New Approach to Find the Exact Solution of Fractional Partial Differential Equation

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Abstract: - In this study, we present the exact solution of certain fractional partial differential equations (FPDE) by using a modified homotopy perturbation method (MHPM).The exact solutions are constructed by choosing an appropriate initial approximation and only one term of the series obtained by MHPM. The exact solutions for initial value problems of FPDE are analytically derived. The methods introduced an efficient tool for solving a wide class of time-fractional partial differential equations.

Key-Words: - Mitting-leffler functions, Green function, Caputo derivative, Backward Klomogorov equation.

1 Introduction

In recent years, fractional calculus has been increasingly used for numerous applications in many scientific and technical fields such as medical sciences, biological research, as well as various chemical, biochemical and physical fields. Fractional calculus can be, for instance, employed to solve a lot of problems within the biomedical research field. Such an important application is studying membrane biophysics and polymer viscoelasticity [17].

Other promising biomedical application fields where fractional calculus can be used is analyzing chaos and nonlinear systems of fractional order. An interesting example of the practical application of fractional order models is to use these models to improve the behavior of bioelectrodes. Such bioelectrodes are usually used for all forms of biopotential recording purposes, such as Electrocardiography (ECG), Electromyo-graphy (EMG) and Electroencephalography (EEG) [26, 27, 18, 5]. In addition to that, theses bioelectrodes are also used for functional electrical stimulation, as in the case of pacemaker and deep brain stimulation [26, 27, 18, 5].

Almost all of the approaches used to solve the problems mentioned above end with solving fractional partial differential equations (FPDE). Therefore, it is natural to see that modeling by using FPDE have interested a wide segment of researchers [39, 21] within numerous application areas in natural and technological sciences.

Srivastava and Rai [36] have used the fractional diffusion equation to construct a mathematical model for oxygen delivery to tissues through a capillary. Mainardi [29] also used fractional partial differential equations to model the propagation of mechanical diffusive waves in viscoelastic media.

Numerous methods and approaches were proposed and used to solve FPDEs. Some of these methods are analytical, such as the Fourier transform method [27], the Fractional Green’s function method [24], the popular Laplace transform method [31, 32], the iteration method [34], the Mellin transform method and the method of orthogonal polynomials [31].

Numerical methods and approaches are also popular and used to obtain approximate solutions of FPDEs. Examples of such numerical methods for solving FPDEs are the Homotopy Perturbation Method (HPM) [6, 9, 11, 14, 15], the Differential Transform Method (DTM) [28] the Variational

Iteration Method (VIM) [8], the New Iterative Method (NIM) [3, 4], the homotopy analysis method (HAM) [1, 17], and the Adomian Decomposition Method (ADM) [2].

Among these numerical methods, the VIM and the ADM are the most popular ones that are used to solve differential and integral equations of integer and fractional order. The HPM is a universal approach which can be used to solve both fractional ordinary differential equations FODEs as well as fractional partial differential equations FPDEs.

This method (the HPM), was originally proposed by He [9, 10]. The HPM is a coupling of homotopy in the topology and the perturbation method. The method is utilized to solve various types of equations such us the Hellmholtz equation, the fifth order KdV [34], the Kleein-Gorden equation [31], the Fokker-Panck equation [38, 39], the nonlinear Kolmogorov-Petrovskii-Piskunov Equation [8] as well as some other types of equations as proposed and used in [12, 13, 14].

Various combinations of the methods mentioned previously have been proposed recently to solve fractional partial differential equations FPDEs. Examples of such combination methods are the Homotopy Analysis Transform Method, the Variational Homotopy Perturbation Method and the Homotopy Perturbation Transformation Method.

The Homotopy Analysis Transform Method is a combination of the Homotopy Analysis Method and the Laplace Decomposition Method [20].

In the case of the Variational Homotopy Perturbation Method, Noor and Mohyud-Din [30] have combined the Variational Iteration Method and the Homotopy Perturbation Method. They used this method for solving higher dimensional initial boundary value problems.

