Some New Generalized Volterra-Fredholm Type Nonlinear Discrete Inequalities

Bin Zheng
Shandong University of Technology
School of Science
Zhangzhou Road 12, Zibo, 255049
China
zhengbin2601@126.com

Abstract: In this paper, we establish some new generalized Volterra-Fredholm type nonlinear discrete inequalities, based on which we study the boundedness of the solutions of a kind of Volterra-Fredholm type sum-difference equation. As a result, New bounds to the solutions are established under some suitable conditions. The established inequalities are further generalizations of the results by Zheng [Adv. Differ. Equ. Article ID: 30 (2011) 1-16], Ma [J. Comput. Appl. Math. 233 (2010) 2170-2180], and Zheng [Abstr. Appl. Anal. Article ID: 584951 (2011) 1-24].

Key–Words: Nonlinear inequalities; Discrete inequalities; Volterra-Fredholm type inequalities; Gronwall-Bellman type inequalities; Finite difference equations; Boundedness.

1 Introduction

The Gronwall-Bellman inequality [1,2] and its various generalizations which provide explicit bounds play a fundamental role in the research of boundedness of solutions of certain differential and difference equations. Recently, much attention has been paid to such inequalities (for example, see [3-18] and the references therein) including the known Ou-lang's inequality [3] as a case of the generalization of the Gronwall-Bellman inequality. In [19], Ma generalized the discrete version of Ou-lang's inequality in two variables to Volterra-Fredholm form for the first time, which has proved to be very useful in the study of qualitative as well as quantitative properties of solutions of certain Volterra-Fredholm type difference equations. But since then few results on Volterra-Fredholm type discrete inequalities have been established in the literature. Recent results in this direction include the works of Zheng [18], Ma [20], Zheng and Feng [21] to our best knowledge. We notice that the Volterra-Fredholm type discrete inequalities in [18, 20, 21] are constructed uniformly by an explicit function u^p in the left side (see [18, Theorems 2.5, 2.6], [20, Theorems 2.1, 2.5, 2.6, 2.7], [21, Theorems 5, 8, 10, 11]).

Motivated by the works in [18, 20, 21], in this paper, we will establish some new generalized Volterra-Fredholm type discrete inequalities with the right side denoted by an arbitrary function $\phi(u)$, which are of more general forms than the inequalities presented in [20, 21], and provide new bounds for unknown functions concerned. For illustrating the usefulness of the established results, we also present some applications for them, and study the boundedness of solutions of certain Volterra-Fredholm type sum-difference equations.

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$, while $\mathbb{Z}$ denotes the set of integers. Let $\Omega := ([m_0, M] \times [n_0, N]) \bigcap \mathbb{Z}^2$, where $m_0,\ n_0 \in \mathbb{Z}$, and $M,\ N \in \mathbb{Z} \bigcup \{\infty\}$ are two constants. $l_1,\ l_2 \in \mathbb{Z}$ are two constants. If U is a lattice, then we denote the set of all $\mathbb{R}$-valued functions on U by $\wp(U)$, and denote the set of all $\mathbb{R}_+$-valued functions on U by $\wp_+(U)$. Finally, for a function $f \in \wp_+(U)$, we have $\sum_{s=m_0}^{m_1} f = 0$ provided $m_0 > m_1$.

2 Main Results

Lemma 1 *Suppose $u,\ a,\ H \in \wp_+(\Omega)$, $b \in \wp_+(\Omega^2)$, and H, a are nondecreasing in every variable with $H(m, n) > 0$, while b is nondecreasing in the third variable. $\varphi,\ \phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ are strictly increasing with $\varphi(r) > 0,\ \phi(r) > 0$ for $r > 0$. If for $(m, n) \in \Omega$, $u(m, n)$ satisfies the following inequality*

$$u(m,n) \le H(m,n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m,n)\varphi(\phi^{-1}(u(s,t)+a(s,t))), \tag{1}$$

then we have

$$u(m,n) \le G^{-1}[G(H(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m,n)], \tag{2}$$

where

$$G(z) = \int_{z_0}^{z} \frac{1}{\varphi[\phi^{-1}(z+a(m,n))]}dz,\ \ z \ge z_0 > 0. \tag{3}$$

Proof. Fix $(m_1,n_1) \in \Omega$, and let $(m,n) \in ([m_0,m_1]\times[n_0,n_1])\bigcap\Omega$. Then we have

$$u(m,n) \le H(m_1,n_1) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m,n)\varphi[\phi^{-1}(u(s,t)+a(s,t))]. \tag{4}$$

Let the right side of (4) be $v(m,n)$. Then

$$u(m,n) \le v(m,n),\quad (m,n) \in ([m_0,m_1]\times[n_0,n_1])\bigcap\Omega, \tag{5}$$

