Oscillation Criteria For A Class Of Third Order Dynamic Equation With Damping On Time Scales

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Abstract: In this paper, some new oscillation criteria for a class of third order dynamic equations with damping on time scales are established by a generalized Riccati transformation technique. The established oscillation criteria unify continuous and discrete analysis, and are new results so far in the literature.

Key–Words: Oscillation; Dynamic equations; Qualititative properties; Time scales; Third order; Damping.

1 Introduction

The theory of time scale, which was initiated by Hilger [1], trying to treat continuous and discrete analysis in a consistent way, have received a lot of attention in recent years. Various investigations have been done by many authors. Among these investigations, some authors have taken research in the oscillation of dynamic equations on time scales, and there has been increasing interest in obtaining sufficient conditions for the oscillation and asymptotic of solutions of various dynamic equations on time scales (for example, we refer to [2-20]). But we notice that most of the investigations are concerned with oscillation of solutions of first or second order dynamic equations on time scales, while relatively less attention has been paid to oscillation of solutions of third order dynamic equations on time scales. For recent results about the oscillation of solutions of third order dynamic equations on time scales, we refer to [21-34]. Moreover, none of the existing results deal with oscillation of solutions of third order dynamic equations with damping term on time scales to our best knowledge.

In this paper, we are concerned of oscillation of solutions of the third order dynamic equation with damping term on time scales of the following form:

$$\begin{aligned}&(a(t)[r(t)x^{\Delta}(t)]^{\Delta})^{\Delta}+p(t)[r(t)x^{\Delta}(t)]^{\Delta}\\&+q(t)x(t)=0,\quad t\in\mathbb{T}_0,\end{aligned}\tag{1}$$

where $\mathbb{T}$ is an arbitrary time scale, $\mathbb{T}_0 = [t_0,\infty)\bigcap\mathbb{T}$, $a,\ r,\ p,\ q\in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$.

A solution of Eq. (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Eq. (1) is said to be oscillatory in case all its solutions are oscillatory.

We will establish some new oscillation criteria for Eq. (1) by a generalized Riccati transformation technique in Section 2, and present some applications for our results in Section 3. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = (0,\infty)$, while $\mathbb{Z}$ denotes the set of integers. $\mathbb{T}$ denotes an arbitrary time scale and $\mathbb{T}_0 = [t_0,\infty)\bigcap\mathbb{T}$, where $t_0\in\mathbb{T}$. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma\in(\mathbb{T},\mathbb{T})$ and $\rho\in(\mathbb{T},\mathbb{T})$ such that $\sigma(t)=\inf\{s\in\mathbb{T}, s>t\}$, $\rho(t)=\sup\{s\in\mathbb{T}, s<t\}$. A point $t\in\mathbb{T}$ with $t>\inf\mathbb{T}$ is said to be left-dense if $\rho(t)=t$ and right-dense if $\sigma(t)=t$, left-scattered if $\rho(t)<t$ and right-scattered if $\sigma(t)>t$. A function $f\in(\mathbb{T},R)$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1+\mu(t)f(t)\neq 0$, where $\mu(t)=\sigma(t)-t$. C_{rd} denotes the set of rd-continuous functions, while $\Re$ denotes the set of all regressive and rd-continuous functions, and $\Re^+=\{f|f\in\Re,\ 1+\mu(t)f(t)>0,\ \forall t\in\mathbb{T}\}$.

Definition 1: For $p\in\Re$, the *exponential function* is defined by

$$e_p(t,s)=\exp(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau)$$

for $s,t\in\mathbb{T}$.

Remark 2: If $\mathbb{T}=\mathbb{R}$, then

$$e_p(t,s)=\exp(\int_s^t p(\tau)d\tau),\quad for\quad s,\ t\in\mathbb{R},$$

If $\mathbb{T}=\mathbb{Z}$, then

$$e_p(t,s)=\prod_{\tau=s}^{t-1}[1+p(\tau)],$$

for $s,\ t\in\mathbb{Z}$ and $s<t$.

The following two theorems include some known properties on the *exponential function*.

Theorem 3 [**35**, **Theorem 5.1**]: If $p\in\Re$, and fix $t_0\in\mathbb{T}$, then the *exponential function* $e_p(t,t_0)$ is the unique solution of the following initial value problem

$$\begin{cases} y^{\Delta}(t)=p(t)y(t),\\ y(t_0)=1.\end{cases}$$

Theorem 4 [**35**, **Theorem 5.2**]: If $p\in\Re^+$, then $e_p(t,s)>0$ for $\forall s,\ t\in\mathbb{T}$.

For more details about the calculus of time scales, we refer to [36].

2 Main Results

For the sake of convenience, in the rest of this paper, we set $\delta_1(t,t_1)=\int_{t_1}^t\frac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}\Delta s$, and we always assume $t_i\in\mathbb{T},\ i=1,2,...,5$.

Lemma 5. Suppose $-\frac{p}{a}\in\Re_+$, and assume that

$$\int_{t_0}^{\infty}\frac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}\Delta s=\infty, \tag{2}$$

$$\int_{t_0}^{\infty}\frac{1}{r(s)}\Delta s=\infty, \tag{3}$$

and Eq. (1) has a positive solution x $[t_0,\infty)_{\mathbb{T}}$. Then there exists a sufficiently large t_1 such that

$$\Big(\frac{a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)}\Big)^{\Delta}<0,\ [r(t)x^{\Delta}(t)]^{\Delta}>0$$

on $[t_1,\infty)_{\mathbb{T}}$.

