(r)}\right\}$$
$$i.e.,\ \log T(r,P_0[f])\leq\left(\rho_f^{L^*}+\varepsilon\right)\log\left\{\frac{r}{4}e^{L\left(\frac{r}{4}\right)}\right\}+\log 4.\quad(11)$$

Hence from (10) and (11) it follows for all sufficiently large values of r that

$$i.e.,\ \log^{[2]}T(r,f\circ g)\geq$$
$$\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{L^*}+\varepsilon}\right)(\log T(r,P_0[f])-\log 4)+\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{L^*}+\varepsilon}\right)L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)$$

$$i.e.,\ \log^{[2]}T(r,f\circ g)\geq$$
$$\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{L^*}+\varepsilon}\right)\left[\log T(r,P_0[f])+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)\right]-\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{L^*}+\varepsilon}\right)\log 4$$

$$i.e.,\ \frac{\log^{[2]}T(r,f\circ g)}{\log T(r,P_0[f])+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)}\geq\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{L^*}+\varepsilon}\right)-\frac{\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{L^*}+\varepsilon}\right)\log 4}{\log T(r,P_0[f])+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)}.\quad(12)$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (12) that

$$\limsup_{r\to\infty}\frac{\log^{[2]}T(r,f\circ g)}{\log T(r,P_0[f])+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)}\geq\frac{\rho_g^{L^*}}{\rho_f^{L^*}}.$$

This proves the theorem. □

In the line of Theorem 24 the following theorem may be proved :

Theorem 25 *Let f be an entire function of finite order or of non zero lower order and g be an entire function with $0<\lambda_f^{L^*}\leq\rho_f^{L^*}<\infty$, $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$ and $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$. Then*

$$\liminf_{r\to\infty}\frac{\log^{[2]}T(r,f\circ g)}{\log T(r,P_0[f])+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)}\geq\frac{\lambda_g^{L^*}}{\rho_f^{L^*}}.$$

Remark 26 *By Lemma 14, the conclusion of Theorem 24 and Theorem 25 can also be drawn under the hypothesis $\sum_{a\neq\infty}\Theta(a;f)=2$ instead of*

$$\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$$

or $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$.

Theorem 27 *Let f be a transcendental entire function of finite order or of non zero lower order and g be an entire function with $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$, $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Then*

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, P_0[f]) + L\left(\frac{1}{8} M\left(\frac{r}{4}, g\right)\right)} \geq \frac{\rho_g^{L^*}}{\rho_f^{L^*}} .$$

Theorem 28 *Let f be a transcendental entire function of finite order or of non zero lower order and g be an entire function with $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$, $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Then*

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, M[f]) + L\left(\frac{1}{8} M\left(\frac{r}{4}, g\right)\right)} \geq \frac{\lambda_g^{L^*}}{\rho_f^{L^*}} .$$

The proof of the above two theorems can be established in the line of Theorem 24 and Theorem 25 respectively and with the help of Lemma 17 and therefore is omitted.

Theorem 29 *Let f be meromorphic and g be entire of finite order or of non zero lower order with $\rho_f^{L^*} < \infty$, $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ and $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$. Then*

(a) if $L(M(r, g)) = o\{\log T(r, P_0[g])\}$ then

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, P_0[g]) + L(M(r, g))} \leq \frac{\rho_f^{L^*}}{\lambda_g^{L^*}} ,$$

and

(b) if $T(r, P_0[g]) = o\{L(M(r, g))\}$ then

$$\lim_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, P_0[g]) + L(M(r, g))} = 0 .$$

Proof. For all sufficiently large values of r we obtain in view of $T(r, g) \leq \log^+ M(r, g)$ and by Lemma 9 that

$$T(r, f \circ g) \leq \{1 + o(1)\} T(M(r, g), f)$$

$$i.e., \ \log T(r, f \circ g) \leq \log\{1 + o(1)\} + \log T(M(r, g), f)$$

$$i.e., \ \log T(r, f \circ g) \leq o(1) + \log T(M(r, g), f)$$

$$i.e., \ \log T(r, f \circ g) \leq o(1) + \left(\rho_f^{L^*} + \varepsilon\right)\{\log M(r, g) + L(M(r, g))\}$$

$$i.e., \ \log T(r, f \circ g) \leq o(1) + \left(\rho_f^{L^*} + \varepsilon\right) \log M(r, g) \left\{1 + \frac{L(M(r, g))}{\log M(r, g)}\right\}$$

