Hyper Domination in Bipartite Semigraphs

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Abstract: Let S be a bipartite semigraph with $|N_{Xa}(y)| \geq 1$ for every $y \in Y$. A vertex $x \in X$ hyper dominates $y \in Y$ if $y \in N_a(x)$ or $y \in N_a(N_{Ya}(x))$. A subset $D \subseteq X$ is a hyper dominating set of S if every $y \in Y$ is hyper dominated by a vertex of D. A subset $D \subseteq X$ is called a minimal hyper dominating set of S if no proper subset of D is a hyper dominating set of S. The minimum cardinality of a minimal hyper dominating set of S is called hyper domination number of S and is denoted by $\gamma_{ha}(S)$. The concept of hyper independence and hyper irredundant is introduced. Inequalities involving dominating parameters and irredundant parameters are proved.

Key–Words: Semigraphs, Bipartite semigraphs, hyper dominating set, hyper independent set, hyper irredundant set.

1 Introduction

Semigraphs introduced by E.Sampathkumar is an interesting type of generalization of the concept of graph. The semigraphs are closer to graphs in some sense than hyper graphs [1]. Jeyabharathi et al. [4] used the concept of semigraphs for DNA splicing. Das [2] used semigraphs for studying a topological invariant of a compact orientable 2-manifold surface and about the generator of a face of relevant surfaces.

Road networks can be modeled using semigraphs. Traffic routing and density of traffic in junction may be studied through domination in semigraphs. Adjacency domination in semigraphs was defined by Kamath and Bhat [6]. Strong and weak domination was defined by S.S.Kamath[5] and Saroja R.Hebber. Gomathi[3] defined the $e-$domination, $ev-$domination and $(m, e)-$strong domination in semigraphs. The concept of domination in bipartite semigraphs was defined by Venkatakrishnan and Swaminathan [11]. In this sequel, we define the hyper domination in bipartite semigraphs.

2 Preliminaries

We give the definitions as in [7].

Definition 1 *A semigraph S is a pair (V, X) where V is a nonempty set whose elements are called vertices of S, and X is a set of ordered $n-$tuples, called edges of S, of distinct vertices, for various $n \geq 2$, satisfying the following conditions:*

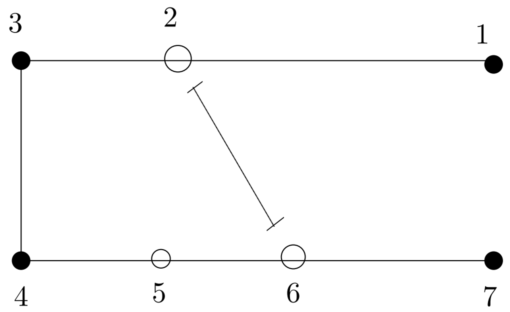
SG1: Any two edges have at most one vertex in common.
SG2: Two edges $E_1 = (u_1, u_2, \cdots, u_m)$ and $E_2 = (v_1, v_2, \cdots, v_n)$ are considered to be equal if and only if
1. $m = n$ and
2. Either $u_i = v_i$ for $1 \leq i \leq n$ or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$. Thus the edge $(u_1, u_2, \cdots, u_m)$ is the same as $(u_m, u_{m-1}, \cdots, u_1)$. The vertices u_1 and u_m are said to be the end vertices of the edge E_1 while $u_2, u_3, \cdots, u_{m-1}$ are said to be the middle vertices of E_1.

From the above definition, it may be noted that the vertices in a semigraph are divided into four types namely end vertices, middle vertices, middle-end vertices and isolated vertices.

A semigraph S may be drawn as a set of points representing the vertices. An edge $E = (v_{i1}, v_{i2}, \cdots, v_{ir})$ is represented by a Jordan curve joining the points corresponding to the vertices $(v_{i1}, v_{i2}, \cdots, v_{ir})$ in the same order as they appear in E. The end points of the curve (i.e the end vertices of E) are denoted by thick dots. The points lying in between the end points (i.e middle vertices of E) are denoted by small circles. If an end vertex v of an edge E is a middle vertex of some edge E^1, a small tangent is drawn to the circle (representing v on E^1) at the end of E.

Example 1: Let $S = (V, X)$ be a semigraph where $V = \{1, 2, 3, 4, 5, 6, 7\}$ and $X = \{(1, 2, 3), (2, 6), (3, 4), (4, 5, 6, 7)\}$. In S, $1, 3, 4, 7$

are end vertices, 5 is middle vertex and $2, 6$ are middle-end vertices.