Proof. By $-\frac{p}{a}\in\Re_+$, we have $e_{-\frac{p}{a}}(t,t_0)>0$. Since x is a positive solution of (1) on $[t_0,\infty)_{\mathbb{T}}$, we obtain that

$$\begin{aligned}&\Big(\frac{a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)}\Big)^{\Delta}\\ &=\frac{(a(t)[r(t)x^{\Delta}(t)]^{\Delta})^{\Delta}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}-\frac{(e_{-\frac{p}{a}}(t,t_0))^{\Delta}a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)e_{-\frac{p}{a}}(\sigma(t),t_0)}\\ &=\frac{(a(t)[r(t)x^{\Delta}(t)]^{\Delta})^{\Delta}+p(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}\\ &=\frac{-q(t)x(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}<0.\end{aligned} \tag{4}$$

Then $\frac{a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)}$ is strictly decreasing on $[t_0,\infty)_{\mathbb{T}}$, and together with $a(t)>0$, $e_{-\frac{p}{a}}(t,t_0)>0$ we deduce that $[r(t)x^{\Delta}(t)]^{\Delta}$ is eventually of one sign. We claim $[r(t)x^{\Delta}(t)]^{\Delta}>0$ on $[t_1,\infty)_{\mathbb{T}}$. Otherwise, assume there exists a sufficiently large t_2 such that $[r(t)x^{\Delta}(t)]^{\Delta}<0$ on $[t_2,\infty)_{\mathbb{T}}$. Then

$$\begin{aligned}&r(t)x^{\Delta}(t)-r(t_2)x^{\Delta}(t_2)\\ &=\int_{t_2}^t\frac{e_{-\frac{p}{a}}(s,t_0)a(s)[r(s)x^{\Delta}(s)]^{\Delta}}{e_{-\frac{p}{a}}(s,t_0)a(s)}\Delta s\\ &\le\frac{a(t_2)[r(t_2)x^{\Delta}(t_2)]^{\Delta}}{e_{-\frac{p}{a}}(t_2,t_0)}\int_{t_2}^t\frac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}\Delta s.\end{aligned} \tag{5}$$

By (2), we have $\lim\limits_{t\to\infty}r(t)x^{\Delta}(t)=-\infty$, and thus there exists a sufficiently large $t_3\in[t_2,\infty)_{\mathbb{T}}$ such that $r(t)x^{\Delta}(t)<0$ on $[t_3,\infty)_{\mathbb{T}}$. By the assumption $[r(t)x^{\Delta}(t)]^{\Delta}<0$ one can see $r(t)x^{\Delta}(t)$ is strictly decreasing on $[t_3,\infty)_{\mathbb{T}}$, and then

$$\begin{aligned}&x(t)-x(t_3)=\int_{t_3}^t\frac{r(s)x^{\Delta}(s)}{r(s)}\Delta s\\ &\le r(t_3)x^{\Delta}(t_3)\int_{t_3}^t\frac{1}{r(s)}\Delta s.\end{aligned}$$

Using (3), we have $\lim\limits_{t\to\infty}x(t)=-\infty$, which leads to a contradiction. So $[r(t)x^{\Delta}(t)]^{\Delta}>0$ on $[t_1,\infty)_{\mathbb{T}}$, and the proof is complete.

Lemma 6. Under the conditions of Lemma 5, furthermore, assume that

$$\begin{aligned}&\lim_{t\to\infty}\sup\int_{t_0}^t\Big[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\Big(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\\ &\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s\Big)\Delta\tau\Big]\Delta\xi=\infty.\end{aligned} \tag{6}$$

Then either there exists a sufficiently large t_4 such that $x^{\Delta}(t) > 0$ on $[t_4, \infty)_{\mathbb{T}}$ or $\lim_{t\to\infty} x(t) = 0$.

Proof. By Lemma 5, we deduce that $x^{\Delta}(t)$ is eventually of one sign. So there exists a sufficiently large t_4 such that either $x^{\Delta}(t) > 0$ or $x^{\Delta}(t) < 0$ on $[t_4, \infty)_{\mathbb{T}}$. If $x^{\Delta}(t) < 0$, together with $x(t)$ is a positive solution of Eq. (1) on $[t_0, \infty)_{\mathbb{T}}$, we obtain $\lim_{t\to\infty} x(t) = \alpha \geq 0$ and $\lim_{t\to\infty} r(t)x^{\Delta}(t) = \beta \leq 0$. We claim $\alpha = 0$. Otherwise, assume $\alpha > 0$. Then there exists t_5 such that $x(t) \geq \alpha$ on $[t_5, \infty)_{\mathbb{T}}$, and an integration for (4) from t to ∞ yields

$$\begin{aligned}
&-\tfrac{a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)} \\
&= -\lim_{t\to\infty} \tfrac{a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)} + \int_t^{\infty} \tfrac{-q(s)x(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s \\
&\leq -\int_t^{\infty} \tfrac{q(s)x(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s \leq -\alpha \int_t^{\infty} \tfrac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s,
\end{aligned}$$

which is followed by

$$\begin{aligned}
&-[r(t)x^{\Delta}(t)]^{\Delta} \\
&\leq -\alpha[\tfrac{e_{-\frac{p}{a}}(t,t_0)}{a(t)} \int_t^{\infty} \tfrac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s].
\end{aligned} \tag{7}$$