$$i.e., \ \log^{[2]} T(r, f \circ g) \leq o(1) + \log\left(\rho_f^{L^*} + \varepsilon\right) + \log^{[2]} M(r, g) + \log\left\{1 + \frac{L(M(r, g))}{\log M(r, g)}\right\}$$

$$i.e., \ \log^{[2]} T(r, f \circ g) \leq o(1) + \log\left(\rho_f^{L^*} + \varepsilon\right) + \left(\rho_g^{L^*} + \varepsilon\right) \log\left\{re^{L(r)}\right\} + \log\left\{1 + \frac{L(M(r, g))}{\log M(r, g)}\right\}$$

$$i.e., \ \log^{[2]} T(r, f \circ g) \leq o(1) + \left(\rho_g^{L^*} + \varepsilon\right)\{\log r + L(r)\} + \frac{L(M(r, g))}{\log M(r, g)} . \quad (13)$$

Again in view of Lemma 15 we get from the definition of L^*-lower order for all sufficiently large values of r that

$$\log T(r, P_0[g]) \geq \left(\lambda_{P_0[g]}^{L^*} - \varepsilon\right) \log\left[re^{L(r)}\right]$$

$$i.e., \ \log T(r, P_0[g]) \geq \left(\lambda_g^{L^*} - \varepsilon\right) \log\left[re^{L(r)}\right]$$

$$i.e., \ \log T(r, P_0[g]) \geq \left(\lambda_g^{L^*} - \varepsilon\right)[\log r + L(r)]$$

$$i.e., \ \log r + L(r) \leq \frac{\log T(r, P_0[g])}{\left(\lambda_g^{L^*} - \varepsilon\right)} . \quad (14)$$

Hence from (13) and (14) it follows for all sufficiently large values of r that

$$\log^{[2]} T(r, f \circ g) \leq o(1) + \left(\frac{\rho_g^{L^*} + \varepsilon}{\lambda_g^{L^*} - \varepsilon}\right) \cdot \log T(r, P_0[g]) + \frac{L(M(r, g))}{\log M(r, g)}$$

$$i.e,\ \frac{\log^{[2]} T(r, f\circ g)}{\log T(r, P_0[g]) + L(M(r,g))}$$
$$\le o(1) + \left(\frac{\rho_f^{L^*}+\varepsilon}{\lambda_g^{L^*}-\varepsilon}\right)\cdot\frac{\log T(r, P_0[g])}{\log T(r, P_0[g]) + L(M(r,g))}$$
$$+\frac{L(M(r,g))}{[\log T(r, P_0[g]) + L(M(r,g))]\log M(r,g)}$$

$$i.e,\ \frac{\log^{[2]} T(r, f\circ g)}{\log T(r, P_0[g]) + L(M(r,g))} \le o(1) + \frac{\left(\frac{\rho_f^{L^*}+\varepsilon}{\lambda_g^{L^*}-\varepsilon}\right)}{1+\frac{L(M(r,g))}{\log T(r,P_0[g])}} + \frac{1}{\left[1+\frac{\log T(r,P_0[g])}{L(M(r,g))}\right]\log M(r,g)} \quad (15)$$

Since $L(M(r,g)) = o\{\log T(r, P_0[g])\}$ as $r\to\infty$ and $\varepsilon(>0)$ is arbitrary, we obtain from (15) that

$$\limsup_{r\to\infty}\frac{\log^{[2]} T(r, f\circ g)}{\log T(r, P_0[g]) + L(M(r,g))} \le \frac{\rho_f^{L^*}}{\lambda_g^{L^*}}. \quad (16)$$

Again if $\log T(r,g) = o\{L(M(r,g))\}$ then from (15) we get that

$$\lim_{r\to\infty}\frac{\log^{[2]} T(r, f\circ g)}{\log T(r, P_0[g]) + L(M(r,g))} = 0\,. \quad (17)$$

Thus from (16) and (17) the theorem is established. □

Corollary 30 *Let f be meromorphic and g be entire of finite order or of non zero lower order with $\rho_f^{L^*} < \infty$, $0 < \lambda_g^{L^*} \le \rho_g^{L^*} < \infty$ and $\delta(\infty;g) = \sum_{a\ne\infty}\delta(a;g) = 1$. Then*

(a) if $L(M(r,g)) = o\{\log T(r, P_0[g])\}$ then

$$\liminf_{r\to\infty}\frac{\log^{[2]} T(r, f\circ g)}{T(r, P_0[g]) + L(M(r,g))} \le 1$$

and

(b) if $T(r, P_0[g]) = o\{L(M(r,g))\}$ then

$$\liminf_{r\to\infty}\frac{\log^{[2]} T(r, f\circ g)}{T(r, P_0[g]) + L(M(r,g))} = 0\,.$$

We omit the proof of Corollary 30 because it can be carried out in the line of Theorem 29.

