Positive solutions of BVPs for some second-order four-point difference systems

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Abstract: This paper is concerned with one type of boundary value problems (BVPs). We first construct Green functions for a second-order four-point difference equation, and try to find delicate conditions for the existence of positive solutions. Our main tool is a nonlinear alternative of Leray-Schauder type, krasnosel'skii's fixed point theorem in a cone and Leggett-Williams fixed point theorem.

Key–Words: Discrete system; Positive solutions; Cone; Nonlinear alternative; Leggett-Williams fixed point theorem; Fixed point.

1 Introduction

In this paper, we consider the following discrete system boundary value problem:

$$\begin{cases} \triangle^2 u_1(k-1) + f_1(k, u_1(k), u_2(k)) = 0, \\ \qquad\qquad k \in [1,T], \\ \triangle^2 u_2(k-1) + f_2(k, u_1(k), u_2(k)) = 0, \\ \qquad\qquad k \in [1,T], \end{cases} \tag{1}$$

with the boundary condition:

$$\begin{array}{ll} \triangle u_1(0) = au_1(l_1), & \triangle u_1(T) = bu_1(l_2), \\ \triangle u_2(0) = au_2(l_1), & \triangle u_2(T) = bu_2(l_2), \end{array} \tag{2}$$

where $T \geq 1$ is a fixed positive integer, $\triangle u(k) = u(k+1) - u(k)$, $\triangle^2 u(k) = \triangle(\triangle u(k))$, $[1,T] = \{1, 2, \dots, T\} \subset Z$ the set of all integers, $l_1,\ l_2 \in [1,T]$, $l_1 < l_2$, $0 \leq a < \frac{1}{l_1}$, $0 \leq b < \frac{1}{l_2}$ and $0 \leq al_1 + bl_2 < 1 + a$, $\delta = a(1 - bl_2) - b(1 - al_1) > 0$.

Many problem in applied mathematics lead to the study of difference system, see [1] and [2] and the references therein. Recently, much attention has been paid to the existence of positive solutions of scalar difference equations [3],[4],[5],[6], [9],[14],[18],[19],[20] and discrete difference systems [8],[13],[16].

In [15], Tian considered the multiplicity for four-point boundary value problems

$$\triangle^2 u(k-1) + q(k) f(k, u(k), \triangle u(k)) = 0,$$

$$k \in N(1,T),$$

$$u(0) = au(l_1), \quad u(T+1) = bu(l_2).$$

Sun and Li [14] investigated the following discrete system

$$\triangle^2 u_1(k) + f_1(k, u_1(k), u_2(k)) = 0, \quad k \in [0,T],$$

$$\triangle^2 u_2(k) + f_2(k, u_1(k), u_2(k)) = 0, \quad k \in [0,T],$$

with the Dirichlet boundary condition

$$u_1(0) = u_1(T+2) = 0, \quad u_2(0) = u_2(T+2) = 0,$$

by using Leggett-Williams fixed point theorem, sufficient conditions are obtained for the existence three positive solutions to the above system.

Motivated by the above works, our purpose in this paper is to study problem (1),(2). Under suitable conditions on f_1 and f_2, we show that the boundary value problems (1),(2) have one or two positive solutions. Since the construction of Green functions for difference equations may be more complicated and overloaded than that for differential equations, the difficulty of this paper is constructing Green functions for (1)(2), which play important roles in the verifying of the existence of positive solutions for the given difference systems . Furthermore, the system (1), (2)consists of two second order difference equations. To the authors best knowledge, no paper has constructed Green functions for a second order four-point difference equation (1), (2). This paper attempts to fill this gap in the literature.

The rest of the paper is organized as follows. First, we shall state two fixed point theorems, the first of which is a nonlinear alternative of Leray-Schauder type, whereas the second is krasnosel'skii's fixed point theorem in a cone. We also present the Green's function for problem (1), (2). In Section 2, criteria for the existence of one and two positive solutions to boundary value problems (1), (2) are established. In Section 3, three positive solution for boundary value problems (1), (2)are obtained. In section 4, we give the conclusion of my paper.

1.1 Several lemmas

In order to prove our main results, the following well-known fixed point theorems are needed.

Lemma 1 ([7]) *Let X be a Banach space with $E \subseteq X$ closed and convex. Assume U is a relatively open ball of E with $0 \in U$ and $T : \bar{U} \to E$ is a continuous and compact map. Then, either*
(a) T has a fixed point in $\bar{U}$, or
(b) there exists $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda T u$.

Lemma 2 ([7]) *Suppose X is a Banach space, $K \subset X$ is a cone. Assume Ω_1, Ω_2 are open subsets of X with $\theta \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Let $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ be a completely continuous operator such that either*
(a) $\|Tu\| \le \|u\|, \quad \forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|, \quad \forall u \in K \cap \partial\Omega_2$, or
(b) $\|Tu\| \ge \|u\|, \quad \forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|, \quad \forall u \in K \cap \partial\Omega_2$.
Then, T has a fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Lemma 3 *Let $\delta := a(1 - bl_2) - b(1 - al_1) \ne 0$, then for $y : [1, T] \to R^+$, the problem*

$$\triangle^2 u(k-1) + y(k) = 0, \quad k \in [1, T], \tag{3}$$

$$\triangle u(0) = au(l_1), \quad \triangle u(T) = bu(l_2), \tag{4}$$

has a unique solution

$$\begin{aligned} u(k) &= \frac{1 - al_1 + ak}{\delta} \sum_{j=1}^{T} y(j) \\ &- + \frac{a(1 - bl_2) + abk}{\delta} \sum_{j=1}^{l_1 - 1} (l_1 - j) y(j) \\ &- \frac{b(1 - al_1) + abk}{\delta} \sum_{j=1}^{l_2 - 1} (l_2 - j) y(j) \\ &- \textstyle\sum_{j=1}^{k-1} (k - j) y(j). \end{aligned}$$