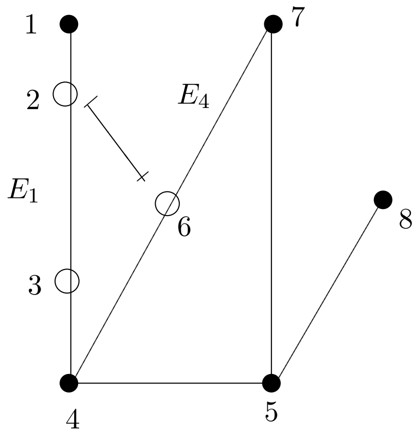


Definition 2 *A sub edge of an edge* $E = (v_{i1}, v_{i2}, \cdots, v_{in})$ *is a* $k-$*tuple* $E^1 = (v_{ij1}, v_{ij2}, \cdots, v_{ijk})$ *where*

$$1 \leq j1 \leq j2 \leq j3 \leq \cdots \leq jk \leq n.$$

We say that E^1 *is the sub edge induced by the set of vertices* $E^1 = (v_{ij1}, v_{ij2}, \cdots, v_{ijk})$.

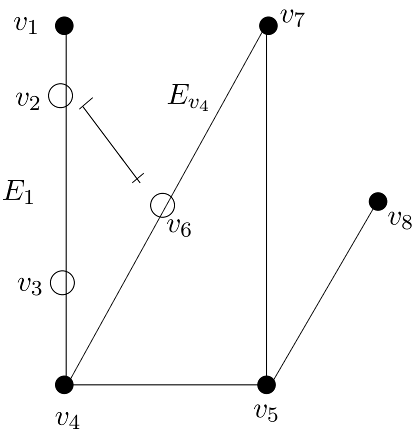
Example 2: Consider the semigraph given below. The edges $E_1 = (1, 2, 3, 4)$ and $E_4 = (4, 6, 7)$.



The set of vertices induced by $(4, 3, 1)$ and $(4, 7)$ are sub edges of the edges E_1 and E_4 respectively.

Definition 3 *An* $fs-$*edge in a semigraph* S *is an edge in* S *or a subedge of an edge in* S. *A* $v_0 - v_n$ *walk in a semigraph* S *is a sequence of vertices* $v_0, v_1, v_2, \cdots, v_n$ *such that* (v_i, v_{i+1}) *for* $i = 0$ *to* $n - 1$ *is an* $fs-$*edge of cardinality two. It is closed if* $v_0 = v_n$ *and open otherwise.The vertices* v_0, v_n *and the vertices* $v_1, v_2, v_3, \cdots, v_{n-1}$ *are end vertices and internal vertices respectively. A* $v_0 - v_n$ *walk is a trail if any two* $fs-$*edges in it are distinct. Vertices may be repeated in a trail. A* $v_0 - v_n$ *path is a trail in which all the vertices are distinct. A cycle is a closed path.*

Example 3: Consider the semigraph given below:



Then
$W_1 : v_1v_3v_4v_7v_5v_4v_6v_2v_3v_4$ is a walk.
$T_1 : v_1v_3v_4v_7v_5v_4v_6v_2v_3$ is a trail.
$P_1 : v_1v_3v_4v_7v_5$ is a path.
$C_1 : v_2v_4v_6v_2$ is a cycle.

2.1 Adjacency of two vertices in a semigraph

There are different types of adjacency of two vertices in a semigraph.

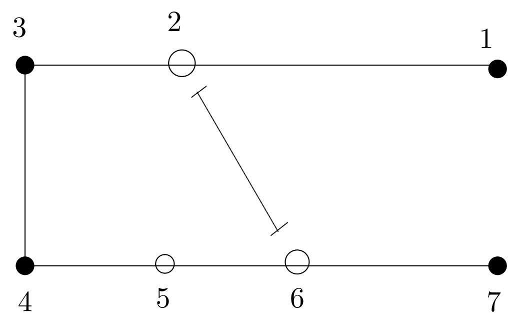
1. Two vertices u and v in a semigraph are said to be adjacent if they belong to the same edge. Let $N_a(u)$ denote the set of all vertices adjacent to u.

2. Two vertices u and v are said to be consecutively adjacent if in addition they are consecutive in order as well.

3. Two vertices u and v are said to be $e-$adjacent if they are the end vertices of edge in the semigraph.

4. Two vertices u and v are said to be $1e-$adjacent if both the vertices u and v belong to the same edge and at least one of them is an end vertex of that edge.

Example 4: In the semigraph S given below the vertices 4 and 6 are adjacent. Vertices 5 and 6 are consecutively adjacent. Vertices 4 and 7 are $e-$adjacent. Vertices 5 and 7 are $1e-$adjacent.



2.2 Graphs associated with a given semigraph

Let S be a given semigraph. The graphs given below are associated with the semigraph S, each having the same vertex set as S:

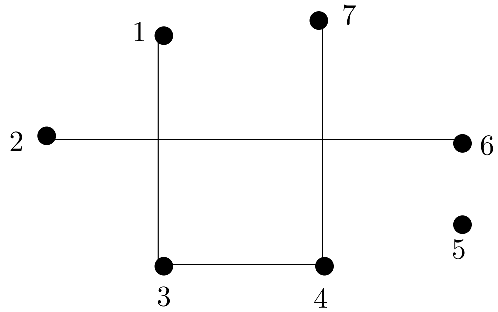
(a) **End vertex graph** S_e**:** Two vertices in S_e are adjacent if they are the end vertices of an edge in S.

