On generalized preconditioned Hermitian and skew-Hermitian splitting methods for saddle point problems

Chunping Pan
Department of Humanities and Social Sciences
Zhejiang Industry Polytechnic College
Shaoxing, Zhejiang 312000
P.R.China
panchunping@126.com

Abstract: In this paper, we study the iterative algorithms for saddle point problems(SPP). Bai, Golub and Pan recently studied a class of preconditioned Hermitian and skew-Hermitian splitting methods(PHSS). By further accelerating it with another parameters, using the Hermitian/skew-Hermitian splitting iteration technique we present the generalized preconditioned Hermitian and skew-Hermitian splitting methods with four parameters(4-GPHSS). Under some suitable conditions, we give the convergence results. Numerical examples further confirm the correctness of the theory and the effectiveness of the method.

Key–Words: Saddle point problems, iterative method, Hermitian and skew-Hermitian splitting method, preconditioned Hermitian and skew-Hermitian splitting method, SOR-like method

1 Introduction

We consider the iterative solutions of large sparse saddle point problems of the form

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \tag{1}$$

where $A \in R^{m\times m}$ is a symmetric positive definite matrix, $B \in R^{m\times n}$ is a matrix of full column rank, $p \in R^m$ and $q \in R^n$ are two given vectors. here $m \geq n$. Denote by B^T the transpose of the matrix B. These assumptions guarantee the existence and uniqueness of the solution of the linear system.

This system arises as the first-order optimality conditions for the following equality-constrained quadratic programming problem:

$$\begin{aligned} \min J(x) &= \frac{1}{2}x^T Ax - p^T x && (2)\\ s.t. \quad Bx &= q && (3) \end{aligned}$$

In this case the variable y represents the vector of Lagrange multipliers. Any solution (x_*, y_*) of (1) is a saddle point for the Lagrangian

$$L(x,y) = \frac{1}{2}x^T Ax - p^T x + (Bx-q)^T y \tag{4}$$

hence the name 'saddle point problem' given to (1) . Recall that a saddle point is a point $(x_*, y_*) \in R^{n+m}$ that satisfies

$$L(x_*, y) \leq L(x_*, y_*) \leq L(x, y_*) \tag{5}$$

for any $x \in R^n$ and $y \in R^m$, or, equivalently,

$$\min_x \max_y L(x,y) = L(x_*, y_*) = \max_y \min_x L(x,y) \tag{6}$$

Systems of the form (1) also arise in nonlinearly constrained optimization (sequential quadratic programming and interior point methods), in fluid dynamics ('Stokes' problem), incompressible elasticity, circuit analysis, structural analysis, and so forth[1].

Since the above problem is large and sparse, iterative methods for solving equation (1) are effective because of storage requirements and preservation of sparsity. The best known and the oldest methods is Uzawa algorithms[2]. The well-known SOR method, which is a simple iterative method that is popular in engineering applications, cannot be applied directly to system (1) because of the singularity of the block diagonal part of the coefficient matrix. Recently, several proposals for generalizing the SOR method to system (1) have been proposed[3, 4, 5, 6, 7].

Recently, Benzi and Golub discussed the convergence and the preconditioning property of the Hermitian and skew-Hermitian splitting(HSS) iteration method when it is used to solve the saddle point problem[8].Then, Bai et al.establish a class of preconditioned Hermitian/skew-Hermitian splitting(PHSS) iteration method for saddle point problem[9], and Pan et al.futher proposed its two-parameter acceleration, called the generalized preconditioned Hermitian/skew-Hermitian splitting(GPHSS) iteration method, and studied the convergence of this iterative scheme[10]. Both theory and experiments have shown that these methods are very robust and efficient for solving the saddle-point problem when they are used as either solvers or preconditioners(for the Krylov subspace iteration methods).

In this paper, By further accelerating PHSS iteration method with another parameter, we present the generalized Preconditioned Hermitian and skew-Hermitian splitting methods with four parameters(4-GPHSS). Under some suitable conditions, we give the convergence results. Numerical results show that the new methods are very effective.

2 The Four-parameter PHSS iteration methods

In this section, we review the HSS and PHSS iteration method for solving the saddle-point problems presented by Bai,Golub and Ng,Bai,Golub and Pan[8, 9].

Let $A \in C^{n\times n}$ be a positive definite matrix. Given an initial guess $x^{(0)} \in C^n$. For $k = 0, 1, 2, \cdots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b \end{cases} \tag{7}$$

where α is a given positive constant.

In matrix-vector form, the above HSS iteration method can be equivalently rewritten as

$$x^{(k+1)} = M(\alpha)x^{(k)} + N(\alpha)b, k = 0, 1, 2, \cdots \tag{8}$$

where

$$M(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)$$

and

$$N(\alpha) = 2\alpha(\alpha I + S)^{-1}(\alpha I + H)^{-1}$$

Here, $M(\alpha)$ is the iteration matrix of the HSS iteration. In fact, (8)may also result from the splitting

$$A = F(\alpha) - G(\alpha)$$

of the coefficient matrix A, with

$$\begin{cases} F(\alpha) = \dfrac{1}{2\alpha}(\alpha I + H)(\alpha I + S) \\ G(\alpha) = \dfrac{1}{2\alpha}(\alpha I - H)(\alpha I - S) \end{cases} \tag{9}$$

The following theorem describes the convergence property of the HSS iteration.

Theorem 1. *[8]Let $A \in C^{n\times n}$ be a positive definite matrix, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, respectively, and α be a positive constant. Then the spectral radius $\rho(M(\alpha))$ of the iteration matrix $M(\alpha)$ of the HSS iteration is bounded by*

$$\sigma(\alpha) = \max_{\lambda_j \in \lambda(H)} \frac{|\alpha - \lambda_j|}{|\alpha + \lambda_j|}$$

where $\lambda(H)$ is the spectral set of the matrix H. Therefore, it follows that

$$\rho(M(\alpha)) \le \sigma(\alpha) < 1, \forall \alpha > 0,$$

i.e., the HSS iteration converges to the exact solution $x^* \in C^n$ of the system of linear equations (1). Moreover, if γ_{min} and γ_{max} are the lower and the upper bounds of the eigenvalues of the matrix H, respectively, then

$$\begin{aligned} \alpha^* &= arg\min_{\alpha} \left\{ \max_{\gamma_{min} \le \lambda \le \gamma_{max}} \left| \frac{\alpha - \lambda}{\alpha + \lambda} \right| \right\} \\ &= \sqrt{\gamma_{min}\gamma_{max}} \end{aligned}$$

and

$$\begin{aligned} \sigma(\alpha^*) &= \frac{\sqrt{\gamma_{max}} - \sqrt{\gamma_{min}}}{\sqrt{\gamma_{max}} + \sqrt{\gamma_{min}}} \\ &= \frac{\sqrt{\kappa(H)} - 1}{\sqrt{\kappa(H)} + 1}, \end{aligned}$$

where $\kappa(H)$ is the spectral condition number of H.