On the other hand, the Homotopy Perturbation Transformation Method was constructed by combining two powerful methods; namely the Homotopy Perturbation Method and the Laplace Transform Method [23]. In the latter case, this successful combination could compensate for the limitation of the Laplace Transform Method which was totally incapable of handling nonlinear equations.

In general, there exists no method that gives an exact solution for FPDEs and most obtained solutions are only approximations. However, Yang [37] used the modified homotopy perturbation method (MHPM) to obtain the exact solution of the Fokker-Plank equation which is a PDE of integer order.

In this paper, the MHPM is used to derive the exact solution of various types of FPDEs instead of PDEs. We present an elegant fast approach by designing and utilizing a proper initial approximation which satisfies the initial condition of the HPM. Successful design results are obtained when this step is performed by means of separation of variables.

The structure of this paper is as follows: In section 1, we begin with an introduction to some necessary definitions of fractional calculus theory. In section 2, the basic idea of the MHPM is presented. In section 3, we present four examples to show the efficiency of using the MHPM to solve FPDEs. Finally, relevant conclusions are drawn in section 4.

2 Basic Definitions of Fractional Calculus

In this section, we present the basic definitions and properties of the fractional calculus theory, which are used further in this paper.

Definition 2.1. A real function $f(t), t > 0$, is said to be in the space $C_\sigma, \sigma \in \mathbb{R}$ if there exists a real number $p > \sigma$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_σ^m if $f^m \in C_\sigma, m \in \mathbb{N}$.

Definition 2.2. The left sided Riemann–Liouville fractional integral of order $\alpha \geq 0$, of a function $f \in C_\sigma, \sigma \geq -1$ is defined as:

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} f(\zeta) d\zeta, \quad (1)$$

where $\alpha > 0, t > 0$ and $\Gamma(\alpha)$ is the gamma function. Properties of the operator J_t^α for $f \in C_\mu, \mu \geq -1,\ \alpha, \beta \geq 0,\ \gamma \geq -1$ can be found for instance in [35], and are defined as follows

$$J_t^0 f(t) = f(t). \quad (2)$$

$$J_t^\alpha J_t^\beta f(t) = J_t^{\alpha+\beta} f(t). \quad (3)$$

$$J_t^\alpha J_t^\beta f(t) = J_t^\beta J_t^\alpha f(t). \quad (4)$$

$$J_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}. \quad (5)$$

Definition 2.3. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D_t^\alpha f(t) = J_t^{n-\alpha} D_t^n f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\zeta)^{n-\alpha-1} f^{(n)}(\zeta) d\zeta, \qquad (6)$$

for $n-1 < \alpha \le n, n \in \mathbb{N}, t > 0, f \in C_\mu^n, \mu \ge -1$.

For Caputo's derivative we have

$$D_t^\alpha C = 0, \quad \text{(C is a constant)} \qquad (7)$$

$$D_t^\alpha t^\gamma = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha} & \gamma > \alpha - 1 \\ 0 & \gamma \le \alpha - 1 \end{cases} \qquad (8)$$

$$J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^k(0^+) \frac{t^k}{k!}, \qquad (9)$$

Definition 2.4. The single parameter and the two parameters variants of the Mitting-leffler function are denoted by $E_\alpha(t)$, and $E_{\alpha,\beta}(t)$, respectively, which are relevant for their connection with fractional calculus, and are defined as,

$$E_\alpha(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j + 1)}, \qquad \alpha > 0, t \in \mathbb{C}$$

$$E_{\alpha,\beta}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j + \beta)}, \qquad \alpha, \beta > 0, t \in \mathbb{C}$$

(10)

Their k-th derivatives are

$$E_\alpha^{(k)}(t) = \frac{d^k}{dt^k} E_\alpha(t),$$

$$= \sum_{j=0}^{\infty} \frac{(k+j)!\, t^j}{j!\, \Gamma(\alpha j + \alpha k + 1)}, \quad k = 0,1,2, \ldots,$$

(11)

$$E_{\alpha,\beta}^{(k)}(t) = \frac{d^k}{dt^k} E_{\alpha,\beta}(t),$$

$$= \sum_{j=0}^{\infty} \frac{(k+j)!\, t^j}{j!\, \Gamma(\alpha j + \alpha k + \beta)}, \quad k = 0,1,2 \ldots,$$

