An extended $(G'/G)-$ expansion method and its applications to the (2+1)-dimensional nonlinear evolution equations

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Abstract: By using an extended $(\frac{G'}{G})$-expansion method, we construct the traveling wave solutions of the (2+1)-dimensional Painleve integrable Burgers equations, the (2+1)-dimensional Nizhnik-Novikov-Veselov equations, the (2+1)-dimensional Boiti-Leon-Pempinelli equations and the (2+1)-dimensional dispersive long wave equations, where G satisfies the second order linear ordinary differential equation. By using this method, new exact solutions involving parameters, expressed by hyperbolic and trigonometric function solutions are obtained. When the parameters are taken as special values, some solitary wave solutions are derived from the hyperbolic function solutions.

Key–Words: Extended $(\frac{G'}{G})$-expansion method; Traveling wave solutions; Painleve integrable Burgers equations; Nizhnik-Novikov-Veselov equations; Boiti-Leon-Pempinelli equations; Dispersive long wave equations; Solitary solutions

1 Introduction

In the nonlinear science, many important phenomena in various fields can be described by the nonlinear evolution equations (NLEEs). Searching for exact soliton solutions of NLEEs plays an important and a significant role in the study on the dynamics of those phenomena. With the development of soliton theory, many powerful methods for obtaining the exact solutions of NLEEs have been presented by many authors (see for example [1-33]), such as the homogeneous balance method [17], the tanh method [5, 20, 28], the sinh-cosh and sine-cosine method [16], the inverse scattering transform [1], the exp-function expansion method [2, 7], the Backlund transform [11, 12], the modified extended Fan subequation method [24], the Jacobi elliptic function expansion method [22], the generalized projective Riccati equation expansion method [10, 19, 23], the truncated Painlev expansion method [8, 33], The auxiliary equation method [13, 14] and so on. More recently, the (G'/G)-expansion method [18, 26, 27, 29-31] has been proposed to obtain traveling wave solutions. This method is firstly proposed by Wang et al [18] for which the traveling wave solutions of the nonlinear evolution equations are obtained. This method has been extended to solve difference-differential equations [9, 32]. The improved (G'/G)-expansion method has been used in [25, 29]. Recently, Guo et al [6], Zayed et al [30] have obtained the exact traveling wave solutions of some nonlinear PDEs using an extended (G'/G)-expansion method.

In the present article, we use the extended (G'/G)-expansion method which is proposed in [6, 30] to derive traveling wave solutions for the (2+1)-dimensional Painleve integrable Burgers equations, the (2+1)-dimensional Nizhnik-Novikov-Veselov equations, the (2+1)-dimensional Boiti-Leon-Pempinelli equations and the (2+1)-dimensional dispersive long wave equations. The extended (G'/G)-expansion method used in this article can be applied to further equations such as difference-differential equations which can be done in forthcoming articles.

The rest of this article is organized as follows: In section 2, we describe the extended (G'/G)-expansion method. In section 3, we apply this method to four interesting nonlinear evolution systems. In section 4, some conclusions are given.

2 Description of an extended $(\frac{G'}{G})$-Expansion method

Consider the nonlinear partial differential equation in the form

$$F = F(u, u_t, u_x, u_y, u_{xy}, u_{tt}, ...) = 0, \quad (1)$$

where $u = u(x, y, t)$ is unknown functions, F is a polynomial in $u(x, y, t)$ and its partial derivatives. In the following, we give the main steps for solving Eq.(1) using an extended (G'/G)-expansion method [6, 30]:

Step 1. The traveling wave variable

$$u(x, y, t) = u(\xi), \xi = x + y - Vt, \tag{2}$$

where V is a constant to be determined later, permits us reducing Eq.(1) to an ODE in the form

$$P(u, u', u'', ...) = 0, \tag{3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives.

Step 2. Suppose the solution of Eq.(3) can be expressed in (G'/G) as follows:

$$\begin{aligned} u(\xi) &= a_0 + \sum_{i=1}^{n} a_i \left(\frac{G'}{G}\right)^i \\ &+ b_i \left(\frac{G'}{G}\right)^{i-1} \sqrt{\sigma \left[1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right]}, \end{aligned} \tag{4}$$

where $G = G(\xi)$ satisfies the following second order linear ODE :

$$G''(\xi) + \mu G(\xi) = 0, \tag{5}$$

where a_0, a_i, b_i, σ and μ are constants, such that $\sigma = \pm 1$ and $\mu \neq 0$. The positive integer n can be determined by balancing the highest order derivatives with the nonlinear terms appearing in Eq.(3).

Step 3. Substituting (4) into Eq.(3) and using Eq.(5), collecting all terms with the same powers of $(G'/G)^k$ and $(G'/G)^k \sqrt{\sigma[1 + \frac{1}{\mu}(G'/G)^2]}$ together and equating each coefficient of them to zero, yield a set of algebraic equations for a_0, a_i, b_i and V.