Proof: We proceed from (3) and obtain

$$\triangle^2 u(k-1) = -y(k),$$

after adding from 1 to $i - 1$, we have

$$\triangle u(i-1) = \triangle u(0) - \sum_{j=1}^{i-1} y(j), \tag{5}$$

and then adding (5) from 1 to k,

$$\begin{aligned} u(k) &= u(0) + k\triangle u(0) - \textstyle\sum_{i=1}^{k} \sum_{j=1}^{i-1} y(j) \\ &= u(0) + k\triangle u(0) - \textstyle\sum_{j=1}^{k-1} (k - j) y(j). \end{aligned} \tag{6}$$

From (4) and (6), we can see that

$$\begin{aligned} u(k) &= \frac{1 - al_1 + ak}{\delta} \sum_{j=1}^{T} y(j) \\ &+ \frac{a(1 - bl_2) + abk}{\delta} \sum_{j=1}^{l_1 - 1} (l_1 - j) y(j) \\ &- \frac{b(1 - al_1) + abk}{\delta} \sum_{j=1}^{l_2 - 1} (l_2 - j) y(j) \\ &- \sum_{j=1}^{k-1} (k - j) y(j). \end{aligned}$$

□

Lemma 4 *Let $\delta \ne 0$, the Green's function for the boundary value problem*

$$-\triangle^2 u(k-1) = 0, \quad k \in [1, T], \tag{7}$$

$$\triangle u(0) = au(l_1), \quad \triangle u(T) = bu(l_2), \tag{8}$$

is given by

$$G(k,j) = \begin{cases} \frac{1}{\delta}(1 + kb - bl_2), \\ \qquad 1 \le j \le \min\{k-1, l_1 - 1\} \le T; \\ \frac{1}{\delta}[(bj + 1 - bl_2) \\ +(j-k)(abl_2 - abl_1 - a)], \\ \qquad 0 \le k \le j \le l_1 - 1; \\ \frac{1}{\delta}(bk + 1 - bl_2)(1 + aj - al_1), \\ \qquad l_1 \le j \le \min\{k-1, l_2 - 1\} \le T; \\ \frac{1}{\delta}(bj + 1 - bl_2)(1 + ak - al_1), \\ \qquad \max\{k, l_1\} \le j \le l_2 - 1; \\ \frac{1}{\delta}(1 + ak - al_1) + (j - k), \\ \qquad l_2 \le j \le k - 1 \le T; \\ \frac{1}{\delta}(1 + ak - al_1), \\ \qquad \max\{k, l_2\} \le j \le T. \end{cases}$$

Lemma 5 *Suppose $1 < l_1 < l_2 < T + 1$, $0 < a < \frac{1}{l_1}$, $0 < b < \frac{1}{l_2}$, $0 < al_1 + bl_2 \le 1 + a$, $\delta > 0$. The Green's function $G(k,j)$ satisfies*

$$G(k,j) > 0, \text{ for } 0 < j, k < T + 1, \tag{9}$$

$$\begin{aligned} G(k,j) &\geq \gamma \max_{0\leq k\leq T+1} G(k,j), \text{ for} \\ & l_1 \leq k \leq l_2,\ 1 < j < T+1, \end{aligned} \tag{10}$$

where γ is defined as

$$\gamma = \min\left\{ \frac{1+bl_1-bl_2}{(b(T+1)+1-bl_2)(1+al_2-al_1)}, \frac{1+bl_1-bl_2}{a(T+1)+1-al_1} \right\}, \tag{11}$$

Proof: Notice that

$$\min_{k\in[l_1,l_2]} G(k,j) = \min\left\{ \frac{1}{\delta}(1+bl_1-bl_2), \frac{1}{\delta}(1+bl_1-bl_2)(1+aj-al_1), \frac{1}{\delta} \right\} = \frac{1}{\delta}(1+bl_1-bl_2).$$

$$\begin{aligned} \max_{k\in[0,T+1]} G(k,j) = \max\Big\{ &\frac{1}{\delta}(1+b(T+1)-bl_2), \\ &\frac{1}{\delta}(1+b(T+1)-bl_2)(1+aj-al_1), \\ &\frac{1}{\delta}(bj+1-bl_2)(1+aj-al_1), \\ &\frac{1}{\delta}(a(T+1)+1-al_1)+(j-k), \\ &\frac{1}{\delta}(aj+1-al_1)\Big\} \\ = \max\Big\{ &\frac{1}{\delta}(1+b(T+1)-bl_2)(1+al_2-al_1), \\ &\frac{1}{\delta}(a(T+1)+1-al_1)\Big\}. \end{aligned}$$

Let

$$\gamma = \min\left\{ \frac{1+bl_1-bl_2}{(b(T+1)+1-bl_2)(1+al_2-al_1)}, \frac{1+bl_1-bl_2}{a(T+1)+1-al_1} \right\},$$

it is obvious that $0 < \gamma < 1$. Therefore, we have

$$\begin{aligned} G(k,j) &\geq \gamma \max_{0\leq k\leq T+1} G(k,j), \\ & \text{for } l_1 \leq k \leq l_2,\ 1 < j < T+1. \end{aligned}$$

□

2 Main result

Let the Banach space $B = \{u : [0, T+1] \to R\}$ be endowed with norm,

$$\|u\|_0 = \max_{k\in[0,T+1]} |u(k)|,$$

and $X = B \times B$ with norm

$$\|(u_1, u_2)\| = \max\{\|u_1\|_0, \|u_2\|_0\},$$

and

$$K = \Big\{ (u_1,u_2) \in X : u_i(k) \geq 0,\ k \in [0, T+1], \min_{k\in[l_1,l_2]} u_i(k) \geq \gamma\|u_i\|_0,\ i = 1,2 \Big\},$$

where γ is defined as (11), then K is a cone in X.