(b) **Adjacency graph** S_a**:** Two vertices in S_a are adjacent if they are adjacent in S.

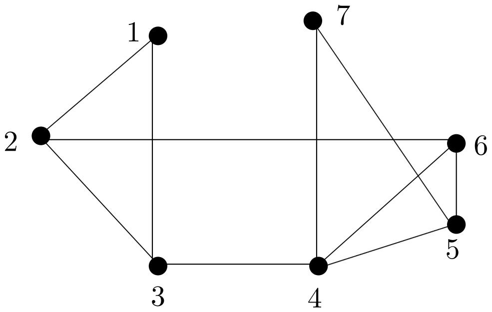
(c) **Consecutive adjacency graph** S_{ca}**:** Two vertices in S_{ca} are adjacent if they are the consecutively adjacent in S.

(d) **One end vertex graph** S_{1e}**:** Two vertices in S_{1e} are adjacent if one of them is an end vertex in S of an edge containing the two vertices.

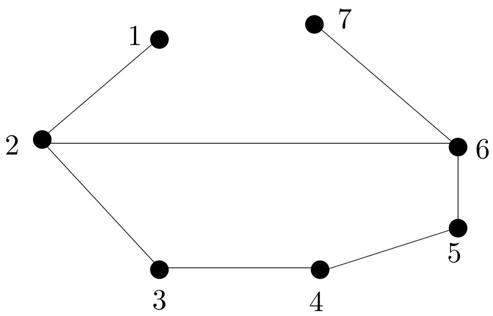
Example 5: The various graphs S_e, S_a, S_{ca} and S_{1e} associated with the semigraph S given in the above example is given below:



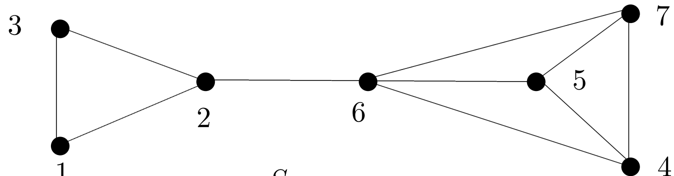
S_e



S_a



S_{ca}



S_{1e}

2.3 Bipartite graph in Semigraphs

There are four types of bipartite semigraphs, namely bipartite semigraph, $e-$bipartite semigraph, strongly bipartite semigraph and edge bipartite semigraph. We consider only bipartite semigraphs.

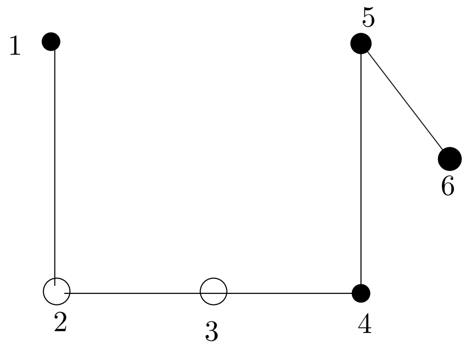
Definition 4 *A set D of vertices in a semigraph is independent if no edge is a subset of D. The semigraph S is* **Bipartite semigraph** *if its vertex set V can be partitioned into sets $\{X, Y\}$ such that X and Y are independent sets.*

Definition 5 *A set D of vertices in a semigraph is $e-$independent if no two end vertices of an edge belong to D. The semigraph S is $e-$***bipartite** *if its vertex set V can be partitioned into sets $\{X, Y\}$ such that both X and Y are $e-$independent.*

Definition 6 *A set D of vertices in a semigraph is strongly independent if no two adjacent vertices belong to D. The semigraph S is* **strongly bipartite** *if V can be partitioned into sets $\{X, Y\}$ such that X and Y are strongly independent.*

Definition 7 *The semigraph S is* **edge bipartite** *if S has no odd cycles.*

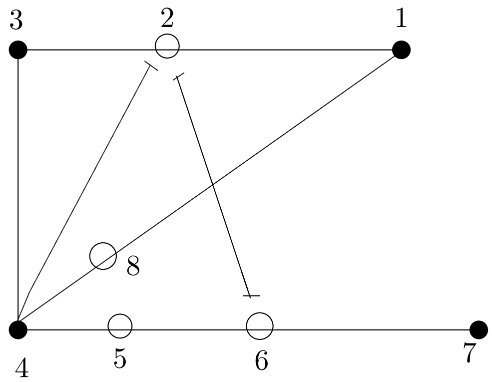
Example 6: Consider the semigraph



Consider the partition of the vertex set as follows: $X = \{1, 4, 6\}$ and $Y = \{2, 3, 5\}$. No edge of the above semigraph is a subset of X and Y. Therefore, the above semigraph is a bipartite semigraph. The above semigraph is not $e-$bipartite, since the end vertices 1 and 4 lie in X.

Consider the semigraph given below.

S :



The vertex set is partitioned into two subsets $X = \{1, 4, 6, 7, 8\}$ and $Y = \{2, 3, 5\}$. No two end vertices of an edge lie in one of the subsets X or Y. Hence, the given graph is $e-$biparatite.