Step 4. Since the general solution of Eq.(5) has been well known for us, then substituting a_0, a_i, b_i, V and the general solution of Eq.(5) into (4), we have the traveling wave solutions of the nonlinear partial differential equation (1).

Remark 1 *It is necessary to point out that by adding the term* $(\frac{G'}{G})^k \sqrt{\sigma[1 + \frac{1}{\mu}(\frac{G'}{G})^2]}$ *into (4), the anstz proposed here is more general than the ansatz in the original* $(\frac{G'}{G})$*-expansion method [18]. Therefore, the extended* $(\frac{G'}{G})$*-expansion method is more powerful than the original* $(\frac{G'}{G})$*-expansion method [18] and some new types of traveling wave solutions and solitary wave solutions would be expected for some NPDEs. If we choose the parameters in (4) and Eq.(5) to take special values, the original* $(\frac{G'}{G})$*-expansion method can be recovered by the extended* $(\frac{G'}{G})$*-expansion method.*

3 Applications

In this section, we will apply the extended $(\frac{G'}{G})$-expansion method to some nonlinear PDEs in mathematical physics as follows:

Example 2 *The (2+1)-dimensional Painleve integrable Burgers equations*

We start with the (2+1)-dimensional Painleve integrable Burgers equations [8] in the form:

$$-u_t + uu_y + \alpha \upsilon u_x + \beta u_{yy} + \alpha\beta u_{xx} = 0, \tag{6}$$

$$u_x - \upsilon_y = 0, \tag{7}$$

where α and β are nonzero constants. This system of equations was derived from the generalized Painleve integrability classification. Some explicitly exact solutions of this system have been obtained by variable separation approach [8] and the original $(\frac{G'}{G})$-expansion method [26]. Let us now solve the system (6) and (7) by the extended $(\frac{G'}{G})$-expansion method. To this end, we see that the following traveling wave variables

$$u(x, y, t) = u(\xi), \upsilon(x, y, t) = \upsilon(\xi), \xi = x + y - Vt, \tag{8}$$

permit us converting Eqs.(6) and (7) into the following ODEs:

$$(u + \alpha\upsilon + V)u' + \beta(1 + \alpha)u'' = 0, \tag{9}$$

$$u' - \upsilon' = 0. \tag{10}$$

Considering the homogeneous balance between the highest derivatives and the nonlinear terms in Eqs.(9) and (10), we get

$$u(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) \sqrt{\sigma[1 + \left(\frac{G'}{G}\right)^2]}, \tag{11}$$

$$\upsilon(\xi) = c_0 + c_1 \left(\frac{G'}{G}\right) + d_1 \sqrt{\sigma[1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2]}, \tag{12}$$

Substituting (11) and (12) along with Eq.(5) into Eqs. (9) and (10), collecting all terms with the same powers of $(\frac{G'}{G})^k$, $(\frac{G'}{G})^k \sqrt{\sigma[1 + \frac{1}{\mu}(\frac{G'}{G})^2]}$ and setting them to zero, we have a system of algebraic equations which can be solved by Maple or Mathematica to obtain the following results:

Case1.

$$\begin{aligned} a_0 &= a_0, a_1=\beta, b_1=\pm\beta\sqrt{\tfrac{\mu}{\sigma}},\\ c_0 &= c_0, c_1=\beta, d_1=\pm\beta\sqrt{\tfrac{\mu}{\sigma}},\\ V &= -\alpha c_0-a_0, \sigma=\sigma\ . \end{aligned} \tag{13}$$

Case2.

$$\begin{aligned} a_0 &= a_0, a_1=2\beta, b_1=0,\\ c_0 &= c_0, c_1=2\beta, d_1=0,\\ V &= -\alpha c_0-a_0, \sigma=\sigma. \end{aligned} \tag{14}$$

where $\sigma=\pm1$. From (11) and (12) and the general solution of Eq.(5), we deduce the traveling wave solutions of Eqs.(6) and (7) as follows:

When $\mu<0$, then **Case (1)** gives the exact traveling wave solutions:

$$\begin{aligned} u(\xi) &= a_0\\ &+ \beta\sqrt{-\mu}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)\\ &\pm \left(\beta\sqrt{\tfrac{\mu}{\sigma}}\right)\\ &\times \sqrt{\sigma\left[1-\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\right]}, \end{aligned} \tag{15}$$

$$\begin{aligned} v(\xi) &= c_0\\ &+ \beta\sqrt{-\mu}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)\\ &\pm \left(\beta\sqrt{\tfrac{\mu}{\sigma}}\right)\\ &\times \sqrt{\sigma\left[1-\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\right]}, \end{aligned} \tag{16}$$

where A and B are arbitrary constants. **Case (2)** gives the exact traveling wave solution:

$$\begin{aligned} u(\xi) &= a_0\\ &+ 2\beta\sqrt{-\mu}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right), \end{aligned} \tag{17}$$

$$\begin{aligned} v(\xi) &= c_0\\ &+ 2\beta\sqrt{-\mu}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right), \end{aligned} \tag{18}$$

where $\xi=x+y+(\alpha c_0+a_0)t$.