The following theorem gives an existence principle for boundary value problems (1),(2). This results is used later to establish the existence of one positive solution of (1),(2).

Theorem 6 *Let $f_i : [1,T] \times R^2 \to R,\ i = 1,2$ be continuous. Suppose that there exists a constant M, independent of λ, such that*

$$\|u\| \neq M, \tag{12}$$

for any solution $u = (u_1, u_2) \in X$ of the boundary value problem

$$\begin{cases} \triangle^2 u_1(k-1) + \lambda f_1(k, u_1(k), u_2(k)) = 0, \\ \qquad k \in [1,T], \\ \triangle^2 u_2(k-1) + \lambda f_2(k, u_1(k), u_2(k)) = 0, \\ \qquad k \in [1,T], \end{cases} \tag{13}$$

and

$$\begin{aligned} \triangle u_1(0) = au_1(l_1), \quad \triangle u_1(T) = bu_1(l_2), \\ \triangle u_2(0) = au_2(l_1), \quad \triangle u_2(T) = bu_2(l_2), \end{aligned} \tag{14}$$

where $\lambda \in (0,1)$. Then, boundary value problems (1),(2) *have at least one solution $u = (u_1,u_2) \in X$ such that $\|u\| \leq M$.*

Proof: Let the operator $T : X \to X$ be defined by

$$T(u_1,u_2) = (U_1(k), U_2(k)) \quad k \in [0, T+1],$$

where

$$U_i(k) = \sum_{j=1}^{T} G(k,j) f_i(j, u_1(j), u_2(j)), \quad i = 1,2.$$

Then, it is noted that T is continuous and completely continuous and that solving (13),(14) is equivalent to finding a $u \in X$ such that $u = \lambda T u$.

In the context of Lemma 1, let

$$U = \{u = (u_1,u_2) \in X : \|u\| < M\}.$$

In view of (12), we cannot have Conclusion (b) of Lemma 1, and hence, Conclusion (a) of lemma 1 holds, i.e., (1),(2) have a solution $u \in \bar{U}$ with $\|u\| \leq M$. The proof is complete. □

Our next results offer the existence of one and two positive solutions of (1), (2). For convenience, the conditions needed are listed as follows.
(H_1) $f_i:\ [1,T]\times[0,\infty)\times[0,\infty)\to[0,\infty)$ is continuous, $i=1,2$;
(H_2) For each $i\in\{1,2\}$, assume that

$$f_i(k,u_1,u_2)\le\alpha_i(k)\omega_{i1}(u_1)\omega_{i2}(u_2),$$

$$\text{for }(k,u_1,u_2)\in[1,T]\times[0,\infty)\times[0,\infty),$$

where $\alpha_i:\ [1,T]\to(0,\infty)$, and $\omega_{il}:\ [0,\infty)\to[0,\infty),\ l=1,2$ are continuous and nondecreasing;
(H_3) There exists $r>0$ such that

$$r>d_i\omega_{i1}(r)\omega_{i2}(r),\quad i=1,2,$$

where

$$d_i=\max_{k\in[0,T+1]}\sum_{j=1}^{T}G(k,j)\alpha_i(j);$$

(H_4) For each $i=\{1,2\}$, there exist $\tau_{il}:\ [l_1,l_2]\to(0,\infty),\ l=1,2$ such that $f_i(k,u_1,u_2)\ge\tau_{il}(k)\omega_{il}(u_l),\ k\in[l_1,l_2],\ u_l>0$;
(H_5) There exists $R>r$ such that for $\forall\ l\in\{1,2\},\ x\in[\gamma R,R]$, the following holds for some i (depending on l) in $\{1,2\}$:

$$x\le\omega_{il}(x)\gamma\sum_{j=l_1}^{l_2}G(\sigma_{il},j)\tau_{il}(j)\tag{15}$$

where $\sigma_{il}\in[0,T+1]$ is defined as

$$\sum_{j=l_1}^{l_2}G(\sigma_{il},j)\tau_{il}(j)=\max_{k\in[0,T+1]}\sum_{j=l_1}^{l_2}G(k,j)\tau_{il}(j);$$

(H_6) There exists $L\in(0,r)$ such that for $\forall l\in\{1,2\},\ x\in[\gamma r,r]$, inequality (15) holds for some i (depending on l) in $1,2$.

Theorem 7 *Suppose that* (H_1) $-$ (H_3) *hold. Then, boundary value problems* (1),(2) *have a positive solution* $u^*=(u_1^*,u_2^*)\in X$ *such that* $\|u^*\|<r$, *where* r *is defined by* (H_3).

Proof: First, we consider the following boundary value problem

$$\begin{cases}\triangle^2u_1(k-1)+\tilde f_1(k,u_1(k),u_2(k))=0,\\ \qquad k\in[1,T],\\ \triangle^2u_2(k-1)+\tilde f_2(k,u_1(k),u_2(k))=0,\\ \qquad k\in[1,T],\end{cases}\tag{16}$$

and

$$\begin{array}{ll}\triangle u_1(0)=au_1(l_1), & \triangle u_1(T)=bu_1(l_2),\\ \triangle u_2(0)=au_2(l_1), & \triangle u_2(T)=bu_2(l_2),\end{array}\tag{17}$$

where $\tilde f_i:[1,T]\times R^2\to R$ is defined by

$$\tilde f_i(k,u_1,u_2)=f_i(k,|u_1|,|u_2|),\quad i=1,2.$$

It is obvious that $\tilde f_i$ is continuous.