Remark 31 *The equality sign in Theorem 29 and Corollary 30 cannot be removed as we see in the following example.*

Example 32 *Let $f = g = \exp z$ and $L(r) = \frac{1}{p}\exp\left(\frac{1}{r}\right)$ where p is any positive real number. Then*

$$\rho_f^{L^*} = \lambda_g^{L^*} = \rho_g^{L^*} = 1$$
$$\text{and } \delta(\infty;g) = \sum_{a\ne\infty}\delta(a;g) = 1.$$

Also let $s=1$, $A_1 = 1$ and

$$n_{i1} = \begin{cases} 1, & \text{for } i = 1\\ 0, & \text{for } i\ne 1.\end{cases}$$

Then

$$P_0[g] = \exp z.$$

Now

$$T(r, f\circ g) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \quad (r\to\infty),$$
$$T(r,g) = \frac{r}{\pi} \text{ and } M(r,g) = \exp r\,.$$

So

$$L(M(r,g)) = L(\exp r) = \frac{1}{p}\exp\left(\frac{1}{\exp r}\right)\,.$$

Hence

$$\liminf_{r\to\infty}\frac{\log^{[2]} T(r, f\circ g)}{\log T(r, P_0[g]) + L(M(r,g))}$$
$$= \limsup_{r\to\infty}\frac{\log^{[2]} T(r, f\circ g)}{\log T(r, P_0[g]) + L(M(r,g))}$$
$$= \limsup_{r\to\infty}\frac{\log\left[r-\frac{1}{2}\log r + O(1)\right]}{\log r + O(1) + \frac{1}{p}\exp\left(\frac{1}{\exp r}\right)}$$
$$= 1\,.$$

Remark 33 *The conclusion of Theorem 29 and Corollary 30 can also be drawn under the hypothesis $\Theta(\infty;g) = \sum_{a\ne\infty}\delta_p(a;g) = 1$ or $\sum_{a\ne\infty}\Theta(a;g) = 2$ instead of $\delta(\infty;g) = \sum_{a\ne\infty}\delta(a;g) = 1$.*

In the line of Theorem 29 and with the help of Lemma 17 we may state the following theorem without proof :

Theorem 34 *Let f be meromorphic and g be transcendental entire of finite order or of non zer lower order with $\rho_f^{L^*} < \infty$, $0 < \lambda_g^{L^*} \le \rho_g^{L^*} < \infty$ and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g) = 4$. Then*

(a) if $L(M(r,g)) = o\{\log T(r,M[g])\}$ then

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, f\circ g)}{\log T(r,M[g]) + L(M(r,g))} \leq \frac{\rho_f^{L^*}}{\lambda_g^{L^*}}$$

and

(b) if $T(r,M[g]) = o\{L(M(r,g))\}$ then

$$\lim_{r\to\infty} \frac{\log^{[2]} T(r, f\circ g)}{\log T(r,M[g]) + L(M(r,g))} = 0 \,.$$

The proof of the following corollary may also be deduced in view of Theorem 34 :

Corollary 35 *Let f be meromorphic and g be transcendental entire of finite order or of non zero lower order with $\rho_f^{L^*} < \infty$, $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$. Then*

(a) if $L(M(r,g)) = o\{\log T(r,M[g])\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, f\circ g)}{T(r,M[g]) + L(M(r,g))} \leq 1$$

and

(b) if $T(r,M[g]) = o\{L(M(r,g))\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, f\circ g)}{T(r,M[g]) + L(M(r,g))} = 0 \,.$$

Theorem 36 *Let f be meromorphic with $\lambda_f^{L^*} < \infty$ and g be entire with finite order or non zero finite lower order and $\sum\limits_{a\neq\infty} \Theta(a;g) = 2$. Also let there exists entire functions $a_i\,(i = 1,2,3,\cdots,n; n\leq\infty)$ such that $T(r,a_i) = o\{T(r,g)\}$ and $\sum\limits_{i=1}^{n} \delta(a_i,g) = 1$. If $L(M(r,g)) = o\{r^{\alpha} e^{\alpha L(r)}\}$ as $r\to\infty$ and for some α with $0 < \alpha < \lambda_g^{L^*}$ then*

$$\liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,P_0[g])} \leq \frac{\pi \lambda_f^{L^*}}{\Gamma_{P_0[g]}} \,,$$

otherwise

$$\liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,P_0[g]) \cdot L(M(r,g))} = 0 \,.$$