Substituting t with τ in (7), an integration for (7) with respect to τ from t to ∞ yields

$$\begin{aligned}
&r(t)x^{\Delta}(t) \\
&= \lim_{t\to\infty} r(t)x^{\Delta}(t) \\
&\quad -\alpha \int_t^{\infty} (\tfrac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)} \int_{\tau}^{\infty} \tfrac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)\Delta\tau \\
&= \beta - \alpha \int_t^{\infty} (\tfrac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)} \int_{\tau}^{\infty} \tfrac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)\Delta\tau \\
&\leq -\alpha \int_t^{\infty} (\tfrac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)} \int_{\tau}^{\infty} \tfrac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)\Delta\tau,
\end{aligned}$$

which implies

$$\begin{aligned}
&x^{\Delta}(t) \\
&\leq -\alpha[\tfrac{1}{r(t)} \int_t^{\infty} (\tfrac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)} \int_{\tau}^{\infty} \tfrac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)\Delta\tau].
\end{aligned} \tag{8}$$

Substituting t with ξ in (8), an integration for (8) with respect to ξ from t_5 to t yields

$$\begin{aligned}
&x(t) - x(t_5) \\
&\leq -\alpha \int_{t_5}^{t} [\tfrac{1}{r(\xi)} \int_{\xi}^{\infty} (\tfrac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)} \\
&\int_{\tau}^{\infty} \tfrac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)\Delta\tau]\Delta\xi.
\end{aligned} \tag{9}$$

By (9) and (6) we have $\lim_{t\to\infty} x(t) = -\infty$, which leads to a contradiction. So we have $\alpha = 0$, and the proof is complete.

Lemma 7. Suppose $-\frac{p}{a} \in \mathfrak{R}_+$, and assume that x is a positive solution of Eq. (1) such that $[r(t)x^{\Delta}(t)]^{\Delta} > 0,\ x^{\Delta}(t) > 0$ on $[t_1, \infty)$, where $t_1 \geq t_0$ is sufficiently large. Then we have

$$x^{\Delta}(t) \geq \tfrac{\delta_1(t,t_1)}{r(t)}[\tfrac{a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)}].$$

Proof. By Lemma 5 we have $\frac{a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)}$ is strictly decreasing on $[t_1, \infty)$. So

$$\begin{aligned}
&r(t)x^{\Delta}(t) \geq r(t)x^{\Delta}(t) - r(t_1)x^{\Delta}(t_1) \\
&= \int_{t_1}^{t} \tfrac{e_{-\frac{p}{a}}(s,t_0)a(s)[r(s)x^{\Delta}(s)]^{\Delta}}{e_{-\frac{p}{a}}(s,t_0)a(s)}\Delta s \\
&\geq \tfrac{a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)} \int_{t_1}^{t} \tfrac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}\Delta s \\
&= \delta_1(t,t_1)\tfrac{a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)},
\end{aligned}$$

and then

$$x^{\Delta}(t) \geq \tfrac{\delta_1(t,t_1)}{r(t)}[\tfrac{a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)}],$$

which is the desired result.

Theorem 8. Suppose $-\frac{p}{a} \in \mathfrak{R}_+$, and assume that (2), (3), (6) hold, and for all sufficiently large t_1, there exists t_2 such that

$$\begin{aligned}
&\lim_{t\to\infty} \sup\{\int_{t_2}^{t} \{q(s)\tfrac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \phi(s)[a(s)\varphi(s)]^{\Delta} \\
&+\tfrac{\phi(s)\delta_1(s,t_1)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)} \\
&-\tfrac{[\phi^{\Delta}(s)r(s)+2\phi(s)\delta_1(s,t_1)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_1)}\}\Delta s\} = \infty,
\end{aligned} \tag{10}$$

where $\phi,\ \varphi$ are two given nonnegative functions on $\mathbb{T}$. Then every solution of Eq. (1) is oscillatory or tends to zero.

Proof. Assume (1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, where t_1 is sufficiently large. By Lemma 6 we have either $x^{\Delta}(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$ for some sufficiently large t_2 or $\lim_{t\to\infty} x(t) = 0$. Now we assume $x^{\Delta}(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Define the generalized Riccati function:

$$\omega(t) = \phi(t)a(t)[\tfrac{(r(t)x^{\Delta}(t))^{\Delta}}{x(t)e_{-\frac{p}{a}}(t,t_0)} + \varphi(t)].$$