We shall show that (16), (17) have a solution. For this, we look at the following problem:

$$\begin{cases}\triangle^2u_1(k-1)+\lambda\tilde f_1(k,u_1(k),u_2(k))=0,\\ \qquad k\in[1,T],\\ \triangle^2u_2(k-1)+\lambda\tilde f_2(k,u_1(k),u_2(k))=0,\\ \qquad k\in[1,T],\end{cases}\tag{18}$$

and

$$\begin{array}{ll}\triangle u_1(0)=au_1(l_1), & \triangle u_1(T)=bu_1(l_2),\\ \triangle u_2(0)=au_2(l_1), & \triangle u_2(T)=bu_2(l_2),\end{array}\tag{19}$$

where $\lambda\in(0,1)$. Let $u=(u_1,u_2)\in X$ be any solution of (18), (19). We claim that

$$\|u\|\ne r.\tag{20}$$

In fact, it is clear that

$$u_1(k)=\lambda\textstyle\sum_{j=1}^{T}G(k,j)\tilde f_1(j,u_1(j),u_2(j)),\quad k\in[0,T+1],\tag{21}$$

$$u_2(k)=\lambda\textstyle\sum_{j=1}^{T}G(k,j)\tilde f_2(j,u_1(j),u_2(j)),\quad k\in[0,T+1].\tag{22}$$

Noting that (21), (H_2) and (H_3), it follows that

$$\begin{aligned}0&\le u_1(k)=\lambda\sum_{j=1}^{T}G(k,j)\tilde f_1(j,u_1(j),u_2(j))\\ &=\lambda\sum_{j=1}^{T}G(k,j)f_1(j,|u_1(j)|,|u_2(j)|)\\ &\le\lambda\sum_{j=1}^{T}G(k,j)\alpha_1(j)\omega_{11}(|u_1(j)|)\omega_{12}(|u_2(j)|)\\ &\le\lambda\sum_{j=1}^{T}G(k,j)\alpha_1(j)\omega_{11}(\|u\|)\omega_{12}(\|u\|)\\ &\le\omega_{11}(\|u\|)\omega_{12}(\|u\|)\max_{k\in[0,T+1]}\sum_{j=1}^{T}G(k,j)\alpha_1(j)\\ &=d_1\omega_{11}(\|u\|)\omega_{12}(\|u\|),\ \ \forall k\in[0,T+1].\end{aligned}$$

This immediately leads to

$$\|u_1\|_0\le d_1\omega_{11}(\|u\|)\omega_{12}(\|u\|).\tag{23}$$

Similarly, from (22), (H_2) and (H_3), we have

$$\|u_2\|_0\le d_2\omega_{21}(\|u\|)\omega_{22}(\|u\|).\tag{24}$$

If

$$\|u\| = \|u_p\|_0, \quad \text{for some } p \in 1, 2,$$

then (23), (24) yield

$$\|u\| \leq d_p \omega_{p1}(\|u\|)\omega_{p2}(\|u\|),$$

from which we conclude, by comparing with (H_3), that $\|u\| \neq r$.

It now follows from Theorem 6 that boundary value problems (16),(17) have a solution $u^* = (u_1^*, u_2^*) \in X$ such that $\|u^*\| \leq r$. Using a similar argument as above, we see that $\|u^*\| \neq r$. Therefore,

$$\|u^*\| < r. \tag{25}$$

Moreover, for $\forall\, k \in [0, T+1]$, we have

$$\begin{aligned} u_1^*(k) &= \sum_{j=1}^{T} G(k,j)\tilde{f}_1(j, u_1^*(j), u_2^*(j)) \\ &= \sum_{j=1}^{T} G(k,j) f_1(j, |u_1^*(j)|, |u_2^*(j)|), \end{aligned} \tag{26}$$

$$\begin{aligned} u_2^*(k) &= \sum_{j=1}^{T} G(k,j)\tilde{f}_2(j, u_1^*(j), u_2^*(j)) \\ &= \sum_{j=1}^{T} G(k,j) f_2(j, |u_1^*(j)|, |u_2^*(j)|). \end{aligned} \tag{27}$$

and it follows immediately that

$$u_i^*(k) \geq 0, \quad k \in [0, T+1], \ \ i = 1, 2, \tag{28}$$

so,

$$\begin{aligned} u_1^*(k) &= \sum_{j=1}^{T} G(k,j) f_1(j, |u_1^*(j)|, |u_2^*(j)|) \\ &= \sum_{j=1}^{T} G(k,j) f_1(j, u_1^*(j), u_2^*(j)), \end{aligned} \tag{29}$$

$$\begin{aligned} u_2^*(k) &= \sum_{j=1}^{T} G(k,j) f_2(j, |u_1^*(j)|, |u_2^*(j)|) \\ &= \sum_{j=1}^{T} G(k,j) f_2(j, u_1^*(j), u_2^*(j)), \end{aligned} \tag{30}$$

i.e., $u^* = (u_1^*, u_2^*) \in X$ is a positive solution of boundary value problems (1) and (2) and satisfies $\|u^*\| < r$. □

Theorem 8 *Suppose that* $(\mathrm{H}_1) - (\mathrm{H}_5)$ *hold. Then, boundary value problems* (1),(2) *have two positive solutions* $u^*, \bar{u} \in X$ *such that*

$$0 \leq \|u^*\| < r < \|\bar{u}\| \leq R,$$

where r *and* R *are defined by* (H_3) *and* (H_5).

Proof: The existence of u^* is guaranteed by Theorem 7. We shall employ Lemma 2 to prove the existence of $\bar{u}$.