and

$$\begin{aligned}
& v(m+1,n) - v(m,n) \\
&= \sum_{s=m_0}^{m}\sum_{t=n_0}^{n-1} b(s,t,m+1,n)\varphi[\phi^{-1}(u(s,t)+a(s,t))] \\
&\quad - \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m,n)\varphi[\phi^{-1}(u(s,t)+a(s,t))] \\
&= \sum_{s=m_0}^{m}\sum_{t=n_0}^{n-1} b(s,t,m+1,n)\varphi[\phi^{-1}(u(s,t)+a(s,t))] \\
&\quad - \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m+1,n)\varphi[\phi^{-1}(u(s,t)+a(s,t))] \\
&\quad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m+1,n)\varphi[\phi^{-1}(u(s,t)+a(s,t))] \\
&\quad - \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m,n)\varphi[\phi^{-1}(u(s,t)+a(s,t))] \\
&= \sum_{t=n_0}^{n-1} b(m,t,m+1,n)\varphi[\phi^{-1}(u(m,t)+a(m,t))] \\
&\quad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} [b(s,t,m+1,n)-b(s,t,m,n)]\times \\
&\qquad \varphi[\phi^{-1}(u(s,t)+a(s,t))] \\
&\le \sum_{t=n_0}^{n-1} b(m,t,m+1,n)\varphi[\phi^{-1}(u(m,t)+a(m,t))] \\
&\quad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} [b(s,t,m+1,n)-b(s,t,m,n)]\times \\
&\qquad \varphi[\phi^{-1}(v(s,t)+a(s,t))] \\
&\le \{\sum_{t=n_0}^{n-1} b(m,t,m+1,n) \\
&\quad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} [b(s,t,m+1,n)-b(s,t,m,n)]\}\times \\
&\qquad \varphi[\phi^{-1}(v(m,n)+a(m,n))],
\end{aligned}$$

that is,

$$\begin{aligned}
& \frac{v(m+1,n)-v(m,n)}{\varphi(\phi^{-1}(v(m,n)+a(m,n)))} \\
&\le \sum_{t=n_0}^{n-1} b(m,t,m+1,n) \\
&\quad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} [b(s,t,m+1,n)-b(s,t,m,n)] \\
&= \sum_{s=m_0}^{m}\sum_{t=n_0}^{n-1} b(s,t,m+1,n) \\
&\quad - \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m+1,n) \\
&\quad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} [b(s,t,m+1,n)-b(s,t,m,n)] \\
&= \sum_{s=m_0}^{m}\sum_{t=n_0}^{n-1} b(s,t,m+1,n) \\
&\quad - \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m,n)
\end{aligned} \tag{6}$$

On the other hand, according to the Mean-Value Theorem for integrals, there exists ξ such that $v(m,n) \le \xi \le v(m+1,n)$, and

$$\begin{aligned}
&\int_{v(m,n)}^{v(m+1,n)} \frac{1}{\varphi(\phi^{-1}(z+a(m,n)))}dz \\
&= \frac{v(m+1,n)-v(m,n)}{\varphi(\phi^{-1}(\xi+a(m,n)))}
\end{aligned}$$

$$\leq \frac{v(m+1,n)-v(m,n)}{\varphi(\phi^{-1}(v(m,n)+a(m,n)))}. \tag{7}$$

So combining (6) and (7) we have

$$\begin{aligned}&\int_{v(m,n)}^{v(m+1,n)} \frac{1}{\varphi(\phi^{-1}(z+a(m,n)))}dz\\ &= G(v(m+1,n)) - G(v(m,n))\\ &\leq \sum_{s=m_0}^{m}\sum_{t=n_0}^{n-1} b(s,t,m+1,n)\\ &\quad -\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m,n),\end{aligned} \tag{8}$$

where G is defined in (3). Setting $m=\eta$ in (8), and a summary with respect to η from m_0 to $m-1$ yields

$$G(v(m,n)) - G(v(m_0,n))$$
$$\leq \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m,n) - 0 = \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m,n).$$

Noticing $v(m_0,n) = H(m_1,n_1)$, and G is increasing, it follows that

$$v(m,n) \leq G^{-1}[G(H(m_1,n_1)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m,n)]. \tag{9}$$

Combining (5) and (9) we obtain

$$\begin{aligned}&u(m,n)\leq\\ &G^{-1}[G(H(m_1,n_1)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m,n)],\\ &(m,n)\in([m_0,m_1]\times[n_0,n_1])\bigcap\Omega.\end{aligned} \tag{10}$$

Setting $m=m_1,\ n=n_1$ in (10), yields

$$\begin{aligned}&u(m_1,n_1)\leq\\ &G^{-1}[G(H(m_1,n_1)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t,m_1,n_1)].\end{aligned} \tag{11}$$

Since (m_1,n_1) is selected from Ω arbitrarily, substituting (m_1,n_1) with (m,n) in (11) we get the desired inequality (2). □

Lemma 2 *[21, Lemma 2.3]. Suppose $u,\ a,\ b\in\wp_+(\Omega)$. If $a(m,n)$ is nondecreasing in the first variable, then for $(m,n)\in\Omega$,*

$$u(m,n)\leq a(m,n) + \sum_{s=m_0}^{m-1} b(s,n)u(s,n)$$

implies

$$u(m,n)\leq a(m,n)\prod_{s=m_0}^{m-1}[1+b(s,n)]. \tag{12}$$

Theorem 3 *Suppose $u\in\wp_+(\Omega)$, $b_i,\ c_i\in\wp_+(\Omega^2)$, $i=1,2,...,l_1$, $d_i,\ e_i\in\wp_+(\Omega^2)$, $i=1,2,...,l_2$ with $b_i,\ c_i,\ d_i,\ e_i$ nondecreasing in the last two variables, and there is at least one function among $d_i,\ e_i,\ i=1,2,...,l_2$ not equivalent to zero, $a,\ \varphi,\ \phi$ are defined as in Lemma 1. If for $(m,n)\in\Omega$, $u(m,n)$ satisfies*