To establish the convergence properties of iterative method for the saddle-point problem, It need to begin by writing the saddle-point problem (1) in nonsymmetric form:

$$\mathcal{A}Z = b \tag{10}$$

where

$$\mathcal{A} = \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}, Z = \begin{pmatrix} x \\ y \end{pmatrix}, b = \begin{pmatrix} p \\ -q \end{pmatrix} \tag{11}$$

define matrices

$$P = \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix} \tag{12}$$

and

$$\overline{B} = A^{-\frac{1}{2}}BQ^{-\frac{1}{2}} \in R^{n\times n} \tag{13}$$

where $Q \in R^{n\times n}$ is a prescribed nonsingular and symmetric matrix, and define

$$\overline{\mathcal{A}} = P^{-\frac{1}{2}}\mathcal{A}P^{-\frac{1}{2}} = \begin{pmatrix} I & \overline{B} \\ -\overline{B}^T & 0 \end{pmatrix} \tag{14}$$

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = P^{\frac{1}{2}} \begin{pmatrix} x \\ y \end{pmatrix} \tag{15}$$

$$\overline{b} = P^{-\frac{1}{2}}b = \begin{pmatrix} \overline{p} \\ -\overline{q} \end{pmatrix} \tag{16}$$

Then the system of linear equations (1) can be transformed into the following equivalent one:

$$\overline{\mathcal{A}} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \overline{b} \tag{17}$$

Evidently, the Hermitian and skew-Hermitian parts of the matrix $\overline{\mathcal{A}} \in R^{(m+n)\times(m+n)}$ are, respectively,

$$\overline{H} = \frac{1}{2}(\overline{\mathcal{A}} + \overline{\mathcal{A}}^T) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{18}$$

and

$$\overline{S} = \frac{1}{2}(\overline{\mathcal{A}} - \overline{\mathcal{A}}^T) = \begin{pmatrix} 0 & \overline{B} \\ -\overline{B}^T & 0 \end{pmatrix} \tag{19}$$

By straightforwardly applying the HSS iteration technique to (17), it then is easy to obtain the HSS iteration scheme [8] :

$$\begin{cases} (\alpha I + \overline{H}) \begin{pmatrix} \overline{x}^{(k+\frac{1}{2})} \\ \overline{y}^{(k+\frac{1}{2})} \end{pmatrix} = (\alpha I - \overline{S}) \begin{pmatrix} \overline{x}^{(k)} \\ \overline{y}^{(k)} \end{pmatrix} + \overline{b} \\ (\alpha I + \overline{S}) \begin{pmatrix} \overline{x}^{(k+1)} \\ \overline{y}^{(k+1)} \end{pmatrix} = (\alpha I - \overline{H}) \begin{pmatrix} \overline{x}^{(k+\frac{1}{2})} \\ \overline{y}^{(k+\frac{1}{2})} \end{pmatrix} + \overline{b} \end{cases} \tag{20}$$

where α is a given positive constant, The iteration matrix of the HSS iteration method is

$$\overline{M}_{(\alpha)} = (\alpha I + \overline{S})^{-1}(\alpha I - \overline{H})(\alpha I + \overline{H})^{-1}(\alpha I - \overline{S})$$

It then follows immediately that in the original variable it is easy to obtain the following preconditioned Hermitian/skew-Hermitian splitting (PHSS) iteration method [9].

$$\begin{cases} (\alpha P + H) \begin{pmatrix} x^{(k+\frac{1}{2})} \\ y^{(k+\frac{1}{2})} \end{pmatrix} = (\alpha P - S) \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + b \\ (\alpha P + S) \begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = (\alpha P - H) \begin{pmatrix} x^{(k+\frac{1}{2})} \\ y^{(k+\frac{1}{2})} \end{pmatrix} + b \end{cases} \tag{21}$$

where

$$H = \frac{1}{2}(\mathcal{A} + \mathcal{A}^T), S = \frac{1}{2}(\mathcal{A} - \mathcal{A}^T)$$

The iteration matrix of the PHSS iteration method is

$$\begin{aligned} &M_{(\alpha)} \\ =&(\alpha P + S)^{-1}(\alpha P - H)(\alpha P + H)^{-1}(\alpha P - S) \end{aligned} \tag{22}$$

Let ω, τ, α and β be four nonzero reals, $I_m \in R^{m\times m}$ and $I_n \in R^{n\times n}$ be the m-by-m and the n-by-n identity matrices, respectively, and

$$\Omega = \begin{pmatrix} \omega I_m & 0 \\ 0 & \tau I_n \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} \alpha I_m & 0 \\ 0 & \beta I_n \end{pmatrix}$$

Then we consider the following four-parameter PHSS iteration scheme:

$$\begin{cases} (\Omega P + H) \begin{pmatrix} x^{(k+\frac{1}{2})} \\ y^{(k+\frac{1}{2})} \end{pmatrix} = (\Omega P - S) \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + b \\ (\Lambda P + S) \begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = (\Lambda P - H) \begin{pmatrix} x^{(k+\frac{1}{2})} \\ y^{(k+\frac{1}{2})} \end{pmatrix} + b \end{cases} \tag{23}$$

More precisely, we have the following algorithmic description of this Four-parameter Preconditioned Hermitian and skew-Hermitian Splitting method(4-GPHSS).

Algorithm 2. *(The 4-GPHSS iteration method)*

Let $Q \in R^{n\times n}$ be a nonsingular and symmetric matrix. Given initial vectors $x^{(0)} \in R^m, y^{(0)} \in R^n$, and four relaxation factors $\omega, \tau, \alpha, \beta \neq 0$. For $k = 0, 1, 2, \ldots$ until the iteration sequence $\{(x^{(k)^T}, y^{(k)^T})^T\}$ is convergent, compute

$$\begin{cases} x^{(k+\frac{1}{2})} &= \frac{\omega}{1+\omega}x^{(k)} + \frac{1}{\omega+1}A^{-1}(p - By^{(k)}) \\ y^{(k+\frac{1}{2})} &= y^{(k)} + \frac{1}{\tau}Q^{-1}(B^T x^{(k)} - q) \\ y^{(k+1)} &= D^{-1}(\beta Q y^{(k+\frac{1}{2})} \\ & \quad +(1 - \frac{1}{\alpha})B^T x^{(k+\frac{1}{2})} + \frac{1}{\alpha}B^T A^{-1}p - q) \\ x^{(k+1)} &= \frac{\alpha-1}{\alpha}x^{(k+\frac{1}{2})} + \frac{1}{\alpha}A^{-1}(p - By^{(k+1)}) \end{cases} \tag{24}$$

here $D = \alpha^{-1}B^T A^{-1}B + \beta Q \in R^{n\times n}$.

Obviously, when $\omega = \alpha,\ \tau = \beta$, Algorithm 2 reduces to `GPHSS` method in[10]; when $\omega = \alpha = \tau = \beta$, it becomes the `PHSS` method[9].

By selecting different matrix Q, we can get some useful 4-GPHSS iterative algorithm. Such as $Q = \theta I, (\theta \neq 0), Q = B^T A^{-1} B, Q = B^T B$, in addition to the above special selection method, as long as you keep Q symmetric positive definite, can also have other method.