When $\mu>0$, then **Case (1)** gives the exact traveling wave solutions:

$$\begin{aligned} u(\xi) &= a_0\\ &+ \beta\sqrt{\mu}\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)\\ &\pm \left(\beta\sqrt{\tfrac{\mu}{\sigma}}\right)\\ &\times \sqrt{\sigma\left[1+\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\right]}, \end{aligned} \tag{19}$$

$$\begin{aligned} v(\xi) &= c_0\\ &+ \beta\sqrt{\mu}\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)\\ &\pm (\beta\sqrt{\tfrac{\mu}{\sigma}})\\ &\times \sqrt{\sigma\left[1+\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\right]}. \end{aligned} \tag{20}$$

Case (2) gives the exact traveling wave solutions:

$$\begin{aligned} u(\xi) &= a_0\\ &+ 2\beta\sqrt{\mu}\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right), \end{aligned} \tag{21}$$

$$\begin{aligned} v(\xi) &= c_0\\ &+ 2\beta\sqrt{\mu}\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right). \end{aligned} \tag{22}$$

In particular, we deduce from (15) and (16) that the solitary wave solutions of Eqs. (6) and (7) are derived as follows:
If $A=0$, $B\neq0$ and $\mu<0$, then we obtain

$$u(\xi)=a_0+\beta\sqrt{-\mu}[\coth(\sqrt{-\mu}\xi)\pm\operatorname{csch}(\sqrt{-\mu}\xi)], \tag{23}$$

$$v(\xi)=c_0+\beta\sqrt{-\mu}[\coth(\sqrt{-\mu}\xi)\pm\operatorname{csch}(\sqrt{-\mu}\xi)], \tag{24}$$

while, if $A\neq0$, $A^2>B^2$ and $\mu<0$, then we obtain

$$\begin{aligned} u(\xi) &= a_0+\beta\sqrt{-\mu}\\ &\times [\tanh(\sqrt{-\mu}\xi+\xi_0)\pm i\operatorname{sech}(\sqrt{-\mu}\xi+\xi_0)], \end{aligned} \tag{25}$$

$$\begin{aligned} v(\xi) &= c_0+\beta\sqrt{-\mu}\\ &\times [\tanh(\sqrt{-\mu}\xi+\xi_0)\pm i\operatorname{sech}(\sqrt{-\mu}\xi+\xi_0)], \end{aligned} \tag{26}$$

where $\xi_0=tanh^{-1}(\frac{B}{A})$ and $i=\sqrt{-1}$. Similarly, we can find more solitary wave solutions of Eqs. (6) and (7) using (17) and (18) but we are omitted them for simplicity.

Example 3 *The (2+1)-dimensional Nizhnik-Novikov-Veselov equations*

In this subsection, we study the following (2+1)-dimensional Nizhnik-Novikov-Veselov equations [16]:

$$\begin{gathered} u_t+ku_{xxx}+ru_{yyy}+su_x+qu_y\\ -3k(uv)_x-3r(uw)_y=0, \end{gathered} \tag{27}$$

$$u_x-v_y=0, \tag{28}$$

$$u_y-w_x=0, \tag{29}$$

where k, r, s and q are constants. Boiti et al [3] solved this system of equations via the inverse scattering transformation, while Zayed [26] has discussed this system using the original $(\frac{G'}{G})$-expansion method. Let us now solve the system (27)-(29) using the extended

$(\frac{G'}{G})$-expansion method. To this end, we see that the traveling wave variables (8) permit us converting the system (27)-(29) into the following ODEs:

$$\begin{aligned}(s+q-V)u'+(k+r)u'''\\-3k(uv)'-3r(uw)'=0,\end{aligned} \tag{30}$$

$$u'-v'=0, \tag{31}$$

$$u'-w'=0, \tag{32}$$

Considering the homogeneous balance between highest order derivatives and nonlinear terms in (30)-(32), we get

$$\begin{aligned}u(\xi) &= a_0+a_1\left(\tfrac{G'}{G}\right)+a_2\left(\tfrac{G'}{G}\right)^2\\&+ b_1\sqrt{\sigma\left[1+\tfrac{1}{\mu}\left(\tfrac{G'}{G}\right)^2\right]}\\&+ b_2\left(\tfrac{G'}{G}\right)\sqrt{\sigma\left[1+\tfrac{1}{\mu}\left(\tfrac{G'}{G}\right)^2\right]},\end{aligned} \tag{33}$$