Let $T : K \to X$ be defined by

$$T(u_1, u_2) = (U_1(k), U_2(k)), \quad k \in [0, T+1],$$

where

$$U_i(k) = \sum_{j=1}^{T} G(k,j) f_i(j, u_1(j), u_2(j)), \quad i = 1, 2.$$

First, we shall show that T maps K into itself. For this, let $u = (u_1, u_2) \in K$. Then, it follows immediately that

$$U_i(k) \geq 0, \quad k \in [0, T+1], \ \ i = 1, 2. \tag{31}$$

Then, we obtain for each $i \in \{1, 2\}$,

$$\begin{aligned} U_i(k) &= \sum_{j=1}^{T} G(k,j) f_i(j, u_1(j), u_2(j)) \\ &\leq \sum_{j=1}^{T} \max\nolimits_{k\in[0,T+1]} G(k,j) f_i(j, u_1(j), u_2(j)), \\ & k \in [0, T+1]. \end{aligned}$$

As a result,

$$\begin{aligned} \|U_i\|_0 &\leq \sum_{j=1}^{T} \max\nolimits_{k\in[0,T+1]} G(k,j) f_i(j, u_1(j), u_2(j)), \\ & i = 1, 2. \end{aligned} \tag{32}$$

Now, in view of (32) and Lemma 5, we have for $\forall k \in [l_1, l_2], \ i = 1, 2,$

$$\begin{aligned} U_i(k) &= \sum_{j=1}^{T} G(k,j) f_i(j, u_1(j), u_2(j)) \\ &\geq \gamma \sum_{j=1}^{T} \max_{k\in[0,T+1]} G(k,j) f_i(j, u_1(j), u_2(j)) \\ &\geq \gamma \|U_i\|_0. \end{aligned}$$

Therefore,

$$\min_{k\in[l_1,l_2]} U_i(k) \geq \gamma \|U_i\|_0, \ \ i = 1, 2. \tag{33}$$

Combining (31) and (33), we obtain $T(K) \subseteq K$. Also, the standard arguments yield that T is completely continuous.

Let

$$\Omega_1 = \{u \in X : \ \|u\| < r\}$$
$$\text{and} \quad \Omega_2 = \{u \in X : \ \|u\| < R\}.$$

We claim that

(i) $\quad \|Tu\| \leq \|u\|, \quad$ for $u \in K \cap \partial\Omega_1,$

(ii) $\quad \|Tu\| \geq \|u\|, \quad$ for $u \in K \cap \partial\Omega_2.$

To justify (i), let $u = (u_1, u_2) \in K \cap \partial\Omega_1$, then $\|u\| = r$ and by (H_2) and (H_3) we have

$$\begin{aligned} 0 \quad &\leq U_i(k) = \sum_{j=1}^{T} G(k,j) f_i(j, u_1(j), u_2(j)) \\ &\leq \sum_{j=1}^{T} G(k,j)\alpha_i(j)\omega_{i1}(u_1(j))\omega_{i2}(u_2(j)) \\ &\leq \omega_{i1}(\|u\|)\omega_{i2}(\|u\|) \sum_{j=1}^{T} G(k,j)\alpha_i(j) \\ &\leq \omega_{i1}(\|r\|)\omega_{i2}(\|r\|) \max_{k\in[0,T+1]} \sum_{j=1}^{T} G(k,j)\alpha_i(j) \\ &= d_i\omega_{i1}(\|r\|)\omega_{i2}(\|r\|) < r = \|u\|, \\ &\ k \in [0, T+1], \ \ i = 1, 2. \end{aligned}$$

Therefore, $\|U_i\|_0 \leq \|u\|, \ \ i = 1, 2$, and so

$$\|Tu\| = \max\{\|U_1\|_0, \|U_2\|_0\} \leq \|u\|.$$

Next, we prove (ii). Let

$$u = (u_1, u_2) \in K \cap \partial\Omega_2.$$

So,

$$\|u\| = \max\{\|u_1\|_0, \|u_2\|_0\} = R = \|u_p\|_0,$$

for some $p \in \{1, 2\}$. Then, it follows that

$$0 \leq u_p(k) \leq R, \quad k \in [0, T+1],$$

and

$$u_p(k) \geq \gamma R, \quad k \in [l_1, l_2].$$

Thus, we have

$$\gamma R \leq u_p(k) \leq R, \quad k \in [l_1, l_2]. \tag{34}$$

In view of (34), (H_4) and (H_5), we find that the following holds for some i (depending on p) in $\{1, 2\}$:

$$\begin{aligned} U_i(\sigma_{ip}) \quad &= \sum_{j=1}^{T} G(\sigma_{ip}, j) f_i(j, u_1(j), u_2(j)) \\ &\geq \sum_{j=l_1}^{l_2} G(\sigma_{ip}, j) f_i(j, u_1(j), u_2(j)) \\ &\geq \sum_{j=l_1}^{l_2} G(\sigma_{ip}, j)\tau_{ip}(j)\omega_{ip}(u_p(j)) \\ &\geq \sum_{j=l_1}^{l_2} G(\sigma_{ip}, j)\tau_{ip}(j)\omega_{ip}(\gamma R) \\ &\geq \sum_{j=l_1}^{l_2} G(\sigma_{ip}, j)\tau_{ip}(j) \\ &\quad \frac{\gamma R}{\gamma \sum_{j=l_1}^{l_2} G(\sigma_{ip}, j)\tau_{ip}(j)} = R \\ &= \|u\|. \end{aligned}$$

Hence, $\|U_i\|_0 \geq \|u\|$, and so $\|Tu\| \geq \|u\|$.

Having obtained (i) and (ii), it follows from Lemma 2 that T has a fixed point $\bar{u} \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$, i.e., $\bar{u}$ is a positive solution of (1), (2) and

$$r \leq \|\bar{u}\| \leq R.$$

Using a similar argument as in the proof of Theorem 7, we see that

$$r < \|\bar{u}\| \leq R.$$

□

It is noted in Theorem 8 that $\|u^*\|$ may be zero. Our next result guarantees that $\|u^*\| \neq 0$.