Proof. In view of the inequality $T(r,g) \leq \log^{+} M(r,g)$ and by Lemma 9 we get for a sequence of values of r tending to infinity that

$$T(r, f\circ g) \leq \{1 + o(1)\} T(M(r,g), f)$$

i.e.,

$$\log T(r, f\circ g) \leq \log\{1 + o(1)\} + \log T(M(r,g), f)$$

i.e.,

$$\log T(r, f\circ g) \leq \left(\lambda_f^{L^*} - \varepsilon\right)(\log M(r,g) + L(M(r,g))) + O(1)$$

i.e.,

$$\frac{\log T(r, f\circ g)}{T(r,P_0[g])} \leq \frac{\left(\lambda_f^{L^*} - \varepsilon\right)(\log M(r,g) + L(M(r,g))) + O(1)}{T(r,P_0[g])}$$

i.e.,

$$\frac{\log T(r, f\circ g)}{T(r,P_0[g])} \leq \left(\lambda_f^{L^*} - \varepsilon\right) \cdot \frac{\log M(r,g) + L(M(r,g))}{T(r,P_0[g])} + O(1). \tag{18}$$

Case I. Let $L(M(r,g)) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\}$ as $r\to\infty$ and for some α with $0 < \alpha < \lambda_g^{L^*}$.

Since $\alpha < \lambda_g^{L^*}$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\alpha < \lambda_g^{L^*} - \varepsilon \,. \tag{19}$$

As $L(M(r,g)) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\}$ as $r\to\infty$ we get in view of (19) that

$$\lim_{r\to\infty} \frac{L(M(r,g))}{\left[re^{L(r)}\right]^{\lambda_g^{L^*} - \varepsilon}} = 0 \,. \tag{20}$$

Again in view of Lemma 13 we obtain for all sufficiently large values of r,

$$\log T(r,P_0[g]) \geq \left(\lambda_{P_0[g]}^{L^*} - \varepsilon\right) \log\left\{re^{L(r)}\right\}$$

$$i.e.,\ \log T(r,P_0[g]) \geq \left(\lambda_g^{L^*} - \varepsilon\right) \log\left\{re^{L(r)}\right\}$$

$$i.e.,\ T(r,P_0[g]) \geq \left[re^{L(r)}\right]^{\lambda_g^{L^*} - \varepsilon} \,. \tag{21}$$

Now from (18) and (21) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log T(r, f\circ g)}{T(r,P_0[g])} \leq$$

$$\left(\lambda_f^{L^*}-\varepsilon\right)\left[\frac{\log M(r,g)}{T(r,P_0[g])}+\frac{L(M(r,g))}{\left[re^{L(r)}\right]^{\lambda_g^{L^*}-\varepsilon}}\right]+O(1)$$

$$i.e., \ \frac{\log T(r,f\circ g)}{T(r,P_0[g])}\leq \left(\lambda_f^{L^*}-\varepsilon\right)\times$$

$$\left[\frac{\log M(r,g)}{T(r,g)}\cdot\frac{T(r,g)}{T(r,P_0[g])}+\frac{L(M(r,g))}{\left[re^{L(r)}\right]^{\lambda_g^{L^*}-\varepsilon}}\right]+O(1). \tag{22}$$

Now combining (20) and (22) and in view of Lemma 11 and Lemma 12, it follows that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[g])}\leq\frac{\pi\lambda_f^{L^*}}{\Gamma_{P_0[g]}}. \tag{23}$$

Case II. If $L(M(r,g))\neq o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r\to\infty$ and for some α with $0<\alpha<\lambda_g^{L^*}$, then from (18) we get for a sequence of values of r tending to infinity that

$$\frac{\log T(r,f\circ g)}{T(r,P_0[g])\,L(M(r,g))}\leq \left(\lambda_f^{L^*}-\varepsilon\right)\cdot\frac{\log M(r,g)}{T(r,P_0[g])\,L(M(r,g))}+\frac{1+O(1)}{T(r,P_0[g])}.$$

$$i.e., \ \liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[g])\,L(M(r,g))}=0.$$

Thus combining Case I and Case II the theorem follows. □

Remark 37 *In view of Lemma* 15 *one can easily verify that the conclusion of Theorem* 35 *can also be deduced if we replace* $\sum_{a\neq\infty}\Theta(a;g)=2$ *by* $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ *or* $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$.