$$\begin{aligned}v(\xi) &= c_0+c_1\left(\tfrac{G'}{G}\right)+c_2\left(\tfrac{G'}{G}\right)^2\\&+ d_1\sqrt{\sigma\left[1+\tfrac{1}{\mu}\left(\tfrac{G'}{G}\right)^2\right]}\\&+ d_2\left(\tfrac{G'}{G}\right)\sqrt{\sigma\left[1+\tfrac{1}{\mu}\left(\tfrac{G'}{G}\right)^2\right]},\end{aligned} \tag{34}$$

$$\begin{aligned}w(\xi) &= f_0+f_1\left(\tfrac{G'}{G}\right)+f_2\left(\tfrac{G'}{G}\right)^2\\&+ g_1\sqrt{\sigma\left[1+\tfrac{1}{\mu}\left(\tfrac{G'}{G}\right)^2\right]}\\&+ g_2\left(\tfrac{G'}{G}\right)\sqrt{\sigma\left[1+\tfrac{1}{\mu}\left(\tfrac{G'}{G}\right)^2\right]},\end{aligned} \tag{35}$$

Substituting (33)-(35) along with Eq.(5) into Eqs.(30)-(32), collecting all terms with the same powers of $(\frac{G'}{G})^k$, $(\frac{G'}{G})^k\sqrt{\sigma[1+\frac{1}{\mu}(\frac{G'}{G})^2]}$ and setting them to zero, we have a system of algebraic equations which can be solved by Maple or Mathematica to obtain the following results:

Case1.

$$\begin{aligned}a_0 &= a_0, a_1=0, a_2=1, b_1=0, b_2=\pm\sqrt{\tfrac{\mu}{\sigma}},\\c_0 &= c_0, c_1=0, c_2=1, d_1=0, d_2=\pm\sqrt{\tfrac{\mu}{\sigma}},\\f_0 &= f_0, f_1=0, f_2=1, g_1=0, g_2=\pm\sqrt{\tfrac{\mu}{\sigma}},\\V &= q+s+(k+r)(5\mu-3a_0)-3kc_0-3rf_0,\\\sigma &= \sigma,\end{aligned} \tag{36}$$

$$\begin{aligned}a_0 &= a_0, a_1=0, a_2=2, b_1=0, b_2=0,\\c_0 &= c_0, c_1=0, c_2=2, d_1=0, d_2=0,\\f_0 &= f_0, f_1=0, f_2=2, g_1=0, g_2=0,\\V &= q+s+(k+r)(8\mu-3a_0)-3kc_0-3rf_0,\\\sigma &= \sigma,\end{aligned} \tag{37}$$

From (33)-(35) and the general solution of Eq.(5), we deduce the traveling wave solutions of Eqs. (27)-(29) as follows:

When $\mu<0$, then case (1) gives the exact traveling wave solution:

$$\begin{aligned}u(\xi) = a_0-\mu\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\pm\\\tfrac{\mu}{\sqrt{\sigma}}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)\times\\\sqrt{\sigma\left[-1+\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\right]},\end{aligned} \tag{38}$$

$$\begin{aligned}v(\xi) = c_0-\mu\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\pm\\\tfrac{\mu}{\sqrt{\sigma}}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)\times\\\sqrt{\sigma\left[-1+\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\right]},\end{aligned} \tag{39}$$

$$\begin{aligned}w(\xi) = f_0-\mu\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\pm\\\tfrac{\mu}{\sqrt{\sigma}}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)\times\\\sqrt{\sigma\left[-1+\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\right]},\end{aligned} \tag{40}$$

where $\xi=x+y-[q+s+(k+r)(5\mu-3a_0)-3kc_0-3rf_0]t$. **Case (2)** gives the exact traveling wave solution:

$$u(\xi)=a_0-2\mu\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2, \tag{41}$$

$$v(\xi)=c_0-2\mu\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2, \tag{42}$$

$$w(\xi)=f_0-2\mu\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2, \tag{43}$$

where $\xi=x+y-[q+s+(k+r)(8\mu-3a_0)-3kc_0-3rf_0]t$. When $\mu>0$, then **Case (1)** gives the exact traveling wave solution:

$$\begin{aligned}u(\xi) = a_0+\mu\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\pm\\\tfrac{\mu}{\sqrt{\sigma}}\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)\times\\\sqrt{\sigma\left[1+\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\right]},\end{aligned} \tag{44}$$

$$\upsilon(\xi)=c_0+\mu\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\pm \frac{\mu}{\sqrt{\sigma}}\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)\times \sqrt{\sigma\left[1+\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\right]}, \tag{45}$$

$$w(\xi)=f_0+\mu\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\pm \frac{\mu}{\sqrt{\sigma}}\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)\times \sqrt{\sigma\left[1+\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\right]}. \tag{46}$$

Case (2) gives the exact traveling wave solution:

$$u(\xi)=a_0+2\mu\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2, \tag{47}$$

$$\upsilon(\xi)=c_0+2\mu\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2, \tag{48}$$

$$w(\xi)=f_0+2\mu\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2. \tag{49}$$