Remark 8 *From the above, it is clear that the only semigraphs which are strongly bipartite are bipartite graphs. Also every $e-$bipartite semigraph is bipartite semigraph but not conversely. The characterization of $e-$bipartite semigraph is given in the following proposition.*

Proposition 9 *A semigraph S is $e-$bipartite if and only if, its $e-$graph S_e is bipartite.*

3 Dominating sets in bipartite semigraphs

Two vertices $x \in X$ and $y \in Y$ are $Ya-$adjacent if x and y belongs to the same edge of the semigraph S. Two vertices u and v in X are $Xa-$adjacent if u and v belongs to the same edge of the semigraph S or an edge E_1 containing u and an edge E_2 containing v are adjacent. Let x belong to X. The set $N_{Ya}(x)$ is the set of vertices $Xa-$adjacent to x in X. We define $\Delta_Y(G) = max\{|N_{Ya}(u)| : u \in X\}$ and the maximum degree of a vertex is denoted by Δ. Let y belong to Y. The set $N_{Xa}(y)$ is the set of vertices in X adjacent to y in Y.

Definition 10 *A subset D of X is called a $Ya-$ dominating set if every vertex $y \in Y$ is $Ya-$adjacent to a vertex of D. The minimum cardinality of a $Ya-$dominating set is called the $Ya-$domination number of S and is denoted by $\gamma_{Ya}(S)$.*

Definition 11 *A subset D of X is called a $Xa-$dominating set if every vertex u in $X - D$ is $Xa-$adjacent to a vertex of D.*

A subset D of X is called a minimal $Xa-$dominating set if no proper subset of D is a $Xa-$dominating set. The minimum cardinality of a minimal $Xa-$dominating set is called the $Xa-$domination number of S and is denoted by $\gamma_{Xa}(S)$. The maximum cardinality of a minimum $Xa-$dominating set is called the upper $Xa-$domination number of a semigraph S and is denoted by $\Gamma_{Xa}(S)$.

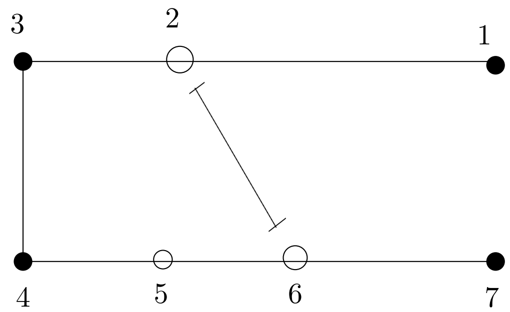
Remark 12 *By a $\gamma_{Xa}-$set, we mean a minimum $Xa-$dominating set of the semigraph S and by $\Gamma_{Xa}-$set, we mean the maximum cardinality of a minimal $Xa-$dominating set.*

The minimal $Xa-$dominating sets was characterized as follows in [11].

Theorem 13 *A $Xa-$dominating set D of a bipartite semigraph is minimal if and only if for every u in D one of the following conditions hold:*
(i) u is a $Ya-$isolate of D.
(ii) There exists a vertex v in $X - D$ such that $N_{Ya}(v) \cup D = \{u\}$.

Proposition 14 *In a semigraph G with no isolates, every $Ya-$dominating set is a $Xa-$dominating set.*

Remark 15 *Converse of the above need not be true. consider the bipartite semigraph.*

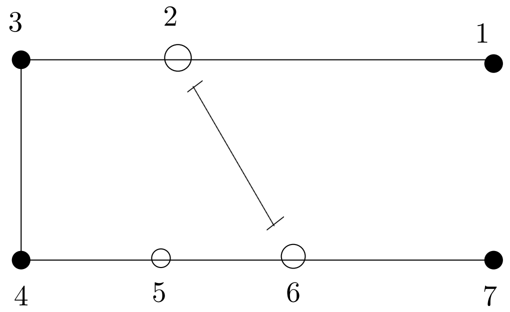


Here $X = \{2, 3, 5, 7\}$ and $Y = \{1, 6, 4\}$. The set $D = \{2, 3\}$ is a $Ya-$dominating set and $D_1 = \{3\}$ is a $Xa-$dominating set.

We now define hyper dominating set in bipartite semigraphs.

Definition 16 *Let S be a bipartite semigraph with $|N_{Xa}(y)| \geq 1$ for every $y \in Y$. A vertex $x \in X$ hyper dominates $y \in Y$ if $y \in N_a(x)$ or $y \in N_a(N_{Ya}(x))$.*

Example 7: Consider the bipartite semigraph.

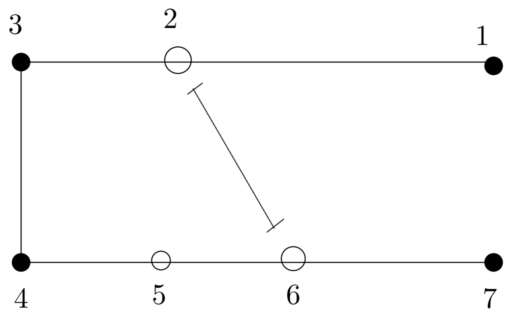


Here $X = \{2,3,5,7\}$ and $Y = \{1,6,4\}$. Vertices 1 and 6 are adjacent to 2 and the vertex $4 \in N_a(N_{Ya}(2))$. The vertex 2 hyper dominates the vertices 1, 6 and 4.