(12)

Some special cases of the Mitting-Leffler function are as follows:

$$E_1(t) = e^t \qquad (13)$$

$$E_{\alpha,1}(t) = E_\alpha(t) \qquad (14)$$

$$\frac{d^k}{dt^k}\left[t^{\beta-1} E_{\alpha,\beta}(at^\alpha)\right] = t^{\beta-k-1} E_{\alpha,\beta-k}(at^\alpha) \qquad (15)$$

Other properties of the Mitting-leffler functions can be found in [21]. These functions are generalizations of the exponential function, because, most linear differential equations of fractional order have solutions that are expressed in terms of these functions.

Theorem 2.5. Consider the following n-term linear fractional differential equation [6]:

$$\left(a_n D_t^{\beta_n} + a_{n-1} D_t^{\beta_{n-1}} + \cdots + a_0 D_t^{\beta_0}\right) u(t) = f(t),$$

(16)

with the constant initial condition:

$$u^{j_i}(0) = C_{ij_i} \quad i = 0,1, \ldots, n, \ \ j_i = 1,2, \ldots, l_i, \qquad (17)$$

where $a_i, C_{ij_i} \in \mathbb{R}$, $n_i - 1 < \beta_i \le n_i, n_i \in \mathbb{N} \cup \{0\}$ and

$$\beta_0 < \beta_1 < \cdots < \beta_{n-1} < n \le \beta_n < n+1$$

Then, we see that the analytical general solution of Eq. (12) is

$$u(t) = \int_0^t G_n(t-\zeta) f(\zeta) d\zeta + \sum_{i=0}^{\infty} \sum_{j_i=0}^{l_i-1} a_i C_{ij_i} G_n^{\beta_i - j_i - 1}(t),$$

(18)

where $G_n(t)$ is the Green function and it is defined as

$$G_n(t) = \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \times \sum_{\substack{k_0, k_1, \ldots, k_{n-2} \ge 0 \\ k_0 + k_1 + \cdots + k_{n-2} = m}} (m; k_0, k_1, \ldots, k_{n-2})$$

$$\times \prod_{p=0}^{n-2} \left(\frac{a_p}{a_n}\right)^{k_p} t^{(\beta_n - \beta_{n-1})m + \beta_n + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j - 1}$$

$$\times E^{(m)}_{\beta_n - \beta_{n-1}, \beta_n + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j - 1} \left(-\frac{a_{n-1}}{a_n} D^{\beta_n - \beta_{n-1}}\right),$$

(19)

where

$$(m; k_0, k_1, \ldots, k_{n-2}) = \frac{m!}{k_0! k_1! \ldots, k_{n-2}!} \qquad (20)$$

and $E^{(m)}_{(.),(.)}$ is the m-th derivative of the Mitting-Leffler function.

In a special case of the latter theorem, the following relaxation-oscillation equation [6] is solved:

$$D_t^{\alpha}u(t) + Au(t) = f(t), \qquad t > 0, \tag{21}$$

$$u^i(0) = b_i, \qquad i = 1,2,\dots,n-1, \tag{22}$$

where b_i are real constants and $n-1 < \alpha \le n$.

By utilizing theorem 2.5 we obtain the solution of Eq. (21) as follows:

$$u(t) = \int_0^t G_2(t-\zeta)\, f(\zeta)d\zeta + \sum_{j=0}^{n-1} b_j D_t^{\alpha-j-1} G_2(t) \tag{23}$$

where $G_2(t) = t^{\alpha-1}E_{\alpha,\alpha}(-At^{\alpha})$.

It is easy to see that if $0 < \alpha \le 1$, then the solution of Eq. (21) becomes as follows:

$$u(t) = \int_0^t G_2(t-\zeta)\, f(\zeta)d\zeta + b_0 D_t^{\alpha-1}G_2(t) \tag{24}$$

which will be used in the coming examples, discussed in this paper.