In the line of Theorem 36 the following theorem can be proved :

Theorem 38 *Let* f *be meromorphic with* $\rho_f^{L^*}<\infty$ *and* g *be entire with finite order or non zero finite lower order and* $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ *or* $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$. *Also let there exists entire functions* a_i ($i=1,2,3,\cdots,n;n\leq\infty$) *such that* $T(r,a_i)=o\{T(r,g)\}$ *and* $\sum_{i=1}^{n}\delta(a_i,g)=1$.

If $L(M(r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ *as* $r\to\infty$ *and for some* α *with* $0<\alpha<\lambda_g^{L^*}$ *then*

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[g])}\leq\frac{\pi\rho_f^{L^*}}{\gamma_{P_0[g]}},$$

otherwise

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[g])\,L(M(r,g))}=0.$$

Remark 39 *In view of Lemma* 14 *one can easily verify that the conclusion of Theorem* 38 *can also be deduced if we replace* $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$*or* $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$ *by* $\sum_{a\neq\infty}\Theta(a;g)=2$.

In the line of Theorem 36 and Theorem 38 and with the help of Lemma 17 we may state the following two theorems without proof :

Theorem 40 *Let* f *be meromorphic with* $\lambda_f^{L^*}<\infty$ *and* g *be transcendental entire with finite order or non zero finite lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Also let there exists entire functions* $a_i\,(i=1,2,3,\cdots,n;n\leq\infty)$ *such that* $T(r,a_i)=o\{T(r,g)\}$ *and* $\sum_{i=1}^{n}\delta(a_i,g)=1$.

If $L(M(r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ *as* $r\to\infty$ *and for some* α *with* $0<\alpha<\lambda_g^{L^*}$ *then*

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,M[g])}\leq\frac{\pi\lambda_f^{L^*}}{\Gamma_M-(\Gamma_M-\gamma_M)\Theta(\infty;g)},$$

otherwise

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,M[g])\cdot L(M(r,g))}=0.$$

Theorem 41 *Let* f *be meromorphic with* $\rho_f^{L^*}<\infty$ *and* g *be transcendental entire with finite order or non zero finite lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Also let there exist entire functions* $a_i\,(i=1,2,3,\cdots,n;n\leq\infty)$ *such that* $T(r,a_i)=o\{T(r,g)\}$ *and* $\sum_{i=1}^{n}\delta(a_i,g)=1$.

If $L(M(r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ *as* $r\to\infty$ *and for some* α *with* $0<\alpha<\lambda_g^{L^*}$ *then*

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[g])}\leq\frac{\pi\rho_f^{L^*}}{\Gamma_M-(\Gamma_M-\gamma_M)\Theta(\infty;g)},$$

otherwise

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,P_0[g])\,L(M(r,g))}=0.$$

Theorem 42 *Let f be a meromorphic function and g be entire of finite order or of non zero lower order such that $\rho_g^{L^*} < \infty$, $\lambda_{f\circ g}^{L^*} = \infty$ and $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$. Then*

$$\lim_{r\to\infty} \frac{\log T(r, f\circ g)}{\log T(r, P_0[g])} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem does not hold.Then we can find a constant $\beta > 0$ such that for a sequence of values of r tending to infinity

$$\log T(r, f\circ g) \le \beta \log T(r, P_0[g]). \tag{24}$$

Again from the definition of $\rho_{P_0[f]}^{L^*}$ it follows that for all sufficiently large values of r and in view of Lemma 6,

$$\log T(r, P_0[g]) \le \left(\rho_{P_0[g]}^{L^*} + \varepsilon\right) \log\left\{re^{L(r)}\right\}$$

$$i.e.,\ \log T(r, P_0[g]) \le \left(\rho_g^{L^*} + \varepsilon\right) \log\left\{re^{L(r)}\right\}. \tag{25}$$

Thus from (24) and (25) we have for a sequence of values of r tending to infinity that

$$\log T(r, f\circ g) \le \beta\left(\rho_g^{L^*} + \varepsilon\right) \log\left\{re^{L(r)}\right\}$$

i.e.,

$$\frac{\log T(r, f\circ g)}{\log\left\{re^{L(r)}\right\}} \le \frac{\beta\left(\rho_g^{L^*} + \varepsilon\right)\log\left\{re^{L(r)}\right\}}{\log\left\{re^{L(r)}\right\}}$$

$$i.e.,\ \liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{\log\left\{re^{L(r)}\right\}} = \lambda_{f\circ g}^{L^*} < \infty.$$

This is a contradiction.