After straightforward computations we obtain the iterration matrix of the 4-GPHSS iteration method

$$\begin{aligned} & M_{(\omega,\tau,\alpha,\beta)} \\ = & (\Lambda P + S)^{-1}(\Lambda P - H)(\Omega P + H)^{-1}(\Omega P - S) \\ = & P^{-\frac{1}{2}} \overline{M}_{(\omega,\tau,\alpha,\beta)} P^{-\frac{1}{2}} \end{aligned}$$

where

$$\begin{aligned} & \overline{M}_{(\omega,\tau,\alpha,\beta)} \\ = & (\Lambda I + \overline{S})^{-1}(\Lambda I - \overline{H})(\Omega I + \overline{H})^{-1}(\Omega I - \overline{S}) \end{aligned}$$

Therefore, we have $\rho(M_{(\omega,\tau,\alpha,\beta)}) = \rho(\overline{M}_{(\omega,\tau,\alpha,\beta)})$.

Evidently, the 4-GPHSS iteration method can be equivalently rewritten as

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \mathcal{L}_{(\omega,\tau,\alpha,\beta)} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \mathcal{N}_{(\omega,\tau,\alpha,\beta)} \begin{pmatrix} p \\ -q \end{pmatrix} \quad (25)$$

where

$$\mathcal{L}_{(\omega,\tau,\alpha,\beta)} = \begin{pmatrix} \alpha A & B \\ -B^T & \beta Q \end{pmatrix}^{-1} \begin{pmatrix} \frac{\omega(\alpha-1)}{\omega+1} A & -\frac{\alpha-1}{\omega+1} B \\ \frac{\beta}{\tau} B^T & \beta Q \end{pmatrix} \quad (26)$$

and

$$\mathcal{N}_{(\omega,\tau,\alpha,\beta)} = \begin{pmatrix} \alpha A & B \\ -B^T & \beta Q \end{pmatrix}^{-1} \begin{pmatrix} \frac{\alpha+\omega}{\omega+1} I_m & 0 \\ 0 & \frac{\beta+\tau}{\tau} I_n \end{pmatrix} \quad (27)$$

Here, $\mathcal{L}_{(\omega,\tau,\alpha,\beta)}$ is the iteration matrix of the 4-GPHSS iteration. In fact, (25) may also result from the splitting

$$\mathcal{A} = M_{(\omega,\tau,\alpha,\beta)} - N_{(\omega,\tau,\alpha,\beta)} \quad (28)$$

of the coefficient matrix $\mathcal{A}$, with

$$M_{(\omega,\tau,\alpha,\beta)} = \begin{pmatrix} \frac{\alpha(\omega+1)}{\omega+\alpha} A & \frac{\omega+1}{\alpha+\omega} B \\ -\frac{\tau}{\beta+\tau} B^T & \frac{\beta\tau}{\beta+\tau} Q \end{pmatrix} \quad (29)$$

$$N_{(\omega,\tau,\alpha,\beta)} = \begin{pmatrix} \frac{\omega(\alpha-1)}{\omega+\alpha} A & -\frac{\alpha-1}{\alpha+\omega} B \\ \frac{\beta}{\beta+\tau} B^T & \frac{\beta\tau}{\beta+\tau} Q \end{pmatrix} \quad (30)$$

3 Convergence analysis

By straightforward computations, we can obtain an explicit expression of the iteration matrix $\mathcal{L}_{(\omega,\tau,\alpha,\beta)}$ in (25).

Lemma 3. *Consider the system of linear equations (10). Let $A \in R^{m\times m}$ be symmetric positive definite matrix, $B \in R^{m\times n}$ be of full column rank, and $\omega, \tau, \alpha, \beta > 0$ four given constants. Assume that $Q \in R^{m\times m}$ is a symmetric positive definite matrix. Then we partition $\mathcal{L}_{(\omega,\tau,\alpha,\beta)}$ in (25) as*

$$\mathcal{L}_{(\omega,\tau,\alpha,\beta)} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} \quad (31)$$

where

$$\begin{aligned} \mathcal{L}_{11} &= \frac{(\alpha-1)\omega}{\alpha(\omega+1)} I - T_{(\alpha,\beta,\omega,\tau)} A^{-1} B S_{(\alpha,\beta)}^{-1} B^T \\ \mathcal{L}_{12} &= -\frac{\beta(\alpha+\omega)}{\alpha(\omega+1)} A^{-1} B S_{(\alpha,\beta)}^{-1} Q \\ \mathcal{L}_{21} &= (\frac{\omega(\alpha-1)}{\alpha(\omega+1)} + \frac{\beta}{\tau}) Q S_{(\alpha,\beta)}^{-1} B^T A^{-1} \\ \mathcal{L}_{22} &= -\frac{\alpha-1}{\omega+1} I + \frac{\beta(\omega+\alpha)}{\omega+1} S_{(\alpha,\beta)}^{-1} Q \end{aligned}$$

and

$$T_{(\alpha,\beta,\omega,\tau)} = \frac{(\alpha-1)\omega}{\alpha^2(\omega+1)} + \frac{\beta}{\alpha\tau} \quad (32)$$

$$S_{(\alpha,\beta)}^{-1} = \beta Q + \frac{1}{\alpha} B^T A^{-1} B \quad (33)$$

Proof. Let

$$\begin{aligned} \overline{M}_{(\omega,\tau,\alpha,\beta)} = & \quad P^{-\frac{1}{2}} M_{(\omega,\tau,\alpha,\beta)} P^{-\frac{1}{2}} & (34) \\ = & \begin{pmatrix} \frac{\alpha(\omega+1)}{\omega+\alpha} I & \frac{\omega+1}{\alpha+\omega} \overline{B} \\ -\frac{\tau}{\beta+\tau} \overline{B}^T & \frac{\beta\tau}{\beta+\tau} I \end{pmatrix} & (35) \end{aligned}$$

$$\begin{aligned} \overline{N}_{(\omega,\tau,\alpha,\beta)} = & \quad P^{-\frac{1}{2}} N_{(\omega,\tau,\alpha,\beta)} P^{-\frac{1}{2}} & (36) \\ = & \begin{pmatrix} \frac{\omega(\alpha-1)}{\omega+\alpha} I & -\frac{\alpha-1}{\alpha+\omega} \overline{B} \\ \frac{\beta}{\beta+\tau} \overline{B}^T & \frac{\beta\tau}{\beta+\tau} I \end{pmatrix} & (37) \end{aligned}$$

where the matrices P and $\overline{B}$ are define in (12)(13). Then

$$\overline{\mathcal{L}}_{(\omega,\tau,\alpha,\beta)} = \begin{pmatrix} \overline{\mathcal{L}}_{11} & \overline{\mathcal{L}}_{12} \\ \overline{\mathcal{L}}_{21} & \overline{\mathcal{L}}_{22} \end{pmatrix} = \overline{M}_{(\omega,\tau,\alpha,\beta)}^{-1} \overline{N}_{(\omega,\tau,\alpha,\beta)} \quad (38)$$