Theorem 9 *Suppose that* $(\mathrm{H}_1) - (\mathrm{H}_6)$ *hold. Then, boundary value problems* (1),(2) *have two positive solutions* $u^*, \bar{u} \in X$ *such that*

$$0 < L \leq \|u^*\| < r < \|\bar{u}\| \leq R,$$

where $L,\ r$ *and* R *are defined by* $(\mathrm{H}_3),\ (\mathrm{H}_5)$ *and* (H_6), *respectively.*

Proof: The existence of $\bar{u}$ is guaranteed by Theorem 8. We shall employ Lemma 2 to show the existence of u^*. Suppose that the set Ω_1 and the map $T : K \to K$ are the same as in the proof of Theorem 8.

Let

$$\Omega_3 = \{u \in X : \ \|u\| < L\}.$$

From the proof of Theorem 8, we see that
(i) $\quad \|Tu\| \leq \|u\|$, for $u \in K \cap \partial\Omega_1$;
thus, it remains to prove that
(ii) $\quad \|Tu\| \geq \|u\|$, for $u \in K \cap \partial\Omega_3$.
For this, let

$$u = (u_1, u_2) \in K \cap \partial\Omega_3.$$

Assume that

$$\|u\| = \|u_p\|_0 = L, \quad \text{for some } p \in \{1, 2\}.$$

Then, we have

$$\gamma L \leq u_p(k) \leq L, \quad k \in [l_1, l_2]. \tag{35}$$

In view of (35), (H_4) and (H_6), we find that the following holds for some i (depending on p) in $\{1,2\}$:

$$\begin{aligned} U_i(\sigma_{ip}) &= \sum_{j=1}^{T} G(\sigma_{ip},j) f_i(j,u_1(j),u_2(j)) \\ &\geq \sum_{j=l_1}^{l_2} G(\sigma_{ip},j) f_i(j,u_1(j),u_2(j)) \\ &\geq \sum_{j=l_1}^{l_2} G(\sigma_{ip},j)\tau_{ip}(j)\omega_{ip}(u_p(j)) \\ &\geq \sum_{j=l_1}^{l_2} G(\sigma_{ip},j)\tau_{ip}(j)\omega_{ip}(\gamma L) \\ &= \sum_{j=l_1}^{l_2} G(\sigma_{ip},j)\tau_{ip}(j) \\ &\frac{\gamma L}{\gamma \sum_{j=l_1}^{l_2} G(\sigma_{ip},j)\tau_{ip}(j)} = L \\ &= \|u\|. \end{aligned}$$

Hence, $\|U_i\|_0 \geq \|u\|$, and so $\|Tu\| \geq \|u\|$.

Having obtained (i) and (ii), we conclude from Lemma 2 that T has a fixed point

$$u^* \in K \cap (\overline{\Omega_1} \backslash \Omega_3),$$

i.e., u^* is a positive solution of boundary value problems (1), (2) and

$$L \leq \|u^*\| \leq r.$$

Using a similar argument as in the proof of Theorem 7, we see that

$$L < \|u^*\| \leq r.$$

□

Example: Consider the following boundary value problem:

$$\begin{cases} \triangle^2 u_1(k-1) + \mu \exp(u_1^{\frac{1}{2}} + u_2^{\frac{1}{8}}) = 0, \\ \qquad k \in [1,20], \\ \triangle^2 u_2(k-1) + \mu \exp(u_1^{\frac{1}{6}} + u_2^{\frac{1}{3}}) = 0, \\ \qquad k \in [1,20], \end{cases} \tag{36}$$

with the boundary condition:

$$\begin{aligned} \triangle u_1(0) = \tfrac{1}{6} u_1(5), \quad \triangle u_1(20) = \tfrac{1}{100} u_1(10), \\ \triangle u_2(0) = \tfrac{1}{6} u_2(5), \quad \triangle u_2(20) = \tfrac{1}{100} u_2(10), \end{aligned} \tag{37}$$

where $\mu > 0$. It is easy to prove that $(\mathrm{H}_1) - (\mathrm{H}_5)$ are satisfied when μ is small enough. Hence, it follows from Theorem 8 that boundary value problems (38), (39) have two positive solutions when μ is small enough.

Remark 10 *If conditions (H_2) and (H_3) are replaced by $(H_2)'$ and $(H_3)'$, respectively, where*
$(H_2)'$ for each $i \in \{1,2\}$, assume that

$$f_i(k,u_1,u_2) \leq \alpha_i(k)\omega_{i1}(u_1) + \beta_i(k)\omega_{i2}(u_2),$$

$$\text{for } (k,u_1,u_2) \in [1,T] \times [0,\infty) \times [0,\infty),$$

where $\alpha_i, \beta_i : [1,T] \to (0,\infty)$, and $\omega_{il} : [0,\infty) \to [0,\infty)$, $l = 1,2$ are continuous and nondecreasing;
$(H_3)'$ There exists $r > 0$ such that

$$r > d_i[\omega_{i1}(r) + \omega_{i2}(r)], \quad i = 1,2,$$

where

$$d_i = \max\left\{ \max_{k\in[0,T+1]} \sum_{j=1}^{T} G(k,j)\alpha_i(j), \max_{k\in[0,T+1]} \sum_{j=1}^{T} G(k,j)\beta_i(j) \right\}, \quad i = 1,2,$$

then, similar conclusions are true.

3 Three positive solution for boundary value problems (1), (2)

Let the Banach space $B = \{u : [0,T+1] \to R\}$ be endowed with norm,

$$\|u\|_0 = \max_{k\in[0,T+1]} |u(k)|,$$

and $X = B \times B$ with norm

$$\|(u_1,u_2)\| = \max\{\|u_1\|_0, \|u_2\|_0\},$$

and

$$\begin{aligned} P &= \Big\{ (u_1,u_2) \in X : u_i(k) \geq 0, \ k \in [0,T+1], \\ &\min_{k\in[l_1,l_2]} u_i(k) \geq \gamma \|u_i\|_0, \ i = 1,2 \Big\}, \end{aligned}$$

where γ is defined as (11), then P is a cone in X.