$$\begin{aligned}&\phi(u(m,n))\leq a(m,n)\\ &+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[b_i(s,t,m,n)\varphi(u(s,t))\\ &+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))]\\ &+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)\phi(u(s,t))\\ &+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)\phi(u(\xi,\eta))],\end{aligned} \tag{13}$$

$$\begin{aligned}&u(m,n)\leq\phi^{-1}\{a(m,n)\\ &+G^{-1}\{G(T^{-1}[\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}B(s,t,M,N)])\\ &+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B(s,t,m,n)\}\},\end{aligned} \tag{14}$$

provided that T is increasing, where G is defined in (3), and

$$T(x) = G(\frac{x-\mu_1}{\mu_2}) - G(x),\ x\geq 0, \tag{15}$$

$$\begin{aligned}B(s,t,m,n) &= \sum_{i=1}^{l_1}[b_i(s,t,m,n)\\ &+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)].\end{aligned} \tag{16}$$

$$\begin{aligned}&J(m,n) = \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)a(s,t)\\ &+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)a(\xi,\eta)],\end{aligned} \tag{17}$$

$$\begin{aligned}&\mu_1 = J(M,N),\\ &\mu_2 = \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,M,N)\\ &+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,M,N)].\end{aligned} \tag{18}$$

Proof. Denote

$$v(m,n) = \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[b_i(s,t,m,n)\varphi(u(s,t))$$

$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))]$$

$$+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)\phi(u(s,t))$$
$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)\phi(u(\xi,\eta))].$$

Then we have

$$u(m,n)\le\phi^{-1}(a(m,n)+v(m,n)). \tag{19}$$

So

$$\begin{aligned}
&v(m,n)\le\\
&\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\{b_i(s,t,m,n)\varphi[\phi^{-1}(a(s,t)+v(s,t))]\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\varphi[\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))]\}\\
&+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)(a(s,t)+v(s,t))\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)(a(\xi,\eta)+v(\xi,\eta))]\\
&=H(m,n)\\
&+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\{b_i(s,t,m,n)\varphi[\phi^{-1}(a(s,t)+v(s,t))]\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\varphi[\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))]\},
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
&H(m,n)=J(m,n)\\
&+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)v(s,t)\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)v(\xi,\eta)],
\end{aligned}$$

and $J(m,n)$ is defined in (17). Then using $H(m,n)$ is nondecreasing in every variable, we obtain

$$\begin{aligned}
&v(m,n)\le H(M,N)\\
&+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[b_i(s,t,m,n)\varphi[\phi^{-1}(a(s,t)+v(s,t))]\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\varphi[\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))]\\
&\le H(M,N)+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[b_i(s,t,m,n)\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)]\varphi[\phi^{-1}(a(s,t)+v(s,t))]\\
&=H(M,N)\\
&+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B(s,t,m,n)\varphi[\phi^{-1}(a(s,t)+v(s,t))],
\end{aligned} \tag{21}$$

where $B(s,t,m,n)$ is defined in (16).

Since there is at least one function among $d_i,\ e_i,\ i=1,2,...,l_2$ not equivalent to zero, then $H(M,N)>0$. On the other hand, as $b_i(s,t,m,n),\ c_i(s,t,m,n)$ are both nondecreasing in the last two variables, then $B(s,t,m,n)$ is also nondecreasing in the last two variables, and by a suitable application of Lemma 1 we obtain

$$\begin{aligned}
&v(m,n)\\
&\le G^{-1}[G(H(M,N))+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B(s,t,m,n)].
\end{aligned} \tag{22}$$

Furthermore, by the definition of $H(m,n)$, μ_1, μ_2 and (22) we have

$$\begin{aligned}
&H(M,N)=J(M,N)\\
&+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{d_i(s,t,M,N)v(s,t)\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,M,N)v(\xi,\eta)\}\\
&\le J(M,N)+v(M,N)\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{d_i(s,t,M,N)\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,M,N)\}\\
&=\mu_1+\mu_2v(M,N)\\
&\le\mu_1\\
&+\mu_2G^{-1}[G(H(M,N))+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}B(s,t,M,N)],
\end{aligned}$$

and

$$\begin{aligned}
&G(\tfrac{H(M,N)-\mu_1}{\mu_2})\\
&\le G(H(M,N))+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}B(s,t,M,N),
\end{aligned}$$

which is rewritten by

$$T(H(M,N))\le\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}B(s,t,M,N),$$

where T is defined in (15). By T is increasing, we have

$$H(M,N)\le T^{-1}[\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}B(s,t,M,N)]. \tag{23}$$

Combining (19), (22) and (23) we get the desired result. □

Corollary 4 *Suppose* $g_{1i},\ g_{2i},\ b_{1i},\ c_{1i}\in\wp_+(\Omega),\ i=1,2,...,l_1$ *with* $g_{1i},\ g_{2i}$ *nondecreasing in every variable.* $d_{1i},\ e_{1i}\in\wp_+(\Omega),\ i=$