In particular, we deduce from (38)-(40) that the solitary wave solutions of Eqs.(27)-(29) are derived as follows:

If $A=0$, $B\neq 0$ and $\mu<0$, then we obtain

$$u(\xi)=a_0 -\mu\coth(\sqrt{-\mu}\xi)[\coth(\sqrt{-\mu}\xi)\mp csch(\sqrt{-\mu}\xi)], \tag{50}$$

$$\upsilon(\xi)=c_0 -\mu\coth(\sqrt{-\mu}\xi)[\coth(\sqrt{-\mu}\xi)\mp csch(\sqrt{-\mu}\xi)], \tag{51}$$

$$w(\xi)=f_0 -\mu\coth(\sqrt{-\mu}\xi)[\coth(\sqrt{-\mu}\xi)\mp csch(\sqrt{-\mu}\xi)], \tag{52}$$

while, if $A\neq 0$, $A^2>B^2$ and $\mu<0$, then we obtain

$$\begin{aligned} u(\xi) &= a_0-\mu\tanh(\sqrt{-\mu}\xi+\xi_0)\\ &\times\ [\tanh(\sqrt{-\mu}\xi+\xi_0)\mp isech(\sqrt{-\mu}\xi+\xi_0)], \end{aligned}\tag{53}$$

$$\begin{aligned} \upsilon(\xi) &= c_0-\mu\tanh(\sqrt{-\mu}\xi+\xi_0)\\ &\times\ [\tanh(\sqrt{-\mu}\xi+\xi_0)\mp isech(\sqrt{-\mu}\xi+\xi_0)], \end{aligned}\tag{54}$$

$$\begin{aligned} w(\xi) &= f_0-\mu\tanh(\sqrt{-\mu}\xi+\xi_0)\\ &\times\ [\tanh(\sqrt{-\mu}\xi+\xi_0)\mp isech(\sqrt{-\mu}\xi+\xi_0)], \end{aligned}\tag{55}$$

where $\xi_0=tanh^{-1}(\frac{B}{A})$. Similarly, we can find more solitary solutions of Eqs. (27)-(29) using (41)-(43) but we are omitted them for simplicity.

Example 4 *The (2+1)-dimensional Boiti-Leon-Pempinelli equations*

In this subsection, we consider the following (2+1)-dimensional Boiti-Leon-Pempinelli equations [3]:

$$u_{ty}-(u^2-u_x)_{xy}-2\upsilon_{xxx}=0, \tag{56}$$

$$\upsilon_t-\upsilon_{xx}-2u\upsilon_x=0. \tag{57}$$

The integrability of this system was established by Hong et al [8]. Boiti et al [3] presented the Backlund transformation of this system to find its solutions, while Zayed [26] has solved this system using the original $(\frac{G'}{G})$-expansion method. Let us now solve the system (56) and (57) using the extended $(\frac{G'}{G})$-expansion method. To this end, we see that the traveling wave variables (8) permit us converting Eqs. (56) and (57) into the following ODEs:

$$(V+2u)u''+2(u')^2+2\upsilon'''-u'''=0, \tag{58}$$

$$V\upsilon'+\upsilon''+2u\upsilon'=0. \tag{59}$$

Considering homogeneous balance between highest order derivatives and nonlinear terms in Eqs. (58) and (59), we deduce that the solutions $u(\xi)$ and $\upsilon(\xi)$ have the same forms of (11) and (12). Substituting (11) and (12) along with Eq.(5) into Eqs. (58)-(59), collecting all terms with the same powers of $(\frac{G'}{G})^k$, $(\frac{G'}{G})^k\sqrt{\sigma[1+\frac{1}{\mu}(\frac{G'}{G})^2]}$ and setting them to zero, we have a system of algebraic equations which can be solved by Maple or Mathematica to obtain the following results:

Case1.

$$\begin{aligned} a_0 &= a_0, a_1=\tfrac{1}{2}, b_1=\pm\tfrac{1}{2}\sqrt{\tfrac{\mu}{\sigma}},\\ c_0 &= c_0, c_1=\tfrac{1}{2}, d_1=\pm\tfrac{1}{2}\sqrt{\tfrac{\mu}{\sigma}},\\ V &= -2a_0, \sigma=\sigma. \end{aligned}\tag{60}$$

Case2.

$$\begin{aligned} a_0 &= a_0, a_1=-\tfrac{1}{2}, b_1=\pm\tfrac{1}{2}\sqrt{\tfrac{\mu}{\sigma}},\\ c_0 &= c_0, c_1=0, d_1=0,\\ V &= -2a_0, \sigma=\sigma. \end{aligned}\tag{61}$$

Case3.