A map α is said to be a nonnegative continuous concave functional on P if

$$\alpha : P \to [0,+\infty)$$

is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

For numbers a, b such that $0 < a < b$ and α is a nonnegative continuous concave functional on P we define the following convex sets

$$P_a = \{x \in P : \|x\| < a\},$$

$$P(\alpha, a, b) = \{x \in P : \ a \le \alpha(x), \ \|x\| \le b\}.$$

Leggett-Williams fixed point theorem. Let $A : \overline{P_c} \to \{x \in P : \ \|x\| < a\}$ be completely continuous and α be a nonnegative continuous functional on P such that $\alpha(x) \le \|x\|$ for all $x \in \overline{P_c}$. Suppose there exist $0 < d < a < b \le c$ such that
(i) $\{x \in P(\alpha, a, b) : \ \alpha(x) > a\} \ne \emptyset$ and $\alpha(Ax) > a$ for $x \in P(\alpha, a, b)$;
(ii) $\|Ax\| < d$ for $\|x\| \le d$;
(iii) $\alpha(Ax) > a$ for $x \in P(\alpha, a, c)$ with $\|Ax\| > b$.

Then A has at least three fixed pointsx_1, x_2, x_3 satisfying

$$\|x_1\| < d, \quad a < \alpha(x_2),$$

$$\|x_3\| > d, \ \text{ and } \ \alpha(x_3) < a.$$

Theorem 11 *Suppose that* $f_i : \ [1, T] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ *is continuous and that there exist numbers* a *and* d *with* $0 < d < a$ *such that the following conditions are satisfied:*
(i) *if* $j \in [1, T], \ u_1, \ u_2 \ge 0$ *and* $u_1 + u_2 \le d$; *then* $f_i(j, u_1, u_2) < \dfrac{d}{2D}$, $i = 1, 2$, *where* $D = \max\limits_{k \in [0, T+1]} \sum\limits_{j=1}^{T} G(k, j)$;
(ii) there exists $i_0 \in \{1, 2\}$, *such that* $f_{i_0}(j, u_1, u_2) > \dfrac{a}{C}$, $j \in [1, T], \ u_1, \ u_2 \ge 0$ *and* $u_1 + u_2 \in [a, \dfrac{a}{\gamma}]$, *where* $C = \min\limits_{k \in [l_1, l_2]} \sum\limits_{j=1}^{T} G(k, j)$;
(iii) one of the following conditions holds:

$$(A) \qquad \lim_{u_1+u_2 \to \infty} \max_{j \in [0,T]} \frac{f_i(j,u_1,u_2)}{u_1 + u_2} < \frac{1}{2D}, \quad i = 1, 2;$$

(B) there exists a number c *such that* $c > \dfrac{a}{\gamma}$ *and if* $j \in [1, T], \ u_1, \ u_2 \ge 0, \ u_1 + u_2 \le c$ *then* $f_i(j, u_1, u_2) < \dfrac{c}{2D}, \quad i = 1, 2.$
Then the boundary value problem (1), (2) has at least three positive solutions.

Proof: For $u = (u_1, \ u_2) \in P$, define

$$\alpha(u) = \min_{k \in [l_1, l_2]} u_1(k) + \min_{k \in [l_1, l_2]} u_2(k),$$

$$A(u_1, u_2) = (U_1(k), U_2(k)) \quad k \in [0, T+1],$$

where

$$U_i(k) = \sum_{j=1}^{T} G(k, j) f_i(j, u_1(j), u_2(j)), \quad i = 1, 2,$$

then it is easy to know that α is a nonnegative continuous concave functional on P with $\alpha(x) \le \|x\|$ for $x \in P$ and that $A : \ P \to P$ is completely continuous.

For the sake of convenience, set $b = \frac{a}{\gamma}$.

Claim 1. If there exists a positive number r such that

$$f_i(j, u_1, u_2) < \frac{r}{2D}, \ i = 1, 2$$

for $j \in [1, T], \ u_1, \ u_2 \ge 0, \ u_1 + u_2 \le r$, then

$$A : \ \overline{P_r} \to P_r.$$

Suppose that $u = (u_1, \ u_2) \in \overline{P_r}$, then

$$\begin{aligned} \|U_i\|_0 &= \max_{k \in [0, T+1]} \sum_{j=1}^{T} G(k, j) f_i(j, u_1(j), u_2(j)) \\ &< \frac{r}{2D} D = \frac{r}{2}, \ i = 1, 2. \end{aligned}$$

Thus

$$\|Au\| = \|u_1\|_0 + \|u_2\|_0 < \frac{r}{2} + \frac{r}{2} = r.$$

Then there exists a number c such that $c > b$ and $A : \overline{P_c} \to P_c$. From Claim 1 with $r = d$ and (i) that $A : \overline{P_d} \to P_d$.

Claim 2. We show that $\{u \in P(\alpha, a, b) : \ \alpha(u) > a\} \ne \emptyset$ and $\alpha(Au) > a$ for $u \in P(\alpha, a, b)$
In fact,

$$u = (u_1(k), u_2(k)) = \left(\frac{a+b}{4}, \frac{a+b}{4} \right)$$

$$\in \{u \in P(\alpha, a, b) : \ \alpha(u) > a\}.$$

For $u = (u_1(k), u_2(k)) \in P(\alpha, a, b)$, we have

$$\begin{aligned} b &\ge \|u_1\|_0 + \|u_2\|_0 \ge u_1(k) + u_2(k) \\ &\ge \min_{k \in [l_1, l_2]} u_1(k) + \min_{k \in [l_1, l_2]} u_2(k) \ge a, \end{aligned}$$

for all $k \in [l_1, l_2]$. Then, in view of (ii), we know that

$$\begin{aligned} \min_{k \in [l_1, l_2]} U_{i_0}(k) &= \min_{k \in [l_1, l_2]} \sum_{j=1}^{T} G(k, j) f_{i_0}(j, u_1(j), u_2(j)) \\ &> \frac{a}{C} \min_{k \in [l_1, l_2]} \sum_{j=1}^{T} G(k, j) = a, \end{aligned}$$

and so

$$\begin{aligned} \alpha(Au) &= \min_{k \in [l_1, l_2]} U_1(k) + \min_{k \in [l_1, l_2]} U_2(k) \\ &\ge \min_{k \in [l_1, l_2]} U_{i_0}(k) > a. \end{aligned}$$

Claim 3. If $u \in P(\alpha, a, c)$ and $\|Au\| > b$, then $\alpha(Au) > a$.