Then we have

$$\omega^{\Delta}(t)=\frac{\phi(t)}{x(t)}[\frac{a(t)(r(t)x^{\Delta}(t))^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)}]^{\Delta}$$
$$+[\frac{\phi(t)}{x(t)}]^{\Delta}\frac{a(\sigma(t))(r(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}$$
$$+\phi(t)[a(t)\varphi(t)]^{\Delta}+\phi^{\Delta}(t)a(\sigma(t))\varphi(\sigma(t))$$
$$=\frac{\phi(t)}{x(t)}[\frac{(a(t)[r(t)x^{\Delta}(t)]^{\Delta})^{\Delta}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}-\frac{(e_{-\frac{p}{a}}(t,t_0))^{\Delta}a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)e_{-\frac{p}{a}}(\sigma(t),t_0)}]$$
$$+[\frac{x(t)\phi^{\Delta}(t)-x^{\Delta}(t)\phi(t)}{x(t)x(\sigma(t))}]\frac{a(\sigma(t))(r(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}$$
$$+\phi(t)[a(t)\varphi(t)]^{\Delta}+\phi^{\Delta}(t)a(\sigma(t))\varphi(\sigma(t))$$
$$=\frac{\phi(t)}{x(t)}[\frac{(a(t)[r(t)x^{\Delta}(t)]^{\Delta})^{\Delta}+p(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}]$$
$$+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}\omega(\sigma(t))-[\frac{\phi(t)x^{\Delta}(t)}{x(t)}]\frac{a(\sigma(t))(r(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))e_{-\frac{p}{a}}(\sigma(t),t_0)}$$
$$+\phi(t)[a(t)\varphi(t)]^{\Delta}$$
$$=-q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}\omega(\sigma(t))$$
$$+\phi(t)[a(t)\varphi(t)]^{\Delta}-[\frac{\phi(t)x^{\Delta}(t)}{x(t)}]\frac{a(\sigma(t))(r(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))e_{-\frac{p}{a}}(\sigma(t),t_0)}$$
$$\le -q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}\omega(\sigma(t))+\phi(t)[a(t)\varphi(t)]^{\Delta}$$
$$-(\frac{\phi(t)}{x(t)})\frac{\delta_1(t,t_1)}{r(t)}[\frac{a(t)[r(t)x^{\Delta}(t)]^{\Delta}}{e_{-\frac{p}{a}}(t,t_0)}]\frac{a(\sigma(t))(r(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))e_{-\frac{p}{a}}(\sigma(t),t_0)}$$
$$\le -q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}\omega(\sigma(t))+\phi(t)[a(t)\varphi(t)]^{\Delta}$$
$$-(\frac{\phi(t)}{x(\sigma(t))})\frac{\delta_1(t,t_1)}{r(t)}[\frac{a(\sigma(t))[r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}]\times$$
$$\frac{a(\sigma(t))(r(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))e_{-\frac{p}{a}}(\sigma(t),t_0)}$$
$$=-q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}\omega(\sigma(t))+\phi(t)[a(t)\varphi(t)]^{\Delta}$$
$$-[\frac{\phi(t)\delta_1(t,t_1)}{r(t)}][\frac{a(\sigma(t))(r(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))e_{-\frac{p}{a}}(\sigma(t),t_0)}]^2$$
$$=-q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}\omega(\sigma(t))+\phi(t)[a(t)\varphi(t)]^{\Delta}$$
$$-[\frac{\phi(t)\delta_1(t,t_1)}{r(t)}][\frac{\omega(\sigma(t))}{\phi(\sigma(t))}-a(\sigma(t))\varphi(\sigma(t))]^2$$
$$=-q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{\Delta}$$
$$-\frac{\phi(t)\delta_1(t,t_1)a^2(\sigma(t))\varphi^2(\sigma(t))}{r(t)}$$
$$+[\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}+2\frac{\phi(t)\delta_1(t,t_1)a(\sigma(t))\varphi(\sigma(t))}{r(t)\phi(\sigma(t))}]\omega(\sigma(t))$$
$$-\frac{\phi(t)\delta_1(t,t_1)}{r(t)\phi^2(\sigma(t))}\omega^2(\sigma(t))$$
$$\le -q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{\Delta}$$
$$-\frac{\phi(t)\delta_1(t,t_1)a^2(\sigma(t))\varphi^2(\sigma(t))}{r(t)}$$
$$+\frac{[\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}+2\frac{\phi(t)\delta_1(t,t_1)a(\sigma(t))\varphi(\sigma(t))}{r(t)\phi(\sigma(t))}]^2}{4\frac{\phi(t)\delta_1(t,t_1)}{r(t)\phi^2(\sigma(t))}}$$
$$=-q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{\Delta}$$
$$-\frac{\phi(t)\delta_1(t,t_1)a^2(\sigma(t))\varphi^2(\sigma(t))}{r(t)}$$
$$+\frac{[\phi^{\Delta}(t)r(t)+2\phi(t)\delta_1(t,t_1)a(\sigma(t))\varphi(\sigma(t))]^2}{4r(t)\phi(t)\delta_1(t,t_1)}. \quad (11)$$

Substituting t with s in (11), an integration for (11) with respect to s from t_2 to t yields

$$\int_{t_2}^{t}\{q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\phi(s)[a(s)\varphi(s)]^{\Delta}$$
$$+\frac{\phi(s)\delta_1(s,t_1)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{\Delta}(s)r(s)+2\phi(s)\delta_1(s,t_1)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_1)}\}\Delta s$$
$$\le \omega(t_2)-\omega(t)\le\omega(t_2),$$

which contradicts (10). So the proof is complete.