with

$$\overline{\mathcal{L}}_{11} = \frac{(\alpha-1)\omega}{\alpha(\omega+1)}I - (\frac{(\alpha-1)\omega}{\alpha^2(\omega+1)} + \frac{\beta}{\alpha\tau})\overline{B}\,\overline{S}^{-1}_{(\alpha,\beta)}\overline{B}^T$$
$$\overline{\mathcal{L}}_{12} = -\frac{\beta(\alpha+\omega)}{\alpha(\omega+1)}\overline{B}\,\overline{S}^{-1}_{(\alpha,\beta)}$$
$$\overline{\mathcal{L}}_{21} = (\frac{\omega(\alpha-1)}{\alpha(\omega+1)} + \frac{\beta}{\tau})\overline{S}^{-1}_{(\alpha,\beta)}\overline{B}^T$$
$$\overline{\mathcal{L}}_{22} = -\frac{\alpha-1}{\omega+1}I + \frac{\beta(\omega+\alpha)}{\omega+1}\overline{S}^{-1}_{(\alpha,\beta)}$$

where

$$\overline{S}^{-1}_{(\alpha,\beta)} = \beta I + \frac{1}{\alpha}\overline{B}^T\overline{B} \tag{39}$$

Then from (28) we have

$$\begin{aligned}\mathcal{L}_{(\omega,\tau,\alpha,\beta)} &= M^{-1}_{(\omega,\tau,\alpha,\beta)}N_{(\omega,\tau,\alpha,\beta)} &(40)\\ &= P^{-\frac{1}{2}}\overline{M}^{-1}_{(\omega,\tau,\alpha,\beta)}\overline{N}_{(\omega,\tau,\alpha,\beta)}P^{\frac{1}{2}} &(41)\\ &= P^{-\frac{1}{2}}\overline{\mathcal{L}}_{(\omega,\tau,\alpha,\beta)}P^{\frac{1}{2}} &(42)\end{aligned}$$

the result follows immediately. □

Based on Lemma (3), we can further obtain the eigenvalues of the iteration matrix $\mathcal{L}_{(\omega,\tau,\alpha,\beta)}$ of the 4-GPHSS method.

Lemma 4. *Let the conditions in Lemma 3 be satisfied. If $\overline{\sigma}_k(k = 1, 2, \ldots n)$ are the positive singular values of the matrix $\overline{B} = A^{-\frac{1}{2}}BQ^{-\frac{1}{2}}$, then the eigenvalues of the iteration matrix $\mathcal{L}_{(\omega,\tau,\alpha,\beta)}$ of the 4-GPHSS iteration method are $\frac{\omega(\alpha-1)}{\alpha(\omega+1)}$ with multiplicity $m-n$, and*

$$\frac{d_k \pm \sqrt{d_k^2 - 4e_k}}{2\tau(\omega+1)(\alpha\beta+\overline{\sigma}_k^2)}, k = 1, 2, \ldots n. \tag{43}$$

where

$$d_k = \omega\tau\beta(\alpha+1)+\alpha\beta\tau(\omega-1)-[\beta(\omega+1)+\tau(\alpha-1)]\overline{\sigma}_k^2$$

and

$$e_k = \beta\tau(\omega+1)(\alpha-1)(\omega\tau+\overline{\sigma}_k^2)(\alpha\beta+\overline{\sigma}_k^2)$$

Proof. From (42) we know that $\mathcal{L}_{(\omega,\tau,\alpha,\beta)}$ is similar to $\overline{\mathcal{L}}_{(\omega,\tau,\alpha,\beta)}$ of (38). Therefore, we only need to compute the eigenvalues of the matrix $\overline{\mathcal{L}}_{(\omega,\tau,\alpha,\beta)}$.

Let $\overline{B} = \overline{U}\,\overline{\Sigma}_1\overline{V}^T$ be the singular value decomposition of the matrix $\overline{B}$, where $\overline{U} \in R^{m\times m}$ and $\overline{V} \in R^{n\times n}$ are unitary matrices, and

$$\overline{\Sigma}_1 = \begin{pmatrix}\overline{\Sigma}\\0\end{pmatrix}, \overline{\Sigma} = diag(\overline{\sigma}_1, \overline{\sigma}_2, \ldots, \overline{\sigma}_n) \in R^{n\times n}. \tag{44}$$

Then after a few computation, we have

$$\overline{S}_{(\alpha,\beta)} = \overline{V}(\beta I + \frac{1}{\alpha}\overline{\Sigma}^2)\overline{V}^T = \overline{V}D\overline{V}^T$$

and therefore

$$\overline{\mathcal{L}}_{11} = \overline{U}\begin{pmatrix}\frac{(\alpha-1)\omega}{\alpha(\omega+1)}I - tD^{-1}\overline{\Sigma}^2 & 0\\ 0 & \frac{\omega(\alpha-1)}{\alpha(\omega+1)}I\end{pmatrix}\overline{U}^T$$
$$\overline{\mathcal{L}}_{12} = \overline{U}\begin{pmatrix}-\frac{\beta(\alpha+\omega)}{\alpha(\omega+1)}\overline{\Sigma}D^{-1}\\0\end{pmatrix}\overline{V}^T$$
$$\overline{\mathcal{L}}_{21} = \overline{V}((\frac{\omega(\alpha-1)}{\alpha(\omega+1)} + \frac{\beta}{\tau})\overline{\Sigma}D^{-1}, 0)\overline{U}^T$$
$$\overline{\mathcal{L}}_{22} = \overline{V}(-\frac{\alpha-1}{\omega+1}I + \frac{\beta(\omega+\alpha)}{\omega+1}D^{-1})\overline{V}^T$$

Define $\overline{Q} = diag(\overline{U}, \overline{V})$, then

$$\overline{Q}^T\overline{\mathcal{L}}_{(\omega,\tau,\alpha,\beta)}\overline{Q} = \begin{pmatrix}rI - tD^{-1}\overline{\Sigma}^2 & 0 & -\frac{\beta(\alpha+\omega)}{\alpha(\omega+1)}\overline{\Sigma}D^{-1}\\ 0 & \frac{\omega(\alpha-1)}{\alpha(\omega+1)}I & 0\\ (r+\frac{\beta}{\tau})\overline{\Sigma}D^{-1} & 0 & -\frac{\alpha-1}{\omega+1}I + \frac{\beta(\omega+\alpha)}{\omega+1}D^{-1}\end{pmatrix}$$

where $t = \frac{(\alpha-1)\omega}{\alpha^2(\omega+1)} + \frac{\beta}{\alpha\tau}$, $r = \frac{(\alpha-1)\omega}{\alpha(\omega+1)}$.