3 The Basic Idea of the Homotopy Perturbation Method

The Homotopy Perturbation Method (HPM) is a combination of the Homotopy technique and the classical Perturbation Method. The HPM is applied to various nonlinear problems as mentioned in the previous sections of this paper.

In this section, we will briefly present the algorithm of this method. To achieve our goal, we consider the nonlinear differential equation:

$$\begin{cases} \mathcal{A}(u) - f(r) = 0, & r \in \Omega, \\ B\left(u, {\partial u}/{\partial n}\right) = 0, & r \in \Gamma, \end{cases} \tag{25}$$

where B is a boundary operator, Γ is the boundary of the domain Ω, $f(r)$ is a known analytic function and $\mathcal{A}$ is a general differential function operator.

The operator $\mathcal{A}$ can be decomposed into a linear operator, denoted by $\mathcal{L}$, and a nonlinear operator, denoted by $\mathcal{N}$. Therefore, Eq. (25) can be written as follows

$$\mathcal{L}(u) + \mathcal{N}(u) - f(r) = 0, \tag{26}$$

By using the homotopy perturbation technique, we construct a homotopy $v(r,p)\colon \Omega \times [0,1] \to \mathbb{R}$ which satisfies:

$$\mathcal{H}(v,p) = (1-p)[\mathcal{L}(v) - \mathcal{L}(u_0)] + p[\mathcal{A}(u) - f(r)] = 0, \quad 0 \le p \le 1, \tag{27}$$

or

$$\mathcal{H}(v,p) = \mathcal{L}(v) - \mathcal{L}(u_0) + p\mathcal{L}(u_0) + p[\mathcal{N}(v) - f(r)] = 0, \quad 0 \le p \le 1, \tag{28}$$

where $r \in \Omega$, u_0 is an initial approximation for Eq. (25) and p is an embedding parameter. When the value of p is changed from $p = 0$ to $p = 1$, we can easily see that

$$\mathcal{H}(v,0) = \mathcal{L}(v) - \mathcal{L}(u_0) = 0, \tag{29}$$

$$\mathcal{H}(v,1) = \mathcal{L}(v) + \mathcal{N}(v) - f(r) = \mathcal{A}(u) - f(r) = 0, \tag{30}$$

This changing process is called deformation, and Eq. (29) and (30) are called homotopic in topology.

If the p-parameter is considered as small, then the solution of Eq. (27) and (28) can be expressed as a power series in p, as follows:

$$v = \sum_{i=0}^{\infty} p^i v_i = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots \tag{31}$$

The best approximation for the solution of Eq. (25) is:

$$u = \lim_{p\to 1} v = \sum_{i=0}^{\infty} v_i = v_0 + v_1 + v_2 + v_3 + \cdots, \tag{32}$$

Now we are able to apply the HPM to solve the class of time fractional partial differential equations defined as follows:

$$D_t^{\alpha}u(x,t) = \mathcal{L}(u(x,t)) + \mathcal{N}(u(x,t)) + f(x,t), \tag{33}$$

subject to the initial condition

$$u(x,0) = g(x) \tag{34}$$

where $0 < \alpha \le 1$ in D_t^{α}, which is identical to the Cupoto fractional derivative of order α.

Now we construct a homotopy

$v(x,t,p)\colon \Omega \times [0,1] \to \mathbb{R}$, which satisfies

$$\mathcal{H}(v(x,t),p) \equiv (1-p)[D_t^\alpha(v(x,t)) - D_t^\alpha(u_0(x,t))] + p[D_t^\alpha(v(x,t)) - \mathcal{L}(v(x,t)) - \mathcal{L}(v(x,t)) - \mathcal{N}(v(x,t)) - f(x,t)] = 0, \quad (35)$$

where $p \in [0,1]$ and $u_0(x,t)$ is an initial approximation of the solution of Eq.(33) which also satisfies the initial condition in Eq. (34).

By simplifying Eq. (35) we get

$$D_t^\alpha(v(x,t)) = D_t^\alpha(u_0(x,t)) + p[D_t^\alpha(u_0(x,t)) - \mathcal{L}(v(x,t)) - \mathcal{N}(v(x,t)) - f(x,t)] \quad (36)$$

where the embedding parameter p is considered to be small and applied to the classical perturbation technique.