$1, 2, ..., l_2$. u, a, φ, ϕ are defined as in Theorem 3. If for $(m,n) \in \Omega$, $u(m,n)$ satisfies

$$\begin{aligned}&\phi(u(m,n))\\&\le a(m,n)+\sum_{i=1}^{l_1} g_{1i}(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[b_{1i}(s,t)\varphi(u(s,t))\\&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_{1i}(\xi,\eta)\varphi(u(\xi,\eta))]\\&+\sum_{i=1}^{l_2} g_{2i}(m,n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_{1i}(s,t)\phi(u(s,t))\\&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_{1i}(\xi,\eta)\phi(u(\xi,\eta))],\end{aligned}$$

then

$$\begin{aligned}&u(m,n)\le\phi^{-1}\{a(m,n)\\&+G^{-1}\{G(T^{-1}[\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}B(s,t,M,N)])\\&+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B(s,t,m,n)\}\},\end{aligned}$$

provided that T is increasing, where G, T are defined in Theorem 3, and

$$\begin{aligned}&B(s,t,m,n)=\sum_{i=1}^{l_1} g_{1i}(m,n)[b_i(s,t)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta)],\\&J(m,n)=\sum_{i=1}^{l_2} g_{2i}(m,n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t)a(s,t)\\&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta)a(\xi,\eta)],\mu_1=J(M,N),\\&\mu_2=\sum_{i=1}^{l_2} g_{2i}(m,n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta)].\end{aligned}$$

The proof for Corollary 4 can be completed by setting $b_i(s,t,m,n) = g_{1i}(m,n)b_{1i}(s,t)$, $c_i(s,t,m,n) = g_{1i}(m,n)c_{1i}(s,t)$, $d_i(s,t,m,n) = g_{2i}(m,n)d_{1i}(s,t)$, $e_i(s,t,m,n) = g_{2i}(m,n)e_{1i}(s,t)$ in Theorem 3.

Theorem 5 *Suppose $w \in \wp_+(\Omega)$, u, a, b_i, c_i, d_i, e_i, φ, ϕ are defined as in Theorem 2.3. Furthermore, assume $\varphi\circ\phi^{-1}$ is submultiplicative, that is, $\varphi(\phi^{-1}(\alpha\beta)) \le \varphi(\phi^{-1}(\alpha))\varphi(\phi^{-1}(\beta))$ for $\forall\alpha,\ \beta\in\mathbb{R}_+$. If for $(m,n)\in\Omega$, $u(m,n)$ satisfies*

$$\begin{aligned}&\phi(u(m,n))\le a(m,n)+\sum_{s=m_0}^{m-1} w(s,n)\phi(u(s,n))\\&+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[b_i(s,t,m,n)\varphi(u(s,t))\\&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))]\\&+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)\phi(u(s,t))\\&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)\phi(u(\xi,\eta))],\end{aligned}\tag{24}$$

then

$$\begin{aligned}&u(m,n)\le\phi^{-1}\{\{a(m,n)\\&+G^{-1}\{G(T^{-1}[\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\overline{B}(s,t,M,N)])\\&+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\overline{B}(s,t,m,n)\}\}\overline{w}(m,n)\}\end{aligned}\tag{25}$$

provided that T is increasing, where G is defined in (3), and

$$T(x)=G(\frac{x-\overline{\mu}_1}{\overline{\mu}_2})-G(x),\ x\ge 0,\tag{26}$$

$$\begin{aligned}\overline{B}(s,t,m,n)=&\sum_{i=1}^{l_1}[\overline{b}_i(s,t,m,n)\\&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{c}_i(\xi,\eta,m,n)].\end{aligned}\tag{27}$$

$$\begin{aligned}\overline{J}(m,n)=&\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[\overline{d}_i(s,t,m,n)a(s,t)\\&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{e}_i(\xi,\eta,m,n)a(\xi,\eta)],\end{aligned}\tag{28}$$

$$\begin{aligned}&\overline{b}_i(s,t,m,n)=b_i(s,t,m,n)\varphi[\phi^{-1}(\overline{w}(s,t))],\\&\overline{c}_i(s,t,m,n)=c_i(s,t,m,n)\varphi[\phi^{-1}(\overline{w}(s,t))],\\&i=1,2,...,l_1,\end{aligned}\tag{29}$$

$$\begin{aligned}&\overline{d}_i(s,t,m,n)=d_i(s,t,m,n)\overline{w}(s,t),\\&\overline{e}_i(s,t,m,n)=e_i(s,t,m,n)\overline{w}(s,t),\\&i=1,2,...,l_2,\end{aligned}\tag{30}$$

$$\overline{w}(m,n)=\prod_{s=m_0}^{m-1}[1+w(s,n)],\tag{31}$$

$$\begin{aligned}\overline{\mu}_1=&\ \overline{J}(M,N),\\\overline{\mu}_2=&\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[\overline{d}_i(s,t,M,N)\\&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{e}_i(\xi,\eta,M,N)].\end{aligned}\tag{32}$$

Proof. Denote

$$z(m,n) = a(m,n) + \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[b_i(s,t,m,n)\varphi(u(s,t)) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))] + \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)\phi(u(s,t)) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)\phi(u(\xi,\eta))].$$

Then we have

$$\phi(u(m,n)) \le z(m,n) + \sum_{s=m_0}^{m-1} w(s,n)\phi(u(s,n)). \tag{33}$$

Obviously $z(m,n)$ is nondecreasing in the first variable. So by Lemma 2 we obtain

$$\phi(u(m,n)) \le z(m,n)\prod_{s=m_0}^{m-1}[1+w(s,n)] = z(m,n)\overline{w}(m,n),$$

where $\overline{w}(m,n)$ is defined in (31). Define

$$v(m,n) = \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[b_i(s,t,m,n)\varphi(u(s,t)) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))] + \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)\phi(u(s,t)) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)\phi(u(\xi,\eta))].$$