This proves the theorem. □

Remark 43 *Theorem 42 is also valid with "limit superior" instead of "limit" if $\lambda_{f\circ g}^{L^*} = \infty$ is replaced by $\rho_{f\circ g}^{L^*} = \infty$ and the other conditions remaining the same.*

Corollary 44 *Under the assumptions of Theorem 42 or Remark 43,*

$$\limsup_{r\to\infty} \frac{T(r, f\circ g)}{T(r, P_0[g])} = \infty.$$

Proof. By Theorem 42 or Remark 43 we obtain for all sufficiently large values of r and for $K > 1$,

$$\begin{aligned} \log T(r, f\circ g) &> K \log T(r, P_0[g]) \\ i.e.,\ T(r, f\circ g) &> \{T(r, P_0[g])\}^K, \end{aligned}$$

from which the corollary follows. □

Remark 45 *The condition $\lambda_{f\circ g}^{L^*} = \infty$ is necessary in Theorem 42 and Corollary 44 which is evident from the following example :*

Example 46 *Let $f = z$, $g = \exp z$ and $L(r) = \frac{1}{p}\exp\left(\frac{1}{r}\right)$ where p is any positive real number.*

Also let $s = 1$, $A_1 = 1$ and

$$n_{i1} = \begin{cases} 1, & \text{for } i = 1 \\ 0, & \text{for } i \neq 1. \end{cases}$$

Then

$$P_0[g] = \exp z.$$

Also

$$\begin{aligned} \delta(\infty; g) &= \sum_{a\neq\infty} \delta(a; g) = 1, \\ \rho_g^{L^*} &= 1 < \infty \text{ and } \lambda_{f\circ g}^{L^*} = 1 < \infty. \end{aligned}$$

Now

$$T(r, f\circ g) = T(r, \exp z) = \frac{r}{\pi}$$

and

$$T(r, P_0[g]) = T(r, \exp z) = \frac{r}{\pi}.$$

Therefore

$$\begin{aligned} \lim_{r\to\infty} \frac{\log T(r, f\circ g)}{\log T(r, P_0[g])} &= \lim_{r\to\infty} \frac{\log r + O(1)}{\log r + O(1)} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{r\to\infty} \frac{T(r, f\circ g)}{T(r, P_0[g])} &= \lim_{r\to\infty} \frac{\left(\frac{r}{\pi}\right)}{\left(\frac{r}{\pi}\right)} \\ &= 1. \end{aligned}$$

Remark 47 *Considering*

$$f = z,\ g = \exp z,\ \ A = 1,\ L(r) = \frac{1}{p}\exp\left(\frac{1}{r}\right)$$

where p is any positive real number $s = 1$, $A_1 = 1$ and

$$n_{i1} = \begin{cases} 1, & \text{for } i = 1 \\ 0, & \text{for } i \neq 1. \end{cases}$$

one may also verify that the condition $\rho_{f\circ g}^{L^*} = \infty$ in Remark 43 and Corollary 44 is essential.

Remark 48 *The conclusion of Theorem* 42*, Remark* 43 *and Corollary* 44 *can also be drawn under the hypothesis* $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ *or* $\sum_{a \neq \infty} \Theta(a; g) = 2$ *instead of* $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$.

In the line of Theorem 42 the following theorem may be deduced:

Theorem 49 *Let* f *be meromorphic and* g *be transcendental entire of finite order or non zero lower order such that* $\rho_g^{L^*} < \infty$, $\lambda_{f \circ g}^{L^*} = \infty$ *and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$. *Then*

$$\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r, M[g])} = \infty.$$

Remark 50 *Theorem* 49 *is also valid with "limit superior" instead of "limit" if* $\lambda_{f \circ g}^{L^*} = \infty$ *is replaced by* $\rho_{f \circ g}^{L^*} = \infty$ *and the other conditions remaining the same.*

Corollary 51 *Under the assumptions of Theorem* 49 *or Remark* 50*,*

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, M[g])} = \infty.$$

The proof is omitted because it can be carried out in the line of Corollary 44.

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