In Theorem 8, if we take $\mathbb{T}$ for some special cases, then we can obtain the following corollaries:

Corollary 9. Let $\mathbb{T}=\mathbb{R}$. Assume that

$$\int_{t_0}^{\infty}\frac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}ds=\infty, \quad (12)$$

$$\int_{t_0}^{\infty}\frac{1}{r(s)}ds=\infty, \quad (13)$$

$$\int_{t_0}^{\infty}[\frac{1}{r(\xi)}\int_{\xi}^{\infty}(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(s,t_0)}ds)d\tau]d\xi=\infty, \quad (14)$$

and for all sufficiently large t_1, there exists t_2 such that

$$\lim_{t\to\infty}\sup\{\int_{t_2}^{t}\{q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(s,t_0)}-\phi(s)[a(s)\varphi(s)]'$$
$$+\frac{\phi(s)\delta_1(s,t_1)a^2(s)\varphi^2(s)}{r(s)}$$
$$-\frac{[\phi'(s)r(s)+2\phi(s)\delta_1(s,t_1)a(s)\varphi(s)]^2}{4r(s)\phi(s)\delta_1(s,t_1)}\}ds\}=\infty, \quad (15)$$

where $\phi,\ \varphi$ are two given nonnegative functions on $\mathbb{R}$, and $\delta_1(s,t_1)=\int_{t_1}^{t}\frac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}ds$. Then every solution of Eq. (1) is oscillatory or tends to zero.

Corollary 10. Let $\mathbb{T}=\mathbb{Z}$ and $-\frac{p}{a}\in\Re_+$. Assume that

$$\sum_{s=t_0}^{\infty}\frac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}=\infty, \quad (16)$$

$$\sum_{s=t_0}^{\infty}\frac{1}{r(s)}=\infty, \quad (17)$$

$$\sum_{\xi=t_0}^{\infty}[\frac{1}{r(\xi)}\sum_{\tau=\xi}^{\infty}(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(s+1,t_0)})]=\infty, \quad (18)$$

and for all sufficiently large t_1, there exists t_2 such that

$$\lim_{t\to\infty}\sup\{\sum_{s=t_2}^{t-1}\{q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(s+1,t_0)} -\phi(s)[a(s+1)\varphi(s+1)-a(s)\varphi(s)] +\frac{\phi(s)\delta_1(s,t_1)a^2(s+1)\varphi^2(s+1)}{r(s)} -\frac{[\phi^{\Delta}(s)r(s)+2\phi(s)\delta_1(s,t_1)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_1)}\}\}=\infty, \tag{19}$$

where $\phi,\ \varphi$ are two given nonnegative functions on $\mathbb{Z}$, and $\delta_1(s,t_1)=\sum_{s=t_1}^{t-1}\frac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}$. Then every solution of Eq. (1) is oscillatory or tends to zero.

Theorem 11. Suppose $-\frac{p}{a}\in\mathfrak{R}_+$, and assume that (2), (3), (6) hold, and define $\mathbb{D}=\{(t,s)|t\ge s\ge t_0\}$. If there exists a function $H\in C_{rd}(\mathbb{D},\mathbb{R})$ such that

$$\begin{aligned}&H(t,t)=0,\ for\ t\ge t_0,\\ &H(t,s)>0,\ for\ t>s\ge t_0,\end{aligned} \tag{20}$$

and H has a nonpositive continuous $\Delta-$ partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable, and

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\{\int_{t_0}^{t}H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} -\phi(s)(a(s)\varphi(s))^{\Delta}+\frac{\phi(s)\delta_1(s,t_1)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)} -\frac{[\phi^{\Delta}(s)r(s)+2\phi(s)\delta_1(s,t_1)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_1)}]\Delta s\}=\infty. \tag{21}$$

Then every solution of Eq. (1) is oscillatory or tends to zero.

Proof. Assume (1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we may assume $x(t)>0$ on $[t_1,\infty)_{\mathbb{T}}$, where t_1 is sufficiently large. By Lemma 6 we have either $x^{\Delta}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$ for some sufficiently large t_2 or $\lim_{t\to\infty}x(t)=0$. Now we assume $x^{\Delta}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$. Let $\omega(t)$ be defined as in Theorem 8. By (11) we have

$$\begin{aligned}&q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}-\phi(t)(a(t)\varphi(t))^{\Delta}\\ &+\frac{\phi(t)\delta_1(t,t_1)a^2(\sigma(t))\varphi^2(\sigma(t))}{r(t)}\\ &-\frac{[\phi^{\Delta}(t)r(t)+2\phi(t)\delta_1(t,t_1)a(\sigma(t))\varphi(\sigma(t))]^2}{4r(t)\phi(t)\delta_1(t,t_1)}\\ &\le-\omega^{\Delta}(t).\end{aligned} \tag{22}$$

Substituting t with s in (22), multiplying both sides by $H(t,s)$ and then integrating with respect to s from t_2 to t yields

$$\begin{aligned}&\int_{t_2}^{t}H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{\Delta}\\ &+\frac{\phi(s)\delta_1(s,t_1)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}\\ &-\frac{[\phi^{\Delta}(s)r(s)+2\phi(s)\delta_1(s,t_1)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_1)}]\Delta s\\ &\le-\int_{t_2}^{t}H(t,s)\omega^{\Delta}(s)\Delta s\\ &=H(t,t_2)\omega(t_2)+\int_{t_2}^{t}H^{\Delta_s}(t,s)\omega(\sigma(s))\Delta s\\ &\le H(t,t_2)\omega(t_2)\le H(t,t_0)\omega(t_2).\end{aligned}$$