Then we obtain

$$u(m,n) \le \phi^{-1}[(a(m,n)+v(m,n))\overline{w}(m,n)], \tag{34}$$

and furthermore, using $\varphi\circ\phi^{-1}$ is submultiplicative, (34) and Lemma 1 we have

$$\begin{aligned}
v(m,n) \le{}& \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\{b_i(s,t,m,n) \times\varphi[\phi^{-1}((a(s,t)+v(s,t))\overline{w}(s,t))]\\
&+ \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n) \times\varphi[\phi^{-1}((a(\xi,\eta)+v(\xi,\eta))\overline{w}(\xi,\eta))]\}\\
&+ \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{d_i(s,t,m,n)[a(s,t)+v(s,t)]\overline{w}(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)[a(\xi,\eta)+v(\xi,\eta)]\overline{w}(\xi,\eta)\}\\
\le{}& \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\{b_i(s,t,m,n)\times \varphi[\phi^{-1}(a(s,t)+v(s,t))]\varphi[\phi^{-1}(\overline{w}(s,t))]\\
&+ \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)\times \varphi[\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))]\varphi[\phi^{-1}(\overline{w}(\xi,\eta))]\}\\
&+ \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{d_i(s,t,m,n)[a(s,t)+v(s,t)]\overline{w}(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)[a(\xi,\eta)+v(\xi,\eta)]\overline{w}(\xi,\eta)\}\\
={}& \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\{\overline{b}_i(s,t,m,n)\phi^{-1}[a(s,t)+v(s,t)] + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} \overline{c}_i(\xi,\eta,m,n)\phi^{-1}[a(\xi,\eta)+v(\xi,\eta)]\}\\
&+ \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{\overline{d}_i(s,t,m,n)[a(s,t)+v(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} \overline{e}_i(\xi,\eta,m,n)[a(\xi,\eta)+v(\xi,\eta)]\}\\
={}& \overline{H}(m,n) + \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\{\overline{b}_i(s,t,m,n)\varphi[\phi^{-1}(a(s,t)+v(s,t))]\\
&+ \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)\varphi[\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))]\},
\end{aligned} \tag{35}$$

where

$$\overline{H}(m,n) = \overline{J}(m,n) + \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{\overline{d}_i(s,t,m,n)v(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} \overline{e}_i(\xi,\eta,m,n)v(\xi,\eta)\},$$

and $\overline{J}(m,n)$ is defined in (28). Then similar to the process of (21)-(23), we obtain

$$v(m,n) \le G^{-1}[G(\overline{H}(M,N)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\overline{B}(s,t,m,n)], \tag{36}$$

and

$$\overline{H}(M,N) \le T^{-1}[\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\overline{B}(s,t,M,N)]. \tag{37}$$

Combining (34), (36) and (37) we get the desired result. □

Theorem 6 *Suppose u, a, b_i, c_i, d_i, e_i, φ, ϕ are defined as in Theorem 2.3. L_{1i}, L_{2i}, T_{1i}, T_{2i} : $\Omega\times\mathbb{R}_+\to\mathbb{R}_+$, $i=1,2,...,l_2$ satisfies $0 \le L_{ji}(m,n,u) - L_{ji}(m,n,v) \le T_{ji}(m,n,v)\ \times (u -$*

$v)$, $j = 1, 2$ for $u \geq v \geq 0$. If for $(m,n) \in \Omega$, $u(m,n)$ satisfies

$$\begin{aligned}
&\phi(u(m,n)) \leq a(m,n)\\
&+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[b_i(s,t,m,n)\varphi(u(s,t))\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))]\\
&+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)L_{1i}(s,t,\phi(u(s,t)))\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)L_{2i}(\xi,\eta,\phi(u(\xi,\eta)))],
\end{aligned} \tag{38}$$

then

$$\begin{aligned}
&u(m,n) \leq \phi^{-1}\{a(m,n)\\
&+G^{-1}\{G(T^{-1}[\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\widehat{B}(s,t,M,N)])\\
&+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widehat{B}(s,t,m,n)\}\},
\end{aligned} \tag{39}$$

provided that T is increasing, where G is defined in (3), and

$$T(x) = G(\tfrac{x-\widehat{\mu}_1}{\widehat{\mu}_2}) - G(x),\ x \geq 0, \tag{40}$$

$$\begin{aligned}
\widehat{B}(s,t,m,n) = &\sum_{i=1}^{l_1}[\widehat{b}_i(s,t,m,n)\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\widehat{c}_i(\xi,\eta,m,n)].
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\widehat{J}(m,n) =\\
&\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)L_{1i}(s,t,a(s,t))\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)L_{2i}(\xi,\eta,a(\xi,\eta))],
\end{aligned} \tag{42}$$

$$\begin{aligned}
&\widehat{b}_i(s,t,m,n) = b_i(s,t,m,n),\ \widehat{c}_i(s,t,m,n)\\
&= c_i(s,t,m,n),\ i = 1,2,...,l_1,
\end{aligned} \tag{43}$$

$$\begin{aligned}
&\widehat{d}_i(s,t,m,n) = d_i(s,t,m,n)T_{1i}(s,t,a(s,t)),\\
&\widehat{e}_i(s,t,m,n) = e_i(s,t,m,n)T_{2i}(s,t,a(s,t)),\\
&i = 1,2,...,l_2,
\end{aligned} \tag{44}$$

$$\begin{aligned}
&\widehat{\mu}_1 = \widehat{J}(M,N),\\
&\widehat{\mu}_2 = \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[\widehat{d}_i(s,t,M,N)\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\widehat{e}_i(\xi,\eta,M,N)],
\end{aligned} \tag{45}$$