Definition 17 *A subset $D \subseteq X$ is a hyper dominating set of S if every $y \in Y$ is hyper dominated by a vertex of D. A subset $D \subseteq X$ is called a minimal hyper dominating set of S if no proper subset of D is a hyper dominating set of S. The minimum cardinality of a minimal hyper dominating set of S is called hyper domination number of S and is denoted by $\gamma_{ha}(S)$. The maximum cardinality of a minimal hyper dominating set of S is called upper hyper dominating set of S and is denoted by $\gamma_{ha}(S)$.*

Remark 18 *By a $\gamma_{ha}-$set, we mean a minimum hyper dominating set of a semigraph S and by $\Gamma_{ha}-$set, we mean a maximum minimal hyper dominating set.*

Example 8: Consider the bipartite semigraph given below.



Here $X = \{2,3,5,7\}$ and $Y = \{1,6,4\}$. The set $\{2\}$ is a hyper dominating set.

Theorem 19 *Let S be a bipartite semigraph with $|N_{Xa}(y)| \geq 1$ for every $y \in Y$. Then every $Xa-$dominating set of S is a hyper dominating set of S.*

Proof: Let $D \subseteq X$ be a $Xa-$dominating set. By hypothesis for any $y \in Y$, there exists $u \in D$ such that u and y are $Ya-$adjacent. If $u \in D$, then y is hyper dominated by u. If $u \in X - D$, then there exists $v \in D$ such that u and v are $Xa-$adjacent. Hence, $y \in N_a(N_{Ya}(v))$. That is y is hyper dominated by $v \in D$. □

Remark 20 *Converse of the above need not be true. Consider the semigraph in above example, the set $\{2\}$ is hyper dominating set but not a $Xa-$dominating set.*

For any bipartite semigraph S, $\gamma_{ha}(S) \leq \gamma_{Xa}(S)$.

Theorem 21 *If the bipartite semigraph S has no $Xa-$isolates, then $\gamma_{Xa}(G) \leq \frac{n}{2}$, n is the number of vertices in X.*

Theorem 22 *If the bipartite semigraph S has no $Xa-$isolates, then $\gamma_{ha}(G) \leq \frac{n}{2}$, n is the number of vertices in X.*

Proof: Every $Xa-$dominating set is a hyper dominating set and therefore, $gamma_{ha}(G) \leq \gamma_{Xa}(G) \leq \frac{n}{2}$. □

Definition 23 *Let S be a bipartite semigraph. Let $D \subseteq X$. A vertex $x \in D$ has a private hyper neighbour $y \in Y$ if*

(i) x is $Ya-$adjacent to y or $y \in N_a(N_{Ya}(x))$ and

(ii) for all vertices $x_1 \in D - \{x\}$, x_1 is not $Ya-$adjacent to y and $y \notin N_a(N_{Ya}(x_1))$.

Definition 24 *Let S be a subset of X. Let u belong to D. The vertex u is called an $Ya-$isolate of D if there exists no vertex v in $D - \{u\}$ such that u and v are $Xa-$adjacent.*

We characterize the minimal hyper dominating sets as follows:

Theorem 25 *A set $D \subseteq X$ is minimal hyper dominating set if and only if for every $u \in D$ one of the following conditions is satisfied:*

(i) u is an $Ya-$isolate of D.

(ii) there exists $y \in Y$ such that y is private hyper neighbour of u with respect to D.

Proof: Let $D \subseteq X$ be a minimal hyper dominating set of S. Let $u \in D$. Then, $D - \{u\}$ is not a hyper dominating set. Therefore, some vertex $y \in Y$ is not $Ya-$adjacent to any vertex of $D - \{u\}$ and $y \notin N_a(N_{Ya}(x))$ for every $x \in D - \{u\}$. Then y is a private hyper neighbour of u which is (ii) or y is $Ya-$adjacent to u which is not $Ya-$adjacent to any vertex in $D - \{u\}$. Therefore, u is an $Ya-$isolate of D.

Let us assume that D is not a minimal hyper dominating set, there exists a vertex $u \in D$ such that $D - \{u\}$ is a hyper dominating set. Hence, $y \in N_a(u)$ or $y \in N_a(N_{Ya}(u))$ is hyper dominated by a vertex in $D - \{u\}$. That is, both the conditions (i) and (ii) are not satisfied. □

3.1 Bipartite semigraphs with $\gamma_{ha}(S) = 1$

Theorem 26 *In a bipartite semigraph S, $\gamma_{ha}(S) = 1$ if and only if there exists a vertex $x \in X$ such that $N_{Xa}[x]$ is a $Ya-$dominating set.*

Proof: Suppose there exists $x \in X$ such that $N_{Xa}[x]$ is a $Ya-$dominating set. Then every $y \in Y$ is $Ya-$adjacent to x or $y \in N_a(N_{Ya}(x))$. Hence, $\{x\}$ is hyper dominating set. Therefore, $\gamma_{ha}(S) = 1$.