Then

$$\begin{aligned}&\int_{t_0}^{t}H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{\Delta}\\ &+\frac{\phi(s)\delta_1(s,t_1)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}\\ &-\frac{[\phi^{\Delta}(s)r(s)+2\phi(s)\delta_1(s,t_1)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_1)}]\Delta s\\ &=\int_{t_0}^{t_2}H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{\Delta}\\ &+\frac{\phi(s)\delta_1(s,t_1)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}\\ &-\frac{[\phi^{\Delta}(s)r(s)+2\phi(s)\delta_1(s,t_1)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_1)}]\Delta s\\ &+\int_{t_2}^{t}H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{\Delta}\\ &+\frac{\phi(s)\delta_1(s,t_1)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}\\ &-\frac{[\phi^{\Delta}(s)r(s)+2\phi(s)\delta_1(s,t_1)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_1)}]\Delta s\\ &\le H(t,t_0)\omega(t_2)\\ &+H(t,t_0)\int_{t_0}^{t_2}|q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{\Delta}\\ &+\frac{\phi(s)\delta_1(s,t_1)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}\\ &-\frac{[\phi^{\Delta}(s)r(s)+2\phi(s)\delta_1(s,t_1)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_1)}|\Delta s.\end{aligned}$$

So

$$\begin{aligned}&\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\{\int_{t_0}^{t}H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\\ &-\phi(s)(a(s)\varphi(s))^{\Delta}+\frac{\phi(s)\delta_1(s,t_1)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}\\ &-\frac{[\phi^{\Delta}(s)r(s)+2\phi(s)\delta_1(s,t_1)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_1)}]\Delta s\\ &\le\omega(t_2)+\int_{t_0}^{t_2}|q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{\Delta}\\ &+\frac{\phi(s)\delta_1(s,t_1)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}\\ &-\frac{\phi^{\Delta}(s)r(s)+2\phi(s)\delta_1(s,t_1)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_1)}|\Delta s\\ &<\infty,\end{aligned}$$

which contradicts (21). So the proof is complete.

In Theorem 11, if we take $H(t,s)$ for some special functions such as $(t-s)^m$ or $\ln\frac{t}{s}$, then we can obtain some corollaries. For example, if we take $H(t,s)=(t-s)^m,\ m\geq 1$, then we have the following corollary:

Corollary 12. Suppose $-\frac{p}{a}\in\Re_+$, and assume that (2), (3), (6) hold, and

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)^m}\{\int_{t_0}^t(t-s)^m[q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} -\phi(s)(a(s)\varphi(s))^\Delta+\frac{\phi(s)\delta_1(s,t_1)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)} -\frac{[\phi^\Delta(s)r(s)+2\phi(s)\delta_1(s,t_1)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_1)}]\Delta s\} =\infty. \tag{23}$$

Then every solution of Eq. (1) is oscillatory or tends to zero.

3 Applications

In this section, we will present some applications for the established results above. First we consider the following third order half linear differential equation with damping term:

Example 1.

$$[t(x''(t))]'+\frac{1}{t^2}x''(t)+\frac{1}{t^2}x(t)=0,\ t\in[2,\infty). \tag{24}$$

We have in (1) $\mathbb{T}=\mathbb{R},\ a(t)=t,\ p(t)=q(t)=\frac{1}{t^2},\ r(t)=1,\ t_0=2$. Then $\mu(t)=\sigma(t)-t=0$, and $-\frac{p}{a}\in\Re_+$. So $e_{-\frac{p}{a}}(t,t_0)=e_{-\frac{p}{a}}(t,2)=exp(-\int_2^t\frac{p(s)}{a(s)}ds)$. Moreover, we have

$$1>\exp(-\int_2^t\frac{p(s)}{a(s)}ds)\geq 1-\int_2^t\frac{p(s)}{a(s)}ds =1-\int_2^t\frac{1}{s^3}ds=1+\frac{1}{2}[t^{-2}-2^{-2}]>\frac{7}{8}.$$

Then we have

$$\int_{t_0}^\infty\frac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}ds>\frac{7}{8}\int_2^\infty\frac{1}{s}ds=\infty,$$

and

$$\int_{t_0}^\infty\frac{1}{r(s)}ds=\infty.$$

Furthermore,

$$\int_{t_0}^\infty[\frac{1}{r(\xi)}\int_\xi^\infty(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\int_\tau^\infty\frac{q(s)}{e_{-\frac{p}{a}}(s,t_0)}ds)d\tau]d\xi =\int_2^\infty[\int_\xi^\infty(\frac{e_{-\frac{p}{a}}(\tau,2)}{\tau}\int_\tau^\infty\frac{1}{s^2e_{-\frac{p}{a}}(s,2)}ds)d\tau]d\xi >\frac{7}{8}\int_2^\infty[\int_\xi^\infty(\frac{1}{\tau}\int_\tau^\infty\frac{1}{s^2}ds)d\tau]d\xi =\frac{7}{8}\int_2^\infty[\int_\xi^\infty\frac{1}{\tau^2}d\tau]d\xi=\frac{7}{8}\int_2^\infty\frac{1}{\xi}d\xi=\infty.$$

On the other hand, for a sufficiently large t_1, we have

$$\delta_1(t,t_1)=\int_{t_1}^t\frac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}ds>\frac{7}{8}\int_{t_1}^t\frac{1}{s}ds\to\infty.$$

So there exists a sufficiently large $t_2>t_1$ such that $\delta_1(t,t_1)>1$ for $t\in[t_2,\infty)$. Taking $\phi(t)=t,\ \varphi(t)=0$ in (15), we get that

$$\int_{t_2}^t[q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(s,t_0)}-\frac{(\phi'(s))^2r(s)}{4\phi(s)\delta_1(s,t_1)}]ds >\int_{t_2}^t(\frac{1}{s}-\frac{1}{4s})ds=\int_{t_2}^t\frac{3}{4s}ds\to\infty.$$

So (12)-(15) all hold, and by Corollary 9 we deduce that every solution of Eq. (24) is oscillatory or tends to zero.