The next step is to use this homotopy parameter p to expand the solution into the following form:

$$v(x,t) = v_0(x,t) + pv_1(x,t) + p^2v_2(x,t) + \cdots \quad (37)$$

Eventually, at $p = 1$, we will obtain the approximate solution of Eq. (33). By substituting Eq. (37) into Eq. (36) and equating the terms with identical powers of p, we can obtain a series of equations as the follows

$$\begin{aligned}
p^0&: \quad D_t^\alpha(v_0(x,t)) = D_t^\alpha(u_0(x,t)),\\
p^1&: \quad D_t^\alpha(v_1(x,t)) = D_t^\alpha(v_0(x,t)) - \mathcal{L}(v_0(x,t)) - \mathcal{N}(v_0(x,t)) - f(x,t),\\
p^2&: \quad D_t^\alpha(v_2(x,t)) = D_t^\alpha(v_1(x,t)) - \mathcal{L}(v_1(x,t)) - \mathcal{N}(v_0(x,t),v_1(x,t)),\\
&\vdots
\end{aligned} \quad (38)$$

Applying the operator J_t^α, which is the Riemann–Liouville fractional integral of order $\alpha \geq 0$, on both sides of all cases of Eq.(38), the solution can be given by

$$\begin{aligned}
v_0(x,t) &= u_0(x,t),\\
v_1(x,t) &= J_t^\alpha[D_t^\alpha(v_0(x,t)) - \mathcal{L}(v_0(x,t)) - \mathcal{N}(v_0(x,t)) - f(x,t)],\\
v_2(x,t) &= J_t^\alpha[D_t^\alpha(v_1(x,t)) - \mathcal{L}(v_1(x,t)) - \mathcal{N}(v_0(x,t),v_1(x,t))],\\
&\vdots
\end{aligned} \quad (39)$$

By utilizing the results in Eq. (39), and substituting them into Eq. (37), we get an accurate n^th approximation of the exact solution as follows

$$u_n(x,t) = v_0 + v_1 + \cdots + v_n = \sum_{i=0}^{n} v_i \quad (40)$$

In Eq. (40), if there exists some $v_n = 0$, $n \geq 1$, then the exact solution can be written in the following form

$$u(x,t) = v_0 + v_1 + \cdots + v_{n-1} = \sum_{i=0}^{n-1} v_i \quad (41)$$

Now we can introduce the core of the work in this paper. At first, we consider the initial approximation of Eq.(25) as follows

$$u_0(x,t) = u(x,0)c_1(t) + u(x)c_2(t), \quad (42)$$

where $u(x,0)$ is the initial condition of Eq.(18), and $u(x) = \frac{\partial u(x,0)}{\partial x}$.

The final goal of our new approach is finding $c_1(t)$ and $c_2(t)$. Hence, for simplicity, we assume that $v_1(x,t) \equiv 0$ in Eq. (28), which means that the exact solution in Eq. (28) is $(x,t) = v_0(x,t)$. And when solving Eq. (26), we obtain the result

$$u_0(x,t) = v_0(x,t). \quad (43)$$

Since $u(x,t)$ satisfies the initial condition as well as Eq. (29), we get

$$\begin{aligned}
u(x,0) &= v_0(x,0)\\
&= u(x,0)c_1(0) + u(x)c_2(0) = g(x),
\end{aligned} \quad (44)$$

and

$$c_1(0) = 1, c_2(0) = 0, \quad (45)$$

On the other hand, we have

$$D_t^\alpha(v_1(x,t)) = D_t^\alpha(u_0(x,t)) - \mathcal{L}(v_0(x,t)) - \mathcal{N}(v_0(x,t)) - f(x,t) \equiv 0 \quad (46)$$

By substituting Eq. (43) and (42) into Eq. (46), we obtain

$$u(x,0)D_t^\alpha(c_1(t)) + u(x)D_t^\alpha(c_2(t)) = \mathcal{L}(u(x,0)c_1(t) + u(x)c_2(t)) + \mathcal{N}(u(x,0)c_1(t) + u(x)c_2(t)) + f(x,t) \quad (47)$$

In this case, the fractional partial differential equation is changed into a fractional ordinary differential equation, which simplifies the problem at hand. Furthermore, if we consider α as an integer number in Eq. (47), then we obtain an ordinary differential equation, which means further simplification. Finally, the target unknowns $c_1(t)$ and $c_2(t)$ can be obtained by utilizing Eq. (47) and the initial conditions in Eq. (45).