Conversely, if $\gamma_{ha}(S) = 1$, then every $y \in Y$ is $Ya-$adjacent to a vertex $x \in X$ or $y \in N_a(N_{Ya}(x))$. Hence, $N_{Ya}[x]$ is a $Ya-$dominating set. □

Theorem 27 *In a bipartite semigraph S, $\gamma_{ha}(S) = 1$ if and only if there exists $x \in X(S)$ such that any $y \in Y$ is either adjacent to x or adjacent to $u \in X - \{x\}$ which is $Xa-$adjacent to x.*

Theorem 28 *In a bipartite semigraph, $\gamma_{ha}(S) \geq \frac{|Y|}{\Delta\Delta_Y}$.*

Proof: A vertex in X can hyper dominate at most $\Delta\Delta_Y$ vertices. Hence, $\gamma_{ha}(S) \geq \frac{|Y|}{\Delta\Delta_Y}$. □

3.2 Hyper irredundant set

The idea of $Xa-$irredundant sets was introduced in [11] and the existence of such sets are proved.

Definition 29 *Let G be a bipartite semigraph. Let S be a subset of X. Let u belong to S. A vertex v is a private $Xa-$neighbour of u with respect S if u is the only vertex of S, $Xa-$adjacent to v. A set S is $Xa-$irredundant set if every u in S has a private $Xa-$neighbour. The $Xa-$irredundance number of a semigraph G is the minimum cardinality of a maximal $Xa-$irredundant set of G and is denoted by $ir_{Xa}(G)$. The upper $Xa-$irredundance number of a graph G is the maximum cardinality of a maximal $Xa-$irredundant set of G and is denoted by $IR_{Xa}(G)$.*

Here we define hyper irredundant set and prove the existence of such sets.

Definition 30 *A subset D of X is hyper irredundant set if every $v \in D$ has a private hyper neighbour. A subset D of X is a maximal hyper irredundant set if any super set of D is not a hyper irredundant set. The hyper irredundance number of a semigraph S is the minimum cardinality of a maximal hyper irredundant set of vertices and is denoted by $ir_{ha}(S)$. The upper hyper irredundance number of a semigraph S is the maximum cardinality of a maximal hyper irredundant set and is denoted by $IR_{ha}(S)$.*

Theorem 31 *A subset of a hyper irredundant set of S is a hyper irredundant set.*

Proof: Let $D \subseteq X$ be a hyper irredundant set of S. Let $T \subset D$. Let $x \in D$, x has a private hyper neighbour in Y with respect to D. That is x is $Ya-$adjacent to y or $y \in N_a(N_{Ya}(x))$. Also for all vertices $x_1 \in D - \{x\}$, x_1 is not adjacent to y and $y \notin N_a(N_{Ya}(x))$. Since $T - \{x\} \subset S - \{x\}$ for every $x_1 \in T - \{x\}$, we get x_1 is not adjacent to y and $y \notin N_a(N_{Ya}(x))$. Therefore, T is a hyper irredundant set. □

Remark 32 *Hyper irredundant set is a hereditary property.*

Theorem 33 *In a bipartite semigraph S, every hyper dominating set D is a minimal hyper dominating set if and only if it is hyper dominating and hyper irredundant.*

Proof: Let D be a hyper dominating set. Then D is a minimal hyper dominating set if and only if for every $u \in D$, there exists $y \in Y$ which is not hyper dominated by $D - \{u\}$. Equivalently, D is minimal hyper dominating set if and only if it is hyper irredundant set.

Conversely, Let D be both hyper dominating set and hyper irredundant set.

Claim: D is minimal hyper dominating set.

If D is not minimal hypr dominating set, there exists $v \in D$ for which $D - \{v\}$ is hyper dominating. Since D is hyper irredundant, v has a private hyper nieghbour u. By definition, u is not hyper adjacent to any vertex in $D - \{v\}$. That is, $D - \{v\}$ is not hyper dominating set, a contradiction. Hence, D is a minimal hyper dominating set. □

Remark 34 *By the above theorem, any minimal hyper dominating set is an hyper irredundant set. Therefore, hyper irredundant sets exists.*

Theorem 35 *Every minimal hyper dominating set is a maximal hyper irredundant set.*

Proof: Every minimal hyper dominating set D is hyper irredundant set.

Claim: D is a maximal hyper irredundant set.