Proof. Denote

$$\begin{aligned}
&v(m,n) = \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[b_i(s,t,m,n)\varphi(u(s,t))\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))]\\
&+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)L_{1i}(s,t,\phi(u(s,t)))\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)L_{2i}(\xi,\eta,\phi(u(\xi,\eta)))].
\end{aligned}$$

Then we have

$$u(m,n) \leq \phi^{-1}(a(m,n) + v(m,n)). \tag{46}$$

So

$$\begin{aligned}
&v(m,n) \leq\\
&\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\{b_i(s,t,m,n)\times\varphi[\phi^{-1}(a(s,t)+v(s,t))]\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\varphi[\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))]\}\\
&+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{d_i(s,t,m,n)L_{1i}(s,t,a(s,t)+v(s,t))\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)L_{2i}(\xi,\eta,a(\xi,\eta)+v(\xi,\eta))\}\\
&=\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\{b_i(s,t,m,n)\varphi[\phi^{-1}(a(s,t)+v(s,t))]\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\varphi[\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))]\}\\
&+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{d_i(s,t,m,n)[L_{1i}(s,t,a(s,t)+v(s,t))\\
&-L_{1i}(s,t,a(s,t))+L_{1i}(s,t,a(s,t))]\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)[L_{2i}(\xi,\eta,a(\xi,\eta)+v(\xi,\eta))\\
&-L_{2i}(\xi,\eta,a(\xi,\eta))+L_{2i}(\xi,\eta,a(\xi,\eta))]\}\\
&\leq\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\{b_i(s,t,m,n)\varphi[\phi^{-1}(a(s,t)+v(s,t))]\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\varphi[\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))]\}\\
&+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{d_i(s,t,m,n)\times\\
&[T_{1i}(s,t,a(s,t))v(s,t)+L_{1i}(s,t,a(s,t))]\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)\times\\
&[T_{2i}(\xi,\eta,a(\xi,\eta))v(\xi,\eta)+L_{2i}(\xi,\eta,a(\xi,\eta))]\}\\
&=H(m,n)+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\{b_i(s,t,m,n)\times\\
&\varphi[\phi^{-1}(a(s,t)+v(s,t))]\\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\varphi[\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))]\},
\end{aligned} \tag{47}$$

where

$$\widehat{H}(m,n)=\widehat{J}(m,n)$$
$$+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{\widehat{d}_i(s,t,m,n)v(s,t)$$
$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\widehat{e}_i(\xi,\eta,m,n)v(\xi,\eta)\},$$

and $\widehat{J}(m,n)$ is defined in (42). Then similar to the process of (21)-(23), we obtain

$$v(m,n)\le$$
$$G^{-1}[G(\widehat{H}(M,N))+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widehat{B}(s,t,m,n)], \tag{48}$$

and

$$H(M,N)\le T^{-1}[\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\widehat{B}(s,t,M,N)]. \tag{49}$$

Combining (46), (48) and (49) we get the desired result. □

Theorem 7 *Suppose $w\in\wp_+(\Omega)$, u, a, b_i, c_i, d_i, e_i, φ, ϕ are defined as in Theorem 3, and L_{ji}, T_{ji}, $j=1,2$, $i=1,2,...,l_2$ are defined as in Theorem 6. If for $(m,n)\in\Omega$, $u(m,n)$ satisfies*

$$\phi(u(m,n))\le$$
$$a(m,n)+\sum_{s=m_0}^{m-1}w(s,n)\phi(u(m,n))$$
$$+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[b_i(s,t,m,n)\varphi(u(s,t))$$
$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))]$$
$$+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)L_{1i}(s,t,\phi(u(s,t)))$$
$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)L_{2i}(\xi,\eta,\phi(u(\xi,\eta)))],$$

then

$$u(m,n)\le\phi^{-1}\{a(m,n)$$
$$+G^{-1}\{G(T^{-1}[\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\widetilde{B}(s,t,M,N)])$$
$$+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widetilde{B}(s,t,m,n)\}\},$$

provided that T is increasing, where G is defined in (3), and

$$T(x)=G(\frac{x-\widetilde{\mu}_1}{\widetilde{\mu}_2})-G(x),\ x\ge0,$$

$$\widetilde{B}(s,t,m,n)$$
$$=\sum_{i=1}^{l_1}[\widetilde{b}_i(s,t,m,n)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\widetilde{c}_i(\xi,\eta,m,n)],$$
$$\widetilde{J}(m,n)=$$
$$\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[d_i(s,t,m,n)L_{1i}(s,t,a(s,t)\widetilde{\omega}(s,t))$$
$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)L_{2i}(\xi,\eta,a(\xi,\eta)\widetilde{\omega}(\xi,\eta))],$$

$$\widetilde{b}_i(s,t,m,n)=b_i(s,t,m,n)\varphi[\phi^{-1}\overline{w}(s,t))],$$
$$\widetilde{c}_i(s,t,m,n)=c_i(s,t,m,n)\varphi[\phi^{-1}\overline{w}(s,t))],$$
$$i=1,2,...,l_1,$$

$$\widetilde{d}_i(s,t,m,n)$$
$$=d_i(s,t,m,n)\widetilde{\omega}(s,t)T_{1i}(s,t,a(s,t)\widetilde{\omega}(s,t)),$$

$$\widetilde{e}_i(s,t,m,n)$$
$$=e_i(s,t,m,n)\widetilde{\omega}(s,t)T_{2i}(s,t,a(s,t)\widetilde{\omega}(s,t)),$$
$$i=1,2,...,l_2,$$

$$\widetilde{w}(m,n)=\prod_{s=m_0}^{m-1}[1+w(s,n)],$$
$$\widetilde{\mu}_1=\widetilde{J}(M,N),\widetilde{\mu}_2=\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[\widetilde{d}_i(s,t,M,N)$$
$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\widetilde{e}_i(\xi,\eta,M,N)].$$

The proof for Theorem 7 is similar to the combination of Theorem 5 and Theorem 6, and we omit the details here.