4 Applications

In this section, in order to assess the applicability and the accuracy of the procedure described in the last section, we consider the following five examples. In examples 4.3 and 4.4, two cases of the single-variable time-fractional Fokker-Plank equation are solved. The general forms of these equations are as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial t^2}B(x,t)\right]u$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial t^2}B(x,t,u)\right]u$$

These equations are used in numerous applications within various areas in physics such us plasma, surface, laser and polymer physics. In examples 4.1 and 4.5, the forward and backward Kolmogorov equations are solved.

Example 4.1 Consider the linear time-fractional partial differential equation

$$D_t^\alpha u = u_{xx} + u_x, \tag{48}$$

where $t > 0$, $x\epsilon\mathbb{R}$, $0 < \alpha \leq 1$, subject to the initial condition

$$u(x,0) = x \tag{49}$$

and $u(x) = \frac{\partial}{\partial x}u(x,0) = 1$

Choose the initial approximation

$$u_0(x,t) = u(x,0)c_1(t) + u(x)c_2(t)$$
$$= xc_1(t) + c_2(t), \tag{50}$$

then

$$D_t^\alpha v_1 = D_t^\alpha\big(xc_1(t) + c_2(t)\big) - \frac{\partial^2}{\partial x^2}\big(xc_1(t) + c_2(t)\big) - \frac{\partial}{\partial x}\big(xc_1(t) + c_2(t)\big) \equiv 0,$$

$$D_t^\alpha v_1 = xD_t^\alpha c_1(t) + D_t^\alpha c_2(t) - c_1(t) \equiv 0, \tag{51}$$

We obtain the fractional differential system

$$\begin{cases} D_t^\alpha c_1(t) = 0, \\ c_1(0) = 1, \end{cases} \tag{52}$$

$$\begin{cases} D_t^\alpha c_2(t) - c_1(t) = 0, \\ c_2(0) = 0, \end{cases} \tag{53}$$

Solving Eq. (52) and (53) by applying Eq. (24), we obtain

$$c_1(t) = 1$$
$$c_2(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{54}$$

and the exact solution is

$$u(x,t) = x + \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{55}$$

If we put $\alpha \to 1$ in Eq. (55) or solve Eq. (52) and (53) with $\alpha = 1$, we obtain the exact solution

$$u(x,t) = x + t \tag{56}$$

Example 4.2 Consider the linear time-fractional partial differential equation

$$D_t^\alpha u = u_{xx} + u, \tag{57}$$

where $t > 0$, $x\epsilon\mathbb{R}$, $0 < \alpha \leq 1$, subject to the initial condition

$$u(x,0) = \cos(\pi x)$$

Choose the initial approximation

$$u_0(x,t) = co\,s(\pi x)\,c_1(t) - \pi sin\,(\pi x)c_2(t), \tag{58}$$

then

$$D_t^\alpha v_1 = [D_t^\alpha c_1(t) + \pi^2 c_1(t) - c_1(t)]cos\,(\pi x) - \pi sin\,(\pi x)[D_t^\alpha c_2(t) + \pi^2 c_2(t) - c_2(t)] \equiv 0, \tag{59}$$

We obtain the fractional differential system

$$\begin{cases} D_t^\alpha c_1(t) + \pi^2 c_1(t) - c_1(t) = 0, \\ c_1(0) = 1, \end{cases} \tag{60}$$

$$\begin{cases} D_t^\alpha c_2(t) + \pi^2 c_2(t) - c_2(t) = 0, \\ c_2(0) = 0, \end{cases} \tag{61}$$