$$\begin{aligned} a_0 &= a_0, a_1=1, b_1=0,\\ c_0 &= c_0, c_1=1, d_1=0,\\ V &= -2a_0, \sigma=\sigma. \end{aligned}\tag{62}$$

Case4.

$$\begin{aligned} a_0 &= a_0, a_1=-1, b_1=0,\\ c_0 &= c_0, c_1=0, d_1=0,\\ V &= -2a_0, \sigma=\sigma. \end{aligned}\tag{63}$$

From (11), (12) and the general solution of Eq.(5), we deduce the traveling wave solutions of Eqs.(56) and (57) as follows:

When $\mu < 0$, then **case (1)** gives the exact traveling wave solution:

$$\begin{aligned} u(\xi) &= a_0 \\ &+ \tfrac{1}{2}\sqrt{-\mu}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right) \\ &\pm \left(\tfrac{1}{2}\sqrt{\tfrac{\mu}{\sigma}}\right) \\ &\times \sqrt{\sigma\left[1-\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\right]}, \end{aligned} \tag{64}$$

$$\begin{aligned} v(\xi) &= c_0 \\ &+ \tfrac{1}{2}\sqrt{-\mu}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right) \\ &\pm \left(\tfrac{1}{2}\sqrt{\tfrac{\mu}{\sigma}}\right) \\ &\times \sqrt{\sigma\left[1-\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\right]}. \end{aligned} \tag{65}$$

Case (2) gives the exact traveling wave solution

$$\begin{aligned} u(\xi) &= a_0 \\ &+ \tfrac{1}{2}\sqrt{-\mu}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right) \\ &\pm \left(\tfrac{1}{2}\sqrt{\tfrac{\mu}{\sigma}}\right) \\ &\times \sqrt{\sigma\left[1-\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\right]}, \end{aligned} \tag{66}$$

$$v(\xi) = c_0. \tag{67}$$

Case (3) gives the exact traveling wave solution

$$u(\xi) = a_0 + \sqrt{-\mu}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right), \tag{68}$$

$$v(\xi) = c_0 + \sqrt{-\mu}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right). \tag{69}$$

Case (4) gives the exact traveling wave solution

$$u(\xi) = a_0 - \sqrt{-\mu}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right), \tag{70}$$

$$v(\xi) = c_0, \tag{71}$$

where $\xi = x + y + 2a_0 t$.

When $\mu > 0$, then **Case (1)** gives the exact traveling wave solution:

$$\begin{aligned} u(\xi) &= a_0 \\ &+ \tfrac{1}{2}\sqrt{\mu}\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right) \\ &\pm \left(\tfrac{1}{2}\sqrt{\tfrac{\mu}{\sigma}}\right) \\ &\times \sqrt{\sigma\left[1+\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\right]}, \end{aligned} \tag{72}$$

$$\begin{aligned} v(\xi) &= c_0 \\ &+ \tfrac{1}{2}\sqrt{\mu}\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right) \\ &\pm \left(\tfrac{1}{2}\sqrt{\tfrac{\mu}{\sigma}}\right) \\ &\times \sqrt{\sigma\left[1+\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\right]}. \end{aligned} \tag{73}$$

Case (2) gives the exact traveling wave solution

$$\begin{aligned} u(\xi) &= a_0 \\ &- \tfrac{1}{2}\sqrt{\mu}\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right) \\ &\pm \left(\tfrac{1}{2}\sqrt{\tfrac{\mu}{\sigma}}\right) \\ &\times \sqrt{\sigma\left[1+\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\right]}, \end{aligned} \tag{74}$$

$$v(\xi) = c_0. \tag{75}$$

Case (3) gives the exact traveling wave solution

$$\begin{aligned} u(\xi) &= a_0 + \sqrt{\mu} \\ &\times \left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right), \end{aligned} \tag{76}$$

$$\begin{aligned} v(\xi) &= c_0 + \sqrt{\mu} \\ &\times \left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right). \end{aligned} \tag{77}$$

Case (4) gives the exact traveling wave solution

$$u(\xi) = a_0 - \sqrt{\mu}\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right), \tag{78}$$

$$v(\xi) = c_0. \tag{79}$$

In particular, we deduce from (64) and (65) that the solitary wave solutions of Eqs. (56) and (57) are derived as follows:

If $A = 0$, $B \neq 0$ and $\mu < 0$, then we obtain

$$u(\xi) = a_0 + \tfrac{1}{2}\sqrt{-\mu}[\coth(\sqrt{-\mu}\xi) \pm csch(\sqrt{-\mu}\xi)], \tag{80}$$

$$v(\xi) = c_0 + \tfrac{1}{2}\sqrt{-\mu}[\coth(\sqrt{-\mu}\xi) \pm csch(\sqrt{-\mu}\xi)], \tag{81}$$

while, if $A \neq 0$, $A^2 > B^2$ and $\mu < 0$, then we obtain

$$\begin{aligned} u(\xi) &= a_0 + \tfrac{1}{2}\sqrt{-\mu} \\ &\times [\tanh(\sqrt{-\mu}\xi + \xi_0) \pm isech(\sqrt{-\mu}\xi + \xi_0)], \end{aligned} \tag{82}$$

$$\begin{aligned} v(\xi) &= c_0 + \tfrac{1}{2}\sqrt{-\mu} \\ &\times [\tanh(\sqrt{-\mu}\xi + \xi_0) \pm isech(\sqrt{-\mu}\xi + \xi_0)], \end{aligned} \tag{83}$$

where $\xi_0 = tanh^{-1}(\frac{B}{A})$. Similarly, we can find more solitary solutions of Eqs. (56) and (57) using (66)-(71) but we are omitted them for simplicity.

Example 5 *The (2+1)-dimensional dispersive long wave equations*

In this subsection, we consider the (2+1)-dimensional dispersive long wave equations [16] in the from :

$$u_{ty} + \upsilon_{xx} + \frac{1}{2}(u^2)_{xy} = 0, \quad (84)$$

$$\upsilon_t + (u\upsilon + u + u_{xy})_x = 0. \quad (85)$$

This system was first obtained by Boiti et al [3] as compatibility condition for a weak Lax pair. The solutions of this system including Jacobi elliptic function solutions, soliton-like solutions, periodic formal solutions and rational function solutions are found in [16]. This system also has been solved using the original $(\frac{G'}{G})$-expansion method [26]. Let us now solve the system (84) and (85) using the extended $(\frac{G'}{G})$-expansion method. To this end, we see that the traveling wave variables (8) permit us converting Eqs. (84) and (85) into the following ODEs:

$$(u-V)u'' + \upsilon'' + (u')^2 = 0, \quad (86)$$

$$(u-V)\upsilon' + (1+\upsilon)u' + u''' = 0. \quad (87)$$

Balancing the highest derivatives and nonlinear terms in Eqs. (86) and (87) gives that the solution $u(\xi)$ has the same form of (11) while, $\upsilon(\xi)$ has the same form of (32). Substituting (11) and (32) along with Eq.(5) into Eqs. (84) and (85), collecting all terms with the same powers of $(\frac{G'}{G})^k$, $(\frac{G'}{G})^k\sqrt{\sigma[1+\frac{1}{\mu}(\frac{G'}{G})^2]}$ and setting them to zero, we have a system of algebraic equations which can be solved by Maple or Mathematica to obtain the following results:

Case1.

$$\begin{aligned} a_0 &= a_0, a_1 = \pm 1, b_1 = \pm\sqrt{\tfrac{\mu}{\sigma}}, \\ c_0 &= -(1+\mu), c_1 = 0, c_2 = -1, d_1 = 0, \\ d_2 &= \pm\sqrt{\tfrac{\mu}{\sigma}}, V = a_0, \sigma = \sigma. \end{aligned} \quad (88)$$

Case2.

$$\begin{aligned} a_0 &= a_0, a_1 = 0, b_1 = \pm 2\sqrt{\tfrac{\mu}{\sigma}}, \\ c_0 &= -(1+\mu), c_1 = 0, c_2 = -2, \\ d_1 &= 0, d_2 = 0, V = a_0, \sigma = \sigma. \end{aligned} \quad (89)$$

Case3.

$$\begin{aligned} a_0 &= a_0, a_1 = \pm 2, b_1 = 0, \\ c_0 &= -(1+2\mu), c_1 = 0, c_2 = -2, d_1 = 0, \\ d_2 &= 0, V = a_0, \sigma = \sigma. \end{aligned} \quad (90)$$

From (11), (34) and the general solution of Eq. (5), we deduce the traveling wave solutions of Eqs. (84) and (85) as follows:

When $\mu < 0$, then **case (1)** gives the exact traveling wave solution:

$$\begin{aligned} u(\xi) &= a_0 \\ &\pm \sqrt{-\mu}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right) \\ &\pm \left(\sqrt{\tfrac{\mu}{\sigma}}\right) \\ &\times \sqrt{\sigma\left[1-\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\right]}, \end{aligned} \quad (91)$$

$$\begin{aligned} \upsilon(\xi) &= -(1+\mu) \\ &+ \mu\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2 \\ &\pm \tfrac{\mu}{\sqrt{\sigma}}\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right) \\ &\times \sqrt{\sigma\left[-1+\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\right]}. \end{aligned} \quad (92)$$

Case (2) gives the exact traveling wave solution

$$\begin{aligned} u(\xi) &= a_0 \pm 2\sqrt{\tfrac{\mu}{\sigma}} \\ &\times \sqrt{\sigma\left[1-\left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2\right]}, \end{aligned} \quad (93)$$

$$\begin{aligned} \upsilon(\xi) &= -(1+\mu) + 2\mu \\ &\times \left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2. \end{aligned} \quad (94)$$