Suppose D is not maximal hyper irredundant set. Then there exists a vertex $u \in X - D$ for which $D \cup \{x\}$ is hyper irredundant. There exists at least one vertex $y \in Y$ which is a private hyper neighbour of u with respect to $D \cup \{u\}$. That is no vertex in D is hyper adjacnt to y. Hence, D is not hyper dominating set, a contradiction. Hence, D is a maximal hyper irredundant set. □

Remark 36 *Clearly*

$$ir_{ha}(S) \leq \gamma_{ha}(S) \text{ and } \Gamma_{ha}(S) \leq IR_{ha}(S).$$

Thus we have the hyper dominating chain or hyper dominating sequence

$$ir_{ha}(S) \leq \gamma_{ha}(S) \leq i_{ha}(S) \leq \beta_{iha}(S) \leq \Gamma_{ha}(S) \leq IR_{ha}(S)$$

3.3 Hyper independent set

The independent set $Xa-$independent set, $Xa-$hyper independent set and hyper $Xa-$ independent set was defined in [11].

Definition 37 *Two vertices u and v in X are $Xa-$independent if u and v are not $Xa-$adjacent. A Subset D of X is called a $Xa-$independent set if any two vertices in D are $Xa-$independent. A set D is called a maximal $Xa-$independent set if we cannot find a $Xa-$independent set D_1 containing D. The maximum cardinality of a maximal $Xa-$independent set is called the $Xa-$independence number and is denoted by $\beta_{Xa}(G)$.*

Definition 38 *A subset D of X is called $Xa-$hyper independent set if $N_{Xa}(y)$ is not contained in D, for every $y \in Y$. The maximum cardinality of a $Xa-$hyper independent set of a semigraph is called $Xa-$hyper independence number and is denoted by $\beta_{ha}(S)$.*

Definition 39 *A subset D of X is called hyper $Xa-$independent set if $N_{Ya}(x)$ is not contained in D, for every x in D. The maximum cardinality of a hyper $Xa-$independent set of a semigraph is called hyper $Xa-$independence number and is denoted by $\beta_{hXa}(G)$.*

Theorem 40 *In a semigraph, a subset D is a $Xa-$dominating set if and only if $X - D$ is hyper $Xa-$independent set.*

Theorem 41 *In a semigraph, a subset D is a $Ya-$dominating set if and only if $X-D$ is $Xa-$hyper independent set.*

We define hyper independence in semigraph as follows:

Definition 42 *A subset F of X is called a hyper independent set if every $y \in Y$ satisfies one of the conditions:*

(i) $y \notin N_a(x)$ for every $x \in F$ or

(ii) there exists a neighbour of y say x in F such that $N_{Ya}(x)$ is not contained in F.

The maximum cardinality of a hyper independent set is called the hyper independence number and is denoted by $\beta_{hYa}(S)$.

Theorem 43 *In a semigraph, every $Xa-$independent set is a $Xa-$hyper independent set.*

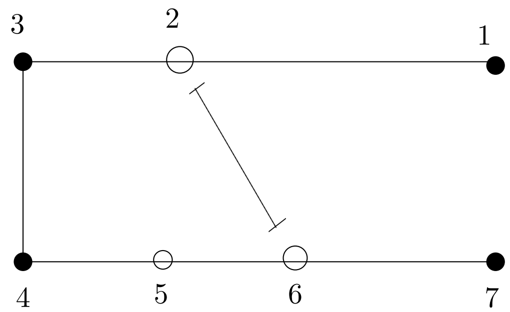
Proof: Let D be a $Xa-$independent set of a semigraph S. Any two vertices in D are not $Xa-$adjacent. Equivalently, for every $y \in Y$, $N_{Xa}(y)$ is not contained in D. Hence, D is a $Xa-$hyper independent set. □

Theorem 44 *In a semigraph, every $Xa-$hyper independent set is a hyper $Xa-$independent set.*

Theorem 45 *In any bipartite semigraph S, every $Xa-$hyper independent set is hyper independent set, but not conversely.*

Proof: Let D be a $Xa-$hyper independent set. Then N_{Xa} is not contained in D for every $y \in Y$. Equivalently, $y \notin N_a(x)$ for every $x \in D$ or there exists a $Xa-$neighbour of y say $x \in D$ such that $N_{Ya}(x)$ is not contained in D. Hence, D is hyper independent set. □

Remark 46 *The converse of the above need not be true. Consider the bipartite semigraph.*



Here $X = \{2, 3, 5, 7\}$ and $Y = \{1, 6, 4\}$. The subset $D = \{3, 5, 7\}$ is a hyper independent set but not a $Xa-$hyper independent set.

A Gallai-type theorem is of the form $x(G) + y(G) = n$ where $x(G)$, $y(G)$ are integer valued minimum or maximum parameters defined on the graph. Gallai type theorems involving $a-$edge covering number and $a-$edge independent number, $ca-$edge covering number and $ca-$edge independence number are proved in semigraphs. In bipartite semigraphs, the parameter involving $Xa-$domination number and hyper $Xa-$independence number, $Ya-$domination number and $Xa-$hyper independence number are also proved.

Theorem 47 *In any bipartite semigraph, D is a hyper dominating set if and only if $X - D$ is hyper independent set.*

Proof: Let D be hyper dominating set. Every $y \in Y$ is hyper dominated by a vertex of D. Equivalently, $y \notin N_a(u)$ where $u \in X - D, Ya-$adjacent to y such that $N_{Xa}(u)$ is not contained in $X - D$. Therefore, $X - D$ is hyper independent set.