It follows immediately that the eigenvalues of the matrix $\overline{\mathcal{L}}_{(\omega,\tau,\alpha,\beta)}$ are just $\frac{\omega(\alpha-1)}{\alpha(\omega+1)}$ with multiplicity $m-n$, and those of the matrix

$$\begin{pmatrix}rI - tD^{-1}\overline{\Sigma}^2 & -\frac{\beta(\alpha+\omega)}{\alpha(\omega+1)}\overline{\Sigma}D^{-1}\\ (r+\frac{\beta}{\tau})\overline{\Sigma}D^{-1} & -\frac{\alpha-1}{\omega+1}I + \frac{\beta(\omega+\alpha)}{\omega+1}D^{-1}\end{pmatrix}$$

which are the same as the matrices $\frac{1}{(\omega+1)(\alpha\beta+\overline{\sigma}_k^2)}\overline{\mathcal{L}}_k$, $k = 1, 2, \ldots, n$. where

$$\overline{\mathcal{L}}_k = \begin{pmatrix}\omega\beta(\alpha-1) - \frac{\beta(\omega+1)}{\tau}\overline{\sigma}_k^2 & -\beta(\omega+\alpha)\overline{\sigma}_k\\ (\omega(\alpha-1) + \frac{\alpha\beta(\omega+1)}{\tau})\overline{\sigma}_k & \alpha\beta(\omega+1) - (\alpha-1)\overline{\sigma}_k^2\end{pmatrix}$$

The two eigenvalues of the matrix $\overline{\mathcal{L}}_k$ are two roots of the quadratic equations

$$\lambda^2 - [2\alpha\beta\omega + (\omega-\alpha)\beta - \frac{\beta(\omega+1)+\tau(\alpha-1)}{\tau}\overline{\sigma}_k^2]\lambda + \frac{\beta}{\tau}(\omega+1)(\alpha-1)(\omega\tau+\overline{\sigma}_k^2)(\alpha\beta+\overline{\sigma}_k^2) = 0$$

or in other words,

$$\lambda = \frac{1}{2\tau}\left(d_k \pm \sqrt{d_k^2 - 4e_k}\right) \tag{45}$$

We know that the eigenvalues of the matrix $\mathcal{L}_{(\omega,\tau,\alpha,\beta)}$ are $\frac{\omega(\alpha-1)}{\alpha(\omega+1)}$ with multiplicity $m-n$, and

$$\frac{d_k \pm \sqrt{d_k^2 - 4e_k}}{2\tau(\omega+1)(\alpha\beta+\overline{\sigma}_k^2)}, k = 1, 2, \dots n. \quad (46)$$

This completes our proof. □

Lemma 5. *[11] Consider the quadratic equation $x^2 - bx + c = 0$, when b and c are real numbers, Both roots of the equation are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

Theorem 6. *Consider the system of linear equations (10). Let $A \in R^{m\times m}$ be symmetric positive definite matrix, $B \in R^{m\times n}$ be of full column rank, and $\omega, \tau, \alpha, \beta > 0$ four given constants. Assume that $Q \in R^{m\times m}$ is a symmetric positive definite matrix, If $\omega\tau = \alpha\beta$, Then*

$$\rho(\mathcal{L}_{(\omega,\tau,\alpha,\beta)}) < 1, \forall\omega > 0, \tau > 0, \alpha > 0, \beta > 0. \quad (47)$$

i.e. the 4-GPHSS iteration converges to the exact solution of the system of linear equations (10) .

Proof. According to lemma 4, we know that the eigenvalues of the iteration matrix $\mathcal{L}_{(\omega,\tau,\alpha,\beta)}$ are $\frac{\omega(\alpha-1)}{\alpha(\omega+1)}$ with multiplicity $m - n$, and roots of the quadratic equations

$$\lambda^2 + b\lambda + c = 0$$

where

$$b = \frac{[\beta(\omega+1) + \tau(\alpha-1)]\overline{\sigma}_k^2 - 2\alpha\beta\omega\tau - (\alpha-\omega)\beta\tau}{\tau(\omega+1)(\alpha\beta+\overline{\sigma}_k^2)}$$

and

$$c = \frac{\beta}{\tau}\frac{\alpha-1}{\omega+1}\frac{\omega\tau+\overline{\sigma}_k^2}{\alpha\beta+\overline{\sigma}_k^2}$$

obviously, we have

$$|\frac{\omega(\alpha-1)}{\alpha(\omega+1}| = |\frac{\alpha-1}{\alpha}||\frac{\omega}{\omega+1}| < 1,$$

and if $\omega\tau = \alpha\beta$, then after a few computation, we have

$$\begin{aligned}|\frac{\beta}{\tau}\frac{\alpha-1}{\omega+1}\frac{\omega\tau+\overline{\sigma}_k^2}{\alpha\beta+\overline{\sigma}_k^2}| &= |\frac{\omega(\alpha-1)}{\alpha(\omega+1}| \\ &= |\frac{\alpha-1}{\alpha}||\frac{\omega}{\omega+1}| < 1\end{aligned}$$

and

$$\begin{aligned}&|\frac{[\beta(\omega+1)+\tau(\alpha-1)]\overline{\sigma}_k^2 - 2\alpha\beta\omega\tau - (\alpha-\omega)\beta\tau}{\tau(\omega+1)(\alpha\beta+\overline{\sigma}_k^2)}| \\ =&|\frac{2\alpha\beta\omega\tau + (\alpha-\omega)\beta\tau + (\tau-\beta-\omega\beta-\alpha\tau)\overline{\sigma}_k^2}{\tau(\omega+1)(\alpha\beta+\overline{\sigma}_k^2)}| \\ <&|\frac{2\alpha\beta\omega\tau + (\alpha-\omega)\beta\tau + (\tau-\beta+\omega\tau+\alpha\beta)\overline{\sigma}_k^2}{\tau(\omega+1)(\alpha\beta+\overline{\sigma}_k^2)}| \\ =&1 + \frac{\beta}{\tau}\frac{\alpha-1}{\omega+1}\frac{\omega\tau+\overline{\sigma}_k^2}{\alpha\beta+\overline{\sigma}_k^2} \\ =&1 + c\end{aligned}$$

By lemma 5, we have $|\lambda| < 1$, in other words, when $\omega\tau = \alpha\beta$, $\rho(\mathcal{L}_{(\omega,\tau,\alpha,\beta)}) < 1, \forall\omega, \tau, \alpha, \beta > 0$. □

Theorem 7. *[10]Consider the system of linear equations (10). Let $A \in R^{m\times m}$ be symmetric positive definite matrix, $B \in R^{m\times n}$ be of full column rank, and $\omega, \tau, \alpha, \beta > 0$ four given constants. Assume that $Q \in R^{m\times m}$ is a symmetric positive definite matrix, If $\omega = \alpha, \tau = \beta$, Then*

$$\rho(\mathcal{L}_{(\omega,\tau)}) < 1, \forall\omega > 0, \tau > 0. \quad (48)$$

i.e. the GPHSS iteration converges to the exact solution of the system of linear equations (10) .

Proof. From (40) we know that $\mathcal{L}_{(\omega,\tau,\alpha,\beta)}$ is similar to $\overline{\mathcal{L}}_{(\omega,\tau,\alpha,\beta)}$ of (38). Therefore, we only need to compute the eigenvalues of the matrix $\overline{\mathcal{L}}_{(\omega,\tau,\alpha,\beta)}$.