Next we consider the following third order half linear difference equation:

Example 2.

$$\Delta[t\Delta^2x(t)]+\frac{1}{t^2}\Delta^2x(t)+\frac{1}{t^2}x(t)=0,\ t\in[2,\infty)_{\mathbb{Z}}, \tag{25}$$

where Δ denotes the difference operator.

We have in (1) $\mathbb{T}=\mathbb{Z},\ a(t)=t,\ p(t)=q(t)=\frac{1}{t^2},\ r(t)=1,\ t_0=2$. Then $\mu(t)=\sigma(t)-t=1$, and

$$1-\mu(t)\frac{p(t)}{a(t)}=1-\frac{1}{t^3}\geq 1-\frac{1}{2^3}>0,$$

which implies $-\frac{p}{a}\in\Re_+$. So by [2, Lemma 2] we obtain

$$e_{-\frac{p}{a}}(t,t_0)=e_{-\frac{p}{a}}(t,2)\geq 1-\int_2^t\frac{p(s)}{a(s)}\Delta s =1-\int_2^t\frac{1}{s^3}\Delta s=1-\sum_{s=2}^{t-1}\frac{1}{s^3} \geq 1-\int_1^{t-1}\frac{1}{s^3}ds=1+\frac{1}{2}[(t-1)^{-2}-1]>\frac{1}{2},$$

and

$$e_{-\frac{p}{a}}(t,t_0)\leq\exp(-\int_2^t\frac{p(s)}{a(s)}\Delta s)<1.$$

Then we have

$$\sum_{s=t_0}^\infty\frac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}=\sum_{s=2}^\infty\frac{e_{-\frac{p}{a}}(s,2)}{a(s)} =\sum_{s=2}^\infty\frac{e_{-\frac{p}{a}}(s,2)}{s}>\frac{1}{2}\sum_{s=2}^\infty\frac{1}{s}=\infty,$$

and

$$\sum_{s=t_0}^\infty\frac{1}{r(s)}=\infty.$$

Furthermore,

$$\begin{aligned}&\sum_{\xi=t_0}^{\infty}[\tfrac{1}{r(\xi)}\sum_{\tau=\xi}^{\infty}(\tfrac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\sum_{s=\tau}^{\infty}\tfrac{q(s)}{e_{-\frac{p}{a}}(s+1,t_0)})]\\&=\sum_{\xi=2}^{\infty}[\tfrac{1}{r(\xi)}\sum_{\tau=\xi}^{\infty}(\tfrac{e_{-\frac{p}{a}}(\tau,2)}{a(\tau)}\sum_{s=\tau}^{\infty}\tfrac{q(s)}{e_{-\frac{p}{a}}(s+1,2)})]\\&>\tfrac12\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}(\tfrac{1}{\tau}\sum_{s=\tau}^{\infty}\tfrac{1}{s^2})]\\&=\tfrac12\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}(\tfrac{1}{\tau}\sum_{s=\tau}^{\infty}\tfrac{1}{s^2})]>\tfrac12\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}(\tfrac{1}{\tau}\sum_{s=\tau}^{\infty}\tfrac{1}{s(s+1)})]\\&=\tfrac12\sum_{\xi=2}^{\infty}\sum_{\tau=\xi}^{\infty}\tfrac{1}{\tau^2}>\tfrac12\sum_{\xi=2}^{\infty}\sum_{\tau=\xi}^{\infty}\tfrac{1}{\tau(\tau+1)}=\tfrac12\sum_{\xi=2}^{\infty}\tfrac{1}{\xi}=\infty.\end{aligned}$$

On the other hand, for a sufficiently large t_1, we have

$$\delta_1(t,t_1)=\sum_{s=t_1}^{t-1}\frac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}>\frac12\sum_{s=t_1}^{t-1}\frac1s\to\infty.$$

So there exists $t_2>t_1$ such that $\delta_1(t,t_1)>1$ for $t\in[t_2,\infty)_{\mathbb{Z}}$. Let $\phi(t)=t,\ \varphi(t)=0$ in (19). Then we have

$$\begin{aligned}&\sum_{s=t_2}^{t-1}[q(s)\tfrac{\phi(s)}{e_{-\frac{p}{a}}(s+1,t_0)}-\tfrac{(\phi(s+1)-\phi(s))^2r(s)}{4\phi(s)\delta_1(s,t_1)}]\\&>\sum_{s=t_2}^{t-1}(\tfrac1s-\tfrac{1}{4s})=\sum_{s=t_2}^{t-1}\tfrac{3}{4s}\to\infty.\end{aligned}$$

So (16)-(19) all hold, and by Corollary 10 we obtain that every solution of Eq. (25) is oscillatory or tends to zero.

Finally we consider the following third order $q-$ difference equation:

Example 3.

$$[t(x^{\Delta\Delta}(t))]^{\Delta}+\frac{1}{t^2}x^{\Delta\Delta}(t)+\frac{1}{t^2}x(t)=0,t\in[q,\infty)_{q^{\mathbb{Z}}},\qquad(26)$$

where $\mathbb{T}=q^{\mathbb{Z}},\ q\ge 2$.