Suppose $u = (u_1, u_2) \in P(\alpha, a, c)$ and $\|Au\| > b$, then, by Lemma 5, we have

$$\begin{aligned}\min_{k\in[l_1,l_2]} U_i(k) &= \min_{k\in[l_1,l_2]} \sum_{j=1}^{T} G(k,j) f_i(j, u_1(j), u_2(j))\\ &= \min_{k\in[l_1,l_2]} \sum_{j=1}^{T} \frac{G(k,j)}{\max_{0\le k\le T+1} G(k,j)}\\ &\quad \max_{0\le k\le T+1} G(k,j) f_i(j, u_1(j), u_2(j)),\\ &\ge \gamma \sum_{j=1}^{T} \max_{0\le k\le T+1} G(k,j) f_i(j,u_1(j),u_2(j))\\ &= \gamma \|U_i(k)\|_0, \ \ i = 1, 2.\end{aligned}$$

Thus

$$\min_{k\in[l_1,l_2]} U_i(k) \ge \gamma \|U_i(k)\|_0, \ \ i = 1, 2,$$

and so

$$\begin{aligned}\alpha(Au) &= \min_{k\in[l_1,l_2]} U_1(k) + \min_{k\in[l_1,l_2]} U_2(k)\\ &\ge \gamma(\|U_1\|_0 + \|U_2\|_0)\\ &= \gamma\|Au\| > \gamma b = a.\end{aligned}$$

Therefore, the hypotheses of Leggett-Williams theorem are satisfied, hence the boundary value problem (1), (2) has at least three positive solutions $u = (u_1, u_2),\ v = (v_1, v_2)$ and $w = (w_1, w_2)$ such that

$$\|u\| < d,$$

$$a < \min_{k\in[l_1,l_2]} v_1(k) + \min_{k\in[l_1,l_2]} v_2(k),$$

$$\|w\| > d \text{ with } \min_{k\in[l_1,l_2]} w_1(k) + \min_{k\in[l_1,l_2]} w_2(k) < a.$$

The proof is complete. □

Example: Consider the following boundary value problem:

$$\begin{cases} \triangle^2 u_1(k-1) + f_1(k, u_1(k), u_2(k)) = 0, \\ \qquad k \in [1, 20], \\ \triangle^2 u_2(k-1) + f_2(k, u_1(k), u_2(k)) = 0, \\ \qquad k \in [1, 20], \end{cases} \tag{38}$$

with the boundary condition:

$$\begin{aligned}\triangle u_1(0) = \tfrac{1}{6} u_1(5), \quad \triangle u_1(20) = \tfrac{1}{100} u_1(10),\\ \triangle u_2(0) = \tfrac{1}{6} u_2(5), \quad \triangle u_2(20) = \tfrac{1}{100} u_2(10),\end{aligned} \tag{39}$$

$$f_i(k, u_1(k), u_2(k)) = \begin{cases} \frac{k}{4485800} + \frac{88}{224290}(u_1 + u_2), \\ \qquad 0 \le u_1 + u_2 \le 1. \\ \frac{k}{4485800} + \frac{88}{224290}[50(\sqrt{u_1+u_2} - 1) + 1], \\ \qquad u_1 + u_2 \ge 1. \end{cases}$$

Then, boundary value problem (38), (39) has at least three positive solutions.

Proof: Choose $d = 1$, $a = 1600$. By computation, we know

$$\gamma = \frac{57}{220}, D = \frac{112145}{89}, C = \frac{11190}{89}.$$

Thus

$$\begin{aligned}f_i(k, u_1(k), u_2(k)) &= \frac{k}{4485800} + \frac{88}{224290}(u_1 + u_2)\\ &< \frac{89}{224290} = \frac{d}{2D},\end{aligned}$$

for $k \in [1, 20]$, $0 \le u_1 + u_2 \le 1$, and

$$\begin{aligned}f_i(k, u_1(k), u_2(k)) &= \frac{k}{4485800} + \frac{88}{224290}[50(\\ &\sqrt{u_1+u_2} - 1) + 1] > \frac{142400}{11190} = \frac{a}{C},\end{aligned}$$

for $k \in [1, 20]$, $1600 \le u_1 + u_2 \le \frac{352000}{57}$.

$$\lim_{u_1+u_2\to\infty} \max_{j\in[0,T]} \frac{f_i(j,u_1,u_2)}{u_1 + u_2} = 0 < \frac{1}{2D}, \quad i = 1, 2.$$

To sum up, by an application of Theorem 11, we know that (38), (39) has at least three positive solutions.

4 Conclusion

In this paper, we first construct Green functions for a second-order four-point difference equation, and try to find suitable conditions on f_1 and f_2, which can guarantee that the boundary value problems (1),(2) have one, two or three positive solutions. Our work presented in this paper has the following new features. Firstly, BVP (1),(2) is a four-point boundary value problem, which is very difficult when constructing Green functions. Secondly, the main tool used in this paper is Leray-Schauder type [7], krasnosel'skii's fixed point theorem in a cone [7] and Leggett-Williams fixed point theorem [17] and the result obtained is the multiple positive solutions of BVP (1),(2). Thirdly, the system (1), (2)consists of two second order difference equations.

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