Based on lemma 4, when $\omega = \alpha, \tau = \beta$, we can obtain that the eigenvalues of the iteration matrix $\overline{\mathcal{L}}_{(\omega,\tau)}$ are $\frac{\omega-1}{\omega+1}$ with multiplicity $m - n$, and

$$\lambda_{1,2}^{(k)} = \frac{\omega(\omega\tau - \overline{\sigma}_k^2) \pm \sqrt{(\tau\omega + \overline{\sigma}_k^2)^2 - 4\omega^3\tau\overline{\sigma}_k^2}}{(\omega+1)(\tau\omega+\overline{\sigma}_k^2)} \quad (49)$$

$k = 1, 2, \dots, n.$

We know that when $\tau\omega + \overline{\sigma}_k^2 > 2\omega\sqrt{\tau\omega}\sigma_k$,

$$\begin{aligned}|\lambda_{1,2}^{(k)}| &= \frac{\omega|\omega\tau - \overline{\sigma}_k^2| + \sqrt{(\tau\omega + \overline{\sigma}_k^2)^2 - 4\omega^3\tau\overline{\sigma}_k^2}}{(\omega+1)(\tau\omega+\overline{\sigma}_k^2)} \\ &= \frac{\omega}{\omega+1}\left(\frac{|\tau\omega+\overline{\sigma}_k^2|}{\tau\omega+\overline{\sigma}_k^2} + \sqrt{\frac{1}{\omega^2} - \frac{4\omega^3\tau\overline{\sigma}_k^2}{(\tau\omega+\overline{\sigma}_k^2)^2}}\right) \\ &< \frac{\omega}{\omega+1}(1 + \frac{1}{\omega}) = 1\end{aligned}$$

when $\tau\omega + \overline{\sigma}_k^2 \le 2\omega\sqrt{\tau\omega}\sigma_k$,

$$|\lambda_{1,2}^{(k)}| = \sqrt{\lambda_1^{(k)}\lambda_2^{(k)}} = \sqrt{\frac{\omega-1}{\omega+1}} < 1$$

Obviously, we have

$$\frac{|\omega - 1|}{\omega + 1} < 1$$

then

$$\rho(M_{(\omega,\tau)}) < 1, \forall \omega > 0, \tau > 0,$$

i.e. the GPHSS iteration converges to the exact solution of the system of linear equations (10). □

Corollary 8. *[9] Consider the system of linear equations (10). Let $A \in R^{m\times m}$ be symmetric positive definite matrix, $B \in R^{m\times n}$ be of full column rank, and $\omega, \tau, \alpha, \beta > 0$ four given constants. Assume that $Q \in R^{m\times m}$ is a symmetric positive definite matrix, If $\omega = \alpha = \tau = \beta$, Then*

$$\rho(\mathcal{L}(\alpha)) < 1, \forall \alpha > 0. \tag{50}$$

i.e. the PHSS iteration converges to the exact solution of the system of linear equations (10).

The optimal iteration parameter and the corresponding asymptotic convergence factor of the GPHSS iteration method are described in the following theorem.

Theorem 9. *Consider the system of linear equations (10). Let $A \in R^{m\times m}$ be symmetric positive definite matrix, $B \in R^{m\times n}$ be of full column rank, and $\omega, \tau, \alpha, \beta > 0$ four given constants. Assume that $Q \in R^{m\times m}$ is a symmetric positive definite matrix, If $\omega = \alpha, \tau = \beta, \overline{\sigma}_k (k = 1, 2, \ldots n)$ are the positive singular values of the matrix $\overline{B} = A^{-\frac{1}{2}} B Q^{-\frac{1}{2}}$, and $\sigma_{min} = \underset{1\le k\le n}{min}\{\overline{\sigma}_k\}, \sigma_{max} = \underset{1\le k\le n}{max}\{\overline{\sigma}_k\}$. Then for the GPHSS iteration converges to the exact solution of the system of linear equations (10), the optimal value of the iteration parameter ω, τ is given by*

$$\begin{aligned}\omega^* &= arg\min_{(\omega,\tau)} \rho(M_{(\omega,\tau)}) \\ &= \frac{\sigma_{max} + \sigma_{min}}{2\sqrt{\sigma_{max}\sigma_{min}}} \\ \tau^* &= arg\min_{(\omega,\tau)} \rho(M_{(\omega,\tau)}) \\ &= \frac{2\sigma_{max}\sigma_{min}\sqrt{\sigma_{max}\sigma_{min}}}{\sigma_{max} + \sigma_{min}}\end{aligned}$$

and correspondingly

$$\rho(M_{(\omega^*,\tau^*)}) = \frac{\sqrt{\sigma_{max}} - \sqrt{\sigma_{min}}}{\sqrt{\sigma_{max}} + \sqrt{\sigma_{min}}}$$

Proof. According to lemma 4, we know

$$|\lambda| = \begin{cases} \frac{|\omega-1|}{\omega+1}, \ or \\ \frac{\omega}{\omega+1}\left(\frac{|\omega\tau - \overline{\sigma}_k^2|}{\omega\tau + \overline{\sigma}_k^2} + \sqrt{\frac{1}{\omega^2} - \frac{4\omega\tau\overline{\sigma}_k^2}{(\omega\tau + \overline{\sigma}_k^2)^2}}\right), \\ for \ \omega\tau + \overline{\sigma}_k^2 > 2\omega\sqrt{\omega\tau}\overline{\sigma}_k \ , or \\ \sqrt{\frac{\omega-1}{\omega+1}}, \ for \ \omega\tau + \overline{\sigma}_k^2 \le 2\omega\sqrt{\omega\tau}\overline{\sigma}_k, \end{cases} \tag{51}$$

for $k = 1, 2, \cdots n$.

We observe that the following two facts hold true:

1. when $\omega \le 1$, $\omega\tau + \overline{\sigma}_k^2 > 2\omega\sqrt{\omega\tau}\overline{\sigma}_k, k = 1, 2, \cdots n$;
2. when $\omega > 1$,
   a. $\omega\tau + \overline{\sigma}_k^2 > 2\omega\sqrt{\omega\tau}\overline{\sigma}_k$, if and only if $0 < \overline{\sigma}_k < \alpha_-$ or $\overline{\sigma}_k > \alpha_+$, $k \in \{1, 2, \cdots n\}$;
   b. $\omega\tau + \overline{\sigma}_k^2 \le 2\omega\sqrt{\omega\tau}\overline{\sigma}_k$, if and only if $\alpha_- \le \overline{\sigma}_k \le \alpha_+$, $k \in \{1, 2, \cdots n\}$;
   c. $\frac{\omega-1}{\omega+1} < \sqrt{\frac{\omega-1}{\omega+1}}$

where

$$\alpha_- = \omega\sqrt{\omega\tau} - \sqrt{\omega\tau(\omega^2 - 1)}$$

$$\alpha_+ = \omega\sqrt{\omega\tau} + \sqrt{\omega\tau(\omega^2 - 1)}$$

Let

$$\begin{aligned} &\Theta(\omega, \tau, \sigma) \\ &= \frac{\omega}{\omega+1}\left(\frac{|\omega\tau - \sigma^2|}{\omega\tau + \sigma^2} + \sqrt{\frac{1}{\omega^2} - \frac{4\omega\tau\sigma^2}{(\omega\tau + \sigma^2)^2}}\right)\end{aligned}$$