Solving Eq. (60) and (61) by applying Eq. (24), we obtain

$$c_1(t) = D_t^{\alpha-1}\left(t^{\alpha-1}E_{\alpha,\alpha}\big((\pi^2-1)t^\alpha\big)\right)$$

$$= E_\alpha\left((\pi^2 - 1)t^\alpha\right),$$

$$c_2(t) = 0, \tag{62}$$

and the exact solution is

$$u(x,t) = cos(\pi x)\, E_\alpha\left((\pi^2 - 1)t^\alpha\right), \tag{63}$$

If we put $\alpha \to 1$ in Eq.(63) or solve Eq. (60) and (61) with $\alpha = 1$, we obtain the exact solution

$$u(x,t) = cos(\pi x)\, e^{-(\pi^2-1)t}, \tag{64}$$

Example 4.3 Consider the linear time-fractional Fokker-Plank equation

$$D_t^\alpha u = -\frac{\partial}{\partial x}\left(\frac{4u}{x} - \frac{x}{3}\right)u + \frac{\partial^2}{\partial x^2}u^2, \tag{65}$$

where $t > 0,\ x\epsilon\mathbb{R},\ 0 < \alpha \leq 1$, subject to the initial condition

$$u(x,0) = x^2 \tag{66}$$

Choose the initial approximation

$$u_0(x,t) = x^2 c_1(t) + 2x c_2(t). \tag{67}$$

then

$$D_t^\alpha v_1 = -x^2\left(D_t^\alpha c_1(t) - c_1(t)\right) - x\left(D_t^\alpha c_2(t) - \frac{2}{3}c_2(t) + 4c_1(t)c_2(t)\right) - 2c_2^2(t) \equiv 0, \tag{68}$$

We obtain the following fractional differential system

$$\begin{cases} D_t^\alpha c_1(t) - c_1(t) = 0, \\ c_1(0) = 1, \end{cases} \tag{69}$$

$$\begin{cases} c_2^2(t) = 0, \\ c_2(0) = 0, \end{cases} \tag{70}$$

Solving Eq. (69), (70) by applying Eq. (24), we obtain

$$c_1(t) = D_t^{\alpha-1}\left(t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha)\right) = E_\alpha(t^\alpha),$$

$$c_2(t) = 0, \tag{71}$$

and the exact solution is

$$u(x,t) = x^2 E_\alpha(t^\alpha), \tag{72}$$

If we put $\alpha \to 1$ in Eq.(50) or solve Eq. (47) and (48) with $\alpha = 1$, we obtain the exact solution

$$u(x,t) = x^2 e^t \tag{73}$$

which is in full agreement with the result in Ref. [39].

Example 4.4 Consider the time-fractional Fokker-Plank equation

$$D_t^\alpha u = -\frac{\partial}{\partial x}\left(e^t\, coth(x)\, cosh(x) + e^t\, sinh(x) - coth(x)\right)u + \frac{\partial^2}{\partial x^2}e^t\, cosh(x)\, u, \tag{74}$$

where $t > 0,\ x\epsilon\mathbb{R},\ 0 < \alpha \leq 1$, subject to the initial condition

$$u(x,0) = \sinh(x), \tag{75}$$

Choose the initial approximation

$$u_0(x,t) = sinh(x)\, c_1(t) + cosh\,(x) c_2(t). \tag{76}$$

then

$$D_t^\alpha v_1 = \frac{1}{sinh^2(x)}\Big[sinh(x)\, cosh^2(x)\left(D_t^\alpha c_1(t) - c_1(t)\right) + sinh(x)\left(D_t^\alpha c_1(t) - c_1(t)\right) + cosh^3(x)\left(D_t^\alpha c_2(t) - c_2(t)\right) + cosh(x)\left(2c_2(t) - D_t^\alpha c_2(t)\right) - e^t D_t^\alpha c_2(t)\Big] \equiv 0, \tag{77}$$