Remark 8 *We note that the established inequalities in Theorems 3, 5, 6, 7 are of more general forms than the results in [20, 21]. In fact, in Theorems 3, 5, 6, 7, the left side of the inequalities is an arbitrary function denoted by $\phi(u)$, while in [21, Theorems 5, 8, 10, 11], the left side of the inequalities is u^p. So, in this point, the established inequalities in this paper are fully different from those in [21], and furthermore, are extension of the results in [21]. On the other hand, as stated in [21, Remark 2.12], the established inequalities in [21] are extension of the results in [20]. So our results are extension of the results in [20, 21].*

3 Applications

In this section, we will present some applications for the established results above, and some new bounds will be derived for solutions of certain difference equations.

Example 9 *Consider the following Volterra-Fredholm type sum-difference equation*

$$u^p(m,n) = \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[F_1(s,t,m,n,u(s,t))$$
$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} F_2(\xi,\eta,m,n,u(\xi,\eta))]$$
$$+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{M-1}[G_1(s,t,m,n,u(s,t))$$
$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} G_2(\xi,\eta,m,n,u(\xi,\eta))], \qquad (50)$$

where $u \in \wp(\Omega)$, $p \geq 1$ is an odd number, $F_i,\ G_i : \Omega^2 \times \mathbb{R} \to \mathbb{R},\ i = 1, 2$.

Theorem 10 *Suppose $u(m,n)$ is a solution of (50), and*

$$|F_1(s,t,m,n,u)| \leq f_1(s,t,m,n)|u|^{\frac{p}{2}},$$
$$|F_2(s,t,m,n,u)| \leq f_2(s,t,m,n)|u|^{\frac{p}{2}},$$
$$|G_1(s,t,m,n,u)| \leq g_1(s,t,m,n)|u|^{p},$$
$$|G_2(s,t,m,n,u)| \leq g_2(s,t,m,n)|u|^{p},$$

where $f_i,\ g_i \in \wp_+(\Omega^2),\ i = 1, 2,\ f_i,\ g_i$ are non-decreasing in the last two variables, $p > 0$ is a constant, and there is at least one function among $g_1,\ g_2$ not equivalent to zero, then we have

$$u(m,n) \leq 4^{-\frac{1}{p}}\{\frac{\sqrt{\mu}}{1-\sqrt{\mu}}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} B(s,t,M,N)$$
$$+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} B(s,t,m,n)\}^{\frac{2}{p}}, \qquad (51)$$

provided that $\mu < 1$, where

$$B(s,t,m,n) = f_1(s,t,m,n) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} f_2(\xi,\eta,m,n),$$
$$\mu = \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[g_1(s,t,M,N)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} g_2(\xi,\eta,M,N)].$$

Proof. From (50) we have

$$|u(m,n)|^p \leq \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[|F_1(s,t,m,n,u(s,t))|$$
$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} |F_2(\xi,\eta,m,n,u(\xi,\eta))|]$$
$$+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{M-1}[|G_1(s,t,m,n,u(s,t))|$$
$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} |G_2(\xi,\eta,m,n,u(\xi,\eta))|]$$
$$\leq |a(m,n)| + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[f_1(s,t,m,n)|u(s,t)|^{\frac{p}{2}}$$
$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} f_2(\xi,\eta,m,n)|u(\xi,\eta)|^{\frac{p}{2}}]$$
$$+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{M-1}[g_1(s,t,m,n)|u(s,t)|^{p}$$
$$+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} g_2(\xi,\eta,m,n)|u(\xi,\eta)|^{p}]. \qquad (52)$$

Define $\phi(u) = u^p,\ \varphi(u) = u^{\frac{p}{2}}$, and

$$G(z) = \int_{z_0}^{z}\frac{1}{\sqrt{z}}dz = 2\sqrt{z} - 2\sqrt{z_0},\ z \geq z_0 > 0, \qquad (53)$$

$$T(x) = G(\frac{x}{\mu}) - G(x) = 2\sqrt{\frac{x}{\mu}} - 2\sqrt{x},\ x \geq 0. \qquad (54)$$

Then by $\mu < 1$ we have T is strictly increasing, and a suitable application of Theorem 2.3 (with $a(m,n) = 0$ and $l_1 = l_2 = 1$) to (52) yields

$$u(m,n) \leq \phi^{-1}\{G^{-1}\{G(T^{-1}[\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} B(s,t,M,N)])$$
$$+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} B(s,t,m,n)\}\}. \qquad (55)$$