Then base on the facts (1) and (2) we easily see that

$$\rho(M_{(\omega,\tau)}) = \begin{cases} \max\{\frac{1-\omega}{1+\omega}, \max\limits_{1\le k\le n} \Theta(\omega, \tau, \overline{\sigma}_k)\}, \\ for \ \omega \le 1, \\ \max\{\sqrt{\frac{\omega-1}{\omega+1}}, \max\limits_{\substack{\overline{\sigma}_k < \alpha_- \\ or \overline{\sigma}_k > \alpha_+}} \Theta(\omega, \tau, \overline{\sigma}_k)\}, \\ for \ \omega > 1. \end{cases} \tag{52}$$

For any fixed $\beta > 0$, we define two functions $\theta_1, \theta_2 : (0, +\infty) \to (0, +\infty)$ by

$$\theta_1(t) = \frac{\beta - t}{\beta + t}, \ \ \theta_2(t) = -\frac{4\beta t}{(\beta + t)^2}.$$

After straightforward computations we obtain

$$\theta_1'(t) = -\frac{2\beta}{(\beta + t)^2}, \ \ \theta_2'(t) = \frac{4\beta(t - \beta)}{(\beta + t)^3}.$$

It then follows that:

i. $\max\limits_{1\le k\le n}\Theta(\omega,\tau,\overline{\sigma}_k)$
$= max\{\Theta(\omega,\tau,\sigma_{min}),\Theta(\omega,\tau,\sigma_{max})\},$
$for\ \omega\le 1,\ and\ \sigma_{min}^2\le\omega\tau\le\sigma_{max}^2$

ii. $\max\limits_{\overline{\sigma}_k<\alpha_-\ or\ \overline{\sigma}_k>\alpha_+}\Theta(\omega,\tau,\overline{\sigma}_k)$
$= max\{\Theta(\omega,\tau,\sigma_{min}),\Theta(\omega,\tau,\sigma_{max})\},$
for $\omega>1$, and $\sigma_{min}<\alpha_-\ or\ \sigma_{max}>\alpha_+$

Therefore, when $\omega\le 1$, the optimal parameter ω^*,τ^* must satisfy $\sigma_{min}^2\le\omega^*\tau^*\le\sigma_{max}^2$, and either of the following three conditions:

1. $\frac{1-\omega^*}{1+\omega^*}=\Theta(\omega^*,\tau^*,\sigma_{min})\ge\Theta(\omega^*,\tau^*,\sigma_{max})$
2. $\frac{1-\omega^*}{1+\omega^*}=\Theta(\omega^*,\tau^*,\sigma_{max})\ge\Theta(\omega^*,\tau^*,\sigma_{min})$
3. $\Theta(\omega^*,\tau^*,\sigma_{min})=\Theta(\omega^*,\tau^*,\sigma_{max})\ge\frac{1-\omega^*}{1+\omega^*}$

and when $\omega>1$, the optimal parameter ω^*,τ^* must satisfy $\sigma_{min}<\alpha_-^*\ or\ \sigma_{max}>\alpha_+^*$, and either of the following three conditions:

a.
$$\sqrt{\frac{\omega^*-1}{\omega^*+1}}=\Theta(\omega^*,\tau^*,\sigma_{min})\ge\Theta(\omega^*,\tau^*,\sigma_{max})$$

b.
$$\sqrt{\frac{\omega^*-1}{\omega^*+1}}=\Theta(\omega^*,\tau^*,\sigma_{max})\ge\Theta(\omega^*,\tau^*,\sigma_{min})$$

c.
$$\Theta(\omega^*,\tau^*,\sigma_{min})=\Theta(\omega^*,\tau^*,\sigma_{max})\ge\sqrt{\frac{\omega^*-1}{\omega^*+1}}$$

where

$$\alpha_-^*=\omega^*\sqrt{\omega^*\tau^*}-\sqrt{\omega^*\tau^*(\omega^{*2}-1)},$$

$$\alpha_+^*=\omega^*\sqrt{\omega^*\tau^*}+\sqrt{\omega^*\tau^*(\omega^{*2}-1)}.$$

By straightforwardly solving the inequalities(1)-(3), and(a)-(c), we can obtain $\omega^*\tau^*=\sigma_{min}\sigma_{max}$, we can further obtain the optimal parameters

$$\omega^*=\frac{\sigma_{max}+\sigma_{min}}{2\sqrt{\sigma_{max}\sigma_{min}}}\tag{53}$$

$$\tau^*=\frac{2\sigma_{max}\sigma_{min}\sqrt{\sigma_{max}\sigma_{min}}}{\sigma_{max}+\sigma_{min}}\tag{54}$$

The by substituting ω^*,τ^*into (52), we obtain

$$\rho(M_{(\omega^*,\tau^*)})=\frac{\sqrt{\sigma_{max}}-\sqrt{\sigma_{min}}}{\sqrt{\sigma_{max}}+\sqrt{\sigma_{min}}}\tag{55}$$

□

Corollary 10. *[9]Consider the system of linear equations (10). Let $A\in R^{m\times m}$ be symmetric positive definite matrix, $B\in R^{m\times n}$ be of full column rank, and $\omega,\tau,\alpha,\beta>0$ four given constants. Assume that $Q\in R^{m\times m}$ is a symmetric positive definite matrix, If $\omega=\alpha=\tau=\beta,\overline{\sigma}_k(k=1,2,\dots n)$ are the positive singular values of the matrix $\overline{B}=A^{-\frac{1}{2}}BQ^{-\frac{1}{2}}$, and $\sigma_{min}=\underset{1\le k\le n}{min}\{\overline{\sigma}_k\},\sigma_{max}=\underset{1\le k\le n}{max}\{\overline{\sigma}_k\}$ Then for the PHSS iteration converges to the exact solution of the system of linear equations (10), the optimal value of the iteration parameter α is given by*

$$\alpha^*=arg\min_{\alpha}\rho(M(\alpha))=\sqrt{\sigma_{min}\sigma_{max}},$$

and correspondingly

$$\rho(M(\alpha^*))=\frac{\sigma_{max}-\sigma_{min}}{\sigma_{max}+\sigma_{min}}\tag{56}$$

4 Numerical examples

In this section, we use a numerical example to further examine the effectiveness and show the advantages of the 4-GPHSS method over the PHSS method and GPHSS method.

This example is a system of purely algebraic equations discussed in[12].The matrices A and B are defined as follows:

$$A=(a_{i,j})_{m\times m}=\begin{cases}i+1, & i=j,\\ 1, & |i-j|=1,\\ 0, & otherwise.\end{cases}$$

$$B=(b_{i,j})_{m\times n}=\begin{cases}j, & i=j+m-n,\\ 0, & otherwise.\end{cases}$$

We report the corresponding the number of iterations(denoted by IT), the spectral radius(denoted by ρ), the time needed for convergence(denoted by CPU)and the norm of absolute error vectors(denoted by RES)by choosing $Q=B^TB$ for all the SOR-like, 4-GPHSS, GPHSS and PHSS methods. The stopping criterion are used in the computations,

$$\frac{||r_k||_2}{||r_0||_2}<10^{-6}$$

where

$$r_k=\begin{pmatrix}p\\-q\end{pmatrix}-\begin{pmatrix}A & B\\-B^T & 0\end{pmatrix}\begin{pmatrix}x^{(k)}\\y^{(k)}\end{pmatrix}$$

and $\{(x^{(k)^T},y^{(k)^T})^T\}$ is the kth iteration for each of the methods.