We have in (1) $a(t)=t,\ p(t)=q(t)=\frac{1}{t^2}$, $r(t)=1,\ t_0=q$. Then $\mu(t)=\sigma(t)-t=t(q-1)$, and

$$\begin{aligned}&1-\mu(t)\tfrac{p(t)}{a(t)}=1-t(q-1)\tfrac{1}{t^3}=1-(q-1)\tfrac{1}{t^2}\\&\ge 1-(q-1)\tfrac{1}{q^2}=\tfrac{q^2-q+1}{q^2}>0,\end{aligned}$$

which implies $-\frac{p}{a}\in\Re_+$. So by [2, Lemma 2] we obtain

$$\begin{aligned}&e_{-\frac{p}{a}}(t,t_0)=e_{-\frac{p}{a}}(t,q)\ge 1-\int_q^t\tfrac{p(s)}{a(s)}\Delta s\\&=1-\int_q^t\tfrac{1}{s^3}\Delta s=1-\tfrac{t^{-2}-q^{-2}}{q^{-2}-1}=\tfrac{1+t^{-2}-2q^{-2}}{1-q^{-2}}\\&>\tfrac{1-2q^{-2}}{1-q^{-2}}\ge\tfrac{1}{2-2q^{-2}}=\tfrac{q^2}{2(q^2-1)},\end{aligned}$$

and

$$e_{-\frac{p}{a}}(t,t_0)\le\exp(-\int_q^t\frac{p(s)}{a(s)}\Delta s)<1.$$

Then we have

$$\begin{aligned}&\int_{t_0}^{\infty}\tfrac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}\Delta s=\int_q^{\infty}\tfrac{e_{-\frac{p}{a}}(s,q)}{a(s)}\Delta s\\&=\int_q^{\infty}\tfrac{e_{-\frac{p}{a}}(s,q)}{s}\Delta s>\tfrac{q^2}{2(q^2-1)}\int_q^{\infty}\tfrac1s\Delta s=\infty,\end{aligned}$$

and

$$\int_{t_0}^{\infty}\frac{1}{r(s)}\Delta s=\infty.$$

Furthermore,

$$\begin{aligned}&\int_{t_0}^{\infty}[\tfrac{1}{r(\xi)}\int_{\xi}^{\infty}(\tfrac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\tfrac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)\Delta\tau]\Delta\xi\\&=\int_q^{\infty}[\int_{\xi}^{\infty}(\tfrac{e_{-\frac{p}{a}}(\tau,q)}{\tau}\int_{\tau}^{\infty}\tfrac{1}{s^2e_{-\frac{p}{a}}(\sigma(s),q)}\Delta s)\Delta\tau]\Delta\xi\\&>\tfrac{q^2}{2(q^2-1)}\int_q^{\infty}[\int_{\xi}^{\infty}(\tfrac1\tau\int_{\tau}^{\infty}\tfrac{1}{s^2}\Delta s)\Delta\tau]\Delta\xi\\&>\tfrac{q^2}{2(q^2-1)}\int_q^{\infty}[\int_{\xi}^{\infty}(\tfrac1\tau\int_{\tau}^{\infty}\tfrac{1}{s\sigma(s)}\Delta s)\Delta\tau]\Delta\xi\\&=\tfrac{q^2}{2(q^2-1)}\int_q^{\infty}[\int_{\xi}^{\infty}(\tfrac1\tau[-\tfrac1s]_{\tau}^{\infty})\Delta\tau]\Delta\xi\\&=\tfrac{q^2}{2(q^2-1)}\int_q^{\infty}[\int_{\xi}^{\infty}\tfrac{1}{\tau^2}\Delta\tau]\Delta\xi\\&>\tfrac{q^2}{2(q^2-1)}\int_q^{\infty}[\int_{\xi}^{\infty}\tfrac{1}{\tau\sigma(\tau)}\Delta\tau]\Delta\xi\\&=\tfrac{q^2}{2(q^2-1)}\int_q^{\infty}\tfrac1\xi\Delta\xi=\infty.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\delta_1(t,t_1)=\int_{t_1}^t\tfrac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}\Delta s\\&>\tfrac{q^2}{2(q^2-1)}\int_{t_1}^t\tfrac1s\Delta s\to\infty.\end{aligned}$$

So there exists $t_2>t_1$ such that $\delta_1(t,t_1)>1$ for $t\in[t_2,\infty)_{q^{\mathbb{Z}}}$. Let $m=1,\ \phi(t)=t,\ \varphi(t)=0$ in (23). Then we have

$$\begin{aligned}&\limsup_{t\to\infty}\tfrac{1}{(t-t_0)}\{\int_{t_2}^t(t-s)[q(s)\tfrac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\\&-\tfrac{(\phi^{\Delta}(s))^2r(s)}{4\phi(s)\delta_1(s,t_1)}]\Delta s\}\\&>\limsup_{t\to\infty}\tfrac{1}{(t-t_0)}\int_{t_2}^t(t-s)\tfrac{3}{4s}\Delta s\\&=\limsup_{t\to\infty}[\tfrac{t}{(t-q)}\int_{t_2}^t\tfrac{3}{4s}\Delta s-\tfrac{3(t-t_2)}{4(t-q)}]=\infty.\end{aligned}$$

So (2), (3), (6) and (23) all hold, and by Corollary 12 we obtain that every solution of Eq. (26) is oscillatory or tends to zero.

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