Combining (53)-(55) we can deduce the desired result. □

Theorem 11 *Suppose $u(m,n)$ is a solution of (50), and*

$$|F_1(s,t,m,n,u)| \leq f_1(s,t,m,n)|u|^{\frac{p}{3}},$$
$$|F_2(s,t,m,n,u)| \leq f_2(s,t,m,n)|u|^{\frac{p}{3}},$$
$$|G_1(s,t,m,n,u)| \leq g_1(s,t,m,n)L_1(s,t,|u|^p),$$
$$|G_2(s,t,m,n,u)| \leq g_2(s,t,m,n)L_2(s,t,|u|^p),$$

where $p,\ f_i,\ g_i,\ i = 1, 2$ are defined as in Theorem 10, $L_1,\ L_2,\ T_1,\ T_2 : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $0 \leq$

$L_i(m,n,u) - L_i(m,n,v) \le T_i(m,n,v)(u-v)$ for $u \ge v \ge 0$ and $L_i(m,n,0) = 0,\ i = 1,\ 2,$ then we have

$$u(m,n) \le \{\frac{2}{3}[\frac{\widehat{\mu}^{\frac{2}{3}}}{1-\widehat{\mu}^{\frac{2}{3}}}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\widehat{B}(s,t,M,N) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widehat{B}(s,t,m,n)]\}^{\frac{3}{2p}}, \tag{56}$$

provided that $\widehat{\mu} < 1$, where

$$\widehat{B}(s,t,m,n) = f_1(s,t,m,n) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} f_2(\xi,\eta,m,n),$$

$$\widehat{\mu} = \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[g_1(s,t,M,N)T_1(s,t,0) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} g_2(\xi,\eta,M,N)T_2(\xi,\eta,0)].$$

Proof. From (50) we have

$$\begin{aligned}|u(m,n)|^p \le& \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[|F_1(s,t,m,n,u(s,t))| + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}|F_2(\xi,\eta,m,n,u(\xi,\eta))|] \\ &+ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{M-1}[|G_1(s,t,m,n,u(s,t))| + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}|G_2(\xi,\eta,m,n,u(\xi,\eta))|] \\ \le& \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[f_1(s,t,m,n)|u(s,t)|^{\frac{p}{3}} + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} f_2(\xi,\eta,m,n)|u(\xi,\eta)|^{\frac{p}{3}}] \\ &+ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{M-1}[g_1(s,t,m,n)L_1(s,t,|u(s,t)|^p) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} g_2(\xi,\eta,m,n)L_1(\xi,\eta,|u(\xi,\eta)|^p)].\end{aligned} \tag{57}$$

Define $\phi(u) = u^p,\ \varphi(u) = u^{\frac{p}{3}}$, and

$$G(z) = \int_{z_0}^{z}\frac{1}{z^{\frac{1}{3}}}dz = \frac{3}{2}[z^{\frac{2}{3}} - z_0^{\frac{2}{3}}],\ z \ge z_0 > 0, \tag{58}$$

$$T(x) = G(\frac{x}{\widehat{\mu}}) - G(x) = \frac{3}{2}(\frac{1-\widehat{\mu}^{\frac{2}{3}}}{\widehat{\mu}^{\frac{2}{3}}})x^{\frac{2}{3}},\ x \ge 0. \tag{59}$$

Then by $\widehat{\mu} < 1$ we have T is strictly increasing, and a suitable application of Theorem 2.3 (with $\phi(u) = u^p,\ \varphi(u) = u^{\frac{p}{3}}$, $a(m,n) = 0$ and $l_1 = l_2 = 1$) to (57) yields

$$u(m,n) \le \phi^{-1}\{G^{-1}\{G(T^{-1}[\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\widehat{B}(s,t,M,N)]) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widehat{B}(s,t,m,n)\}\}. \tag{60}$$

Combining (58)-(60) we can deduce that

$$\begin{aligned}u(m,n) \le& \\ &\phi^{-1}\{G^{-1}\{G([\frac{2\widehat{\mu}^{\frac{2}{3}}}{3(1-\widehat{\mu}^{\frac{2}{3}})}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\widehat{B}(s,t,M,N)]^{\frac{3}{2}}) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widehat{B}(s,t,m,n)\}\} \\ =& \phi^{-1}\{G^{-1}\{\frac{3}{2}[(\frac{2\widehat{\mu}^{\frac{2}{3}}}{3(1-\widehat{\mu}^{\frac{2}{3}})}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\widehat{B}(s,t,M,N)) - z_0^{\frac{2}{3}}] + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widehat{B}(s,t,m,n)\}\} \\ =& \phi^{-1}\{\{\frac{2}{3}[\frac{\widehat{\mu}^{\frac{2}{3}}}{1-\widehat{\mu}^{\frac{2}{3}}}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\widehat{B}(s,t,M,N) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widehat{B}(s,t,m,n)]\}^{\frac{3}{2}}\} \\ =& \{\frac{2}{3}[\frac{\widehat{\mu}^{\frac{2}{3}}}{1-\widehat{\mu}^{\frac{2}{3}}}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\widehat{B}(s,t,M,N) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widehat{B}(s,t,m,n)]\}^{\frac{3}{2p}},\end{aligned}$$

which is the desired result (56). □

4 Conclusions

In this paper, we establish some new Volterra-Fredholm type nonlinear discrete inequalities, which are of more general forms than inequalities of the same kind existing in the literature. In order to illustrate the usefulness of the established inequalities, we apply them to study the boundedness of some certain Volterra-Fredholm type sum-difference equation. As a result, some new bounds are deduced for the solutions of the sum-difference equation.

Acknowledgements: This work is partially supported by National Natural Science Foundation of China (Grant No 11171178). The author would thank the referees very much for their careful comments and valuable suggestions on this paper.

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