We obtain the following fractional differential system

$$\begin{cases} D_t^\alpha c_1(t) - c_1(t) = 0, \\ c_1(0) = 1, \end{cases} \tag{78}$$

$$\begin{cases} D_t^\alpha c_2(t) - c_2(t) = 0, \\ c_2(0) = 0, \end{cases} \tag{79}$$

Solving Eq. (78), (79) by applying Eq. (24), we obtain

$$c_1(t) = D_t^{\alpha-1}\left(t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha)\right) = E_\alpha(t^\alpha),$$

$$c_2(t) = 0, \tag{80}$$

and the exact solution is

$$u(x,t) = cos(\pi x)\, E_\alpha(t^\alpha), \tag{81}$$

If we put $\alpha \to 1$ in Eq.(81) or solve Eq. (78) and (79) with $\alpha = 1$, then we obtain the exact solution

$$u(x,t) = e^t cos(\pi x) \quad (82)$$

which is in full agreement with the result in Ref. [37].

Example 4.5 Consider the non-homogenous time-fractional Backward Klomogorov equation

$$D_t^\alpha u = -x^2 e^t u_{xx} + (x+1)u_x + tx, \quad (83)$$

where $t > 0,\ x\epsilon\mathbb{R},\ 0 < \alpha \leq 1$, subject to the initial condition

$$u(x,0) = x + 1, \quad (84)$$

Choose initial approximation

$$u_0(x,t) = (x+1)c_1(t) + c_2(t). \quad (85)$$

then

$$D_t^\alpha v_1 = x(D_t^\alpha c_1(t) - c_1(t) - t) + \left(D_t^\alpha c_1(t) + D_t^\alpha c_2(t) - c_1(t)\right) \equiv 0, \quad (86)$$

We obtain the fractional differential system

$$\begin{cases} D_t^\alpha c_1(t) - c_1(t) - t = 0, \\ c_1(0) = 1, \end{cases} \quad (87)$$

$$\begin{cases} D_t^\alpha c_2(t) + D_t^\alpha c_1(t) - c_1(t) = 0, \\ c_2(0) = 0, \end{cases} \quad (88)$$

Solving Eq. (87), (88) by applying Eq. (28) and (5), we obtain

$$c_1(t) = \int_0^t G_2(t-\zeta)\zeta d\zeta + D_t^{\alpha-1}\left(t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha)\right) = \sum_{j=0}^{\infty} \frac{t^{\alpha j+\alpha+1}}{(\alpha j+\alpha+1)(\alpha j+\alpha)\Gamma(\alpha j+\alpha)} + E_\alpha(t^\alpha) \quad (89)$$

$$c_2(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \quad (90)$$

and the exact solution is

$$u(x,t) = (x+1)[E_\alpha(t^\alpha) + \sum_{j=0}^{\infty} \frac{t^{\alpha j+\alpha+1}}{(\alpha j+\alpha+1)(\alpha j+\alpha)\Gamma(\alpha j+\alpha)}] + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \quad (91)$$

If we put $\alpha \to 1$ in Eq. (91) or solve Eq. (87) and (88) with $\alpha = 1$, we obtain the exact solution

$$u(x,t) = (x+1)[E_1(t) + \sum_{j=0}^{\infty} \frac{t^{j+2}}{\Gamma(j+1)(j+2)(j+1)}] + \frac{t^2}{\Gamma(3)}, = (x+1)(2e^t - t - 1) - \tfrac{1}{2}t^2, \quad (92)$$

4 Conclusion

It is well known that numerous mathematical models of physical, chemical and biological systems lead to or end up with partial differential equations. These equations may be of fractional or integer order. Finding the exact solutions for these equations is very important for the understanding of many phenomena and processes in natural sciences.

In this paper, we have proposed a new analytical method based on the homotopy perturbation method (HPM) to solve time fractional partial differential equations. This method is intuitive and very easy to understand. It can be easily implemented and it is fast (i.e. computationally efficient) in finding the desired exact solution. Since our new approach converts a partial differential equation (PDE) into an ordinary differential equation (ODE) and after that proceeds to solve the resulting ODE. Thus, it can be used to solve equations with fractional and integer order with respect to time.

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