Case (3) gives the exact traveling wave solution

$$\begin{aligned} u(\xi) &= a_0 \pm 2\sqrt{-\mu} \\ &\times \left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right), \end{aligned} \quad (95)$$

$$\begin{aligned} \upsilon(\xi) &= -(1+2\mu) + 2\mu \\ &\times \left(\tfrac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right)^2, \end{aligned} \quad (96)$$

where $\xi = x + y - a_0 t$. When $\mu > 0$, then **case** (1) gives the exact traveling wave solution

$$\begin{aligned} u(\xi) &= a_0 \\ &\pm \sqrt{\mu}\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right) \\ &\pm \left(\sqrt{\tfrac{\mu}{\sigma}}\right) \\ &\times \sqrt{\sigma\left[1+\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\right]}, \end{aligned} \quad (97)$$

$$\begin{aligned} &\upsilon(\xi) = -(1+\mu) \\ &-\mu\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2 \\ &\pm\left(\tfrac{\mu}{\sqrt{\sigma}}\right)\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right) \\ &\times\sqrt{\sigma\left[1+\left(\tfrac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\right]}. \end{aligned} \quad (98)$$

Case (2) gives the exact traveling wave solution

$$\begin{aligned} u(\xi) &= a_0 \\ &\pm 2\sqrt{\frac{\mu}{\sigma}} \\ &\times \sqrt{\sigma\left[1+\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2\right]}, \end{aligned} \quad (99)$$

$$\begin{aligned} v(\xi) &= -(1+\mu) \\ &- 2\mu\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2. \end{aligned} \quad (100)$$

Case (3) gives the exact traveling wave solution

$$\begin{aligned} u(\xi) &= a_0 \\ &\pm 2\sqrt{\mu}\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right), \end{aligned} \quad (101)$$

$$\begin{aligned} v(\xi) &= -(1+2\mu) \\ &- 2\mu\left(\frac{B\cos(\sqrt{\mu}\xi)-A\sin(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi)+B\sin(\sqrt{\mu}\xi)}\right)^2. \end{aligned} \quad (102)$$

In particular, we deduce from (91) and (92) that the solitary wave solutions of Eqs. (84) and (85) are derived as follows:
If $A = 0$, $B \neq 0$ and $\mu < 0$, then we obtain

$$\begin{aligned} u(\xi) &= a_0 \pm \sqrt{-\mu} \\ &\times[\coth(\sqrt{-\mu}\xi) \pm \operatorname{csch}(\sqrt{-\mu}\xi)], \end{aligned} \quad (103)$$

$$\begin{aligned} v(\xi) &= -(1+\mu) + \mu\coth(\sqrt{-\mu}\xi) \\ &\times[\coth(\sqrt{-\mu}\xi) \pm \operatorname{csch}(\sqrt{-\mu}\xi)], \end{aligned} \quad (104)$$

while, if $A \neq 0$, $A^2 > B^2$ and $\mu < 0$, then we obtain

$$\begin{aligned} &u(\xi) = a_0 \pm \sqrt{-\mu} \\ &\times[\tanh(\sqrt{-\mu}\xi + \xi_0) + i\operatorname{sech}(\sqrt{-\mu}\xi + \xi_0)], \end{aligned} \quad (105)$$

$$\begin{aligned} &v(\xi) = -(1+\mu) + \mu\tanh(\sqrt{-\mu}\xi + \xi_0) \\ &\times[\tanh(\sqrt{-\mu}\xi + \xi_0) \pm i\operatorname{sech}(\sqrt{-\mu}\xi + \xi_0)], \end{aligned} \quad (106)$$

where $\xi_0 = \tanh^{-1}(\frac{B}{A})$. Similarly, we can find more solitary wave solutions of Eqs. (84) and (85) using (93)-(96) but we are omitted them for simplicity.

Remark 6 *All solutions of this article have been checked with the Maple by putting them back into the original equations (6), (7), (27)-(29), (56), (57), (84) and (85).*

4 Conclusion

In this article, we have shown that the traveling wave solutions in terms of hyperbolic and trigonometric functions for the (2+1)-dimensional Painleve integrable Burgers equations, the (2+1)-dimensional Nizhnik-Novikov-Veselov equations, the (2+1)-dimensional Boiti-Leon-Pempinelli equations and the (2+1)-dimensional dispersive long wave equations are successfully found out by using the extended $(\frac{G'}{G})$-expansion method. On comparing the results of these equations with the results obtained in [26] we see that, our results in the present article are more general and have not been reported in previous literature. The performance of this method is reliable, effective and giving many new solutions to many other nonlinear PDEs. Finally, the solutions of the proposed nonlinear evolution equations in this paper have not been obtained elsewhere. These solutions have many applications in the mathematical biology, quantum mechanics, plasma physics, fluid physics, quantum field theory and so on.

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