The optimum parameter for the SOR-like, PHSS and GPHSS method were determined according to had and given the results. We chose the parameters for the 4-GPHSS method by trial and error. All the computations were performed on an Intel E2180 2.0GHZ CPU, 2.0G Memory, Windows XP system using Matlab 7.0.

From the below numerical results, we can see that the iteration number and the time in the 4-GPHSS method and GPHSS method are less than that in the SOR-like, PHSS method. From the IT and CPU two rows, we know that we can decrease the number of iterations and the time needed for convergence by choosing four suitable parameters. However, we only give the two optimum parameters, further theoretical considerations regarding the determination of the four optimum parameters for the 4-GPHSS method and numerical computations are needed before any firm conclusions can be drawn. Further work in this direction is underway.

Table 1: Iteration parameters for the PHSS method.

m	50	200	400
n	40	150	300
m+n	90	350	700
ω_{opt}	0.2037	0.0993	0.0705
IT	99	210	306
CPU(s)	0.2507	21.5841	311.6206
$\rho(M_{(\omega_{opt})})$	0.3653	0.3276	0.3320
RES	9.01E-7	9.47E-7	9.90E-7

Table 2: Iteration parameters for the GPHSS method.

m	50	200	400
n	40	150	300
m+n	90	350	700
ω_{opt}	1.0742	1.0584	1.0601
τ_{opt}	0.0386	0.0093	0.0047
IT	10	9	9
CPU(s)	0.0492	1.1086	9.1771
$\rho(M_{(\omega_{opt},\tau_{opt})})$	0.1892	0.1685	0.1708
RES	3.18E-7	7.52E-7	8.75E-7

Table 3: Iteration number for the SOR-like method.

m	n	m+n	ω_{opt}	IT	CPU(s)
50	40	90	1.8201	292	0.422
200	150	350	1.9533	1032	32.75
400	300	700	1.9759	2066	546.281

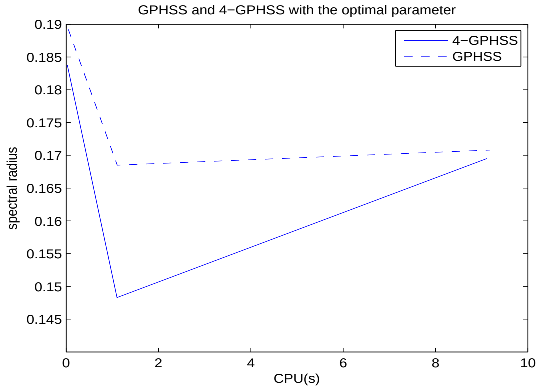

Table 4: Iteration parameters for the 4GPHSS method.

m	50	200	400
n	40	150	300
m+n	90	350	700
ω	1.0742	1.0584	1.0601
τ	0.0386	0.0093	0.0047
α	1.08	1.064	1.064
β	0.0384	0.00925	0.00468
IT	9	9	9
CPU(s)	0.0263	1.1026	9.1113
$\rho(M_{(\omega,\tau,\alpha,\beta)})$	0.1838	0.1483	0.1695
RES	9.63E-7	4.47E-7	5.07E-7

Table 5: Iteration parameters for the GPHSS method without the optimum parameters.

m	50	200	400
n	40	150	300
m+n	90	350	700
ω	1.2	1.2	1.2
τ	0.2	0.2	0.05
IT	55	103	102
CPU(s)	0.144	13.843	110.092
RES	8.89E-7	9.35E-7	9.82E-7

Table 6: Iteration parameters for the 4GPHSS method without the optimum parameters.

m	50	200	400
n	40	150	300
m+n	90	350	700
ω	1.2	1.2	1.2
τ	0.2	0.1	0.05
α	2.6	4.4	4
β	0.0923	0.0273	0.015
IT	26	35	37
CPU(s)	0.0812	4.376	40.355
RES	9.59E-7	5.68E-7	7.81E-7

Acknowledgements: The research was supported by the National Research Subject of Education Information Technology (grant No.126240641), the Research Topic of Higher Vocational Education in Zhejiang Province, P.R. China (grant No.YB1115), the Science and Technology Project of Zhejiang Industry Polytechnic College, P.R. China (grant No.1023401092012014).

References:

[1] M. Benzi, G. H. Golub and J. Liesen, Numerical solution of saddle point problems, *Acta Numerica*, vol.14, 2005, pp. 1-137.

[2] J. H. Bramble, J. E. Pasciak and A. T. Vassilev, Analysis of the inexact Uzawa algorithm for saddle point problem, *SIAM J.Numer.Anal.*, vol.34, 1997, No.3, pp. 1072-1092.

[3] G. H. Golub, X. Wu, J.-Y. Yuan, SOR-like methods for augmented systems, *BIT*, vol. 41, 2001, pp. 71-85.

[4] Z. Z. Bai, B. N. Parlett., Z.-Q. Wang, On generalized successive overrelaxation methods for augmented linear systems,*Numer. Math.*, vol.102, 2005, pp. 1-38.

[5] J. C. Li, X. Kong. Optimum parameters of GSOR-1ike methods for the augmented systems, *Applied Mathematics and Computation*, vol.204, 2008, No.1, 2008, pp. 150-161.

[6] C. P. Pan, H. Y. Wang, The polynomial Acceleration of GSOR Iterative Algorithms for Large Sparse Saddle point problems, *Chinese Journal of Engineering Mathematics*, vol.28, 2011, No.3, pp. 307-314.

[7] C. P. Pan, H. Y. Wang, W. L.Zhao, On generalized SSOR iterative method for Saddle point problems, *Journal of Mathematics*, vol.31, 2011, No.3, pp. 569-574.

[8] Z. Z. Bai, G. H. Golub and M. K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems,*SIAM Journal of Matrix Analysis and Applications*, vol.24, 2003, pp. 603-626.

[9] Z. Z. Bai, G. H. Golub and J. Y. Pan, Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, *Numer. Math.*, vol.98, 2004, pp. 1-32.

[10] C. P. Pan, H. Y. Wang, On generalized Preconditioned Hermitian and skew-Hermitian splitting methods for saddle point problems, *Journal On Numerical Methods and Computer Applications*, vol.32, 2011, No.3 pp. 174-182.

[11] D. M. Young, *Iterative Solutions of Large Linear Systems*, Academic Press, New York, 1971.

[12] Q. Hu, and J. Zou, An iterative method with variable relaxation parameters for saddle-point problems, *SIAM Journal of Matrix Analysis and Applications*, vol. 23, 2001, pp. 317-338.