Conversely, let D be a hyper independent set. Then $y \notin N_a(x)$ for every $x \in D$ or there exists a neighbour of y say x in D such that $N_{Ya}(x)$ is not contained in D. Equivalently, $y \in N_a(u)$ for some $u \in X - D$ or there exists $u \in X - D$ such that $N_a(N_{Ya}(x))$ contains y. Hence, $X - D$ is a hyper dominating set of S. □

Corollary 48 *In a bipartite semigraph S, $\gamma_{ha}(S) + \beta_{ha}(S) = |X|$.*

Proof: Let D be a $\gamma_{ha}-$ set. Then $X - D$ is hyper independent set. Hence, $\gamma_{ha}(S) + \beta_{ha}(S) \geq |X|$.

Conversely, let F be a hyper independent set. Then $X - F$ is a hyper dominating set. Hence, $\gamma_{ha}(S) + \beta_{ha}(S) \leq |X|$. The result follows. □

We now characterize semigraphs for which $\gamma_{ha}(G) = \gamma_{Xa}(G)$.

Theorem 49 *Let G be a bipartite semigraph. Then $\gamma_{ha}(G) = \gamma_{Xa}(G)$ if and only if there exists a $\gamma_{ha}-$set S such that $X - S$ is hyper $Xa-$independent set.*

Proof: Let S be a $\gamma_{ha}-$set such that $X - S$ is hyper $Xa-$independent set. Then S is a $Xa-$dominating set of G. Therefore, $\gamma_{Xa}(G) \leq |S| = \gamma_{ha}(G)$. This is equivalent to $\gamma_{Xa}(G) \leq \gamma_{ha}(G)$. But, $\gamma_{ha}(G) \leq \gamma_{Xa}(G)$. Therefore, $\gamma_{ha}(G) = \gamma_{Xa}(G)$.

Conversely, let us assume that $\gamma_{ha}(G) = \gamma_{Xa}(G)$. Let D be a $\gamma_{Xa}-$set of G. Therefore, $X - D$ is a hyper $Xa-$independent set of G. But every $Xa-$dominating set is a hyper dominating set and $\gamma_{ha}(G) = \gamma_{Xa}(G)$, it follows that D is a minimum hyper dominating set of G. □

We now characterize semigraphs for which $\gamma_{ha}(G) = \gamma_{Ya}(G)$.

Theorem 50 *Let G be a bipartite semigraph. Then $\gamma_{ha}(G) = \gamma_{Ya}(G)$ if and only if there exists a $\gamma_{ha}-$set S such that $X - S$ is $Xa-$hyper independent set.*

Proof: Let S be a $\gamma_{ha}-$set such that $X - S$ is $Xa-$hyper independent set. Then S is a $Ya-$dominating set of G. Therefore, $\gamma_{Ya}(G) \leq |S| = \gamma_{ha}(G)$. This is equivalent to $\gamma_{Ya}(G) \leq \gamma_{ha}(G)$. But, $\gamma_{ha}(G) \leq \gamma_{Xa}(G) \leq \gamma_{Ya}(G)$. Therefore, $\gamma_{ha}(G) = \gamma_{Ya}(G)$.

Conversely, let us assume that $\gamma_{ha}(G) = \gamma_{Ya}(G)$. Let D be a $\gamma_{Ya}-$set of G. Therefore, $X - D$ is a $Xa-$hyper independent set of G. But every $Ya-$dominating set is a $Xa-$dominating set and hence, hyper dominating set and $\gamma_{ha}(G) = \gamma_{Ya}(G)$, it follows that D is a minimum hyper dominating set of G. □

We now find a characterization of $\gamma_{ha}(G) = 1$ in terms of hyper independent set.

Theorem 51 *Let G be a bipartite semigraph with $\gamma_{ha}(G) = 1$. Let $x \in X$ be such that $N_{Ya}[x] = Y$. Then $X - \{x\}$ is a hyper independent set. Conversely, if $X - \{x\}$ is hyper independent set for some $x \in X$, then $\gamma_{ha}(G) = 1$.*

Proof: Let $\gamma_{ha}(G) = 1$. Suppose, S is a $\gamma_{ha}-$set of G. Then, $S = \{x\}$ for some $x \in X$. Therefore, for every $y \in Y$, either $y \in N_a(x)$ or $y \in N_a(N_{Ya}(x))$. Suppose $y \in N_a(x)$ and $y \notin N_a(u)$ for every $u \in X - \{x\}$. Then y satisfies the first condition for hyper independence of $X - \{x\}$.

Suppose $y \in N_a(x)$ and $y \in N_a(u)$ for some $u \in X - \{x\}$. Then x and u are $Xa-$adjacent. That is $x \in N_{Ya}(u)$. That is y satisfies the second condition for hyper independence of $X - \{x\}$. Suppose $y \notin N_a(x)$. Then there exists $u \in N_{Ya}(x)$ such that $y \in N_a(u)$. That is $y \in N_a(u)$, $u \in X - \{x\}$ and $x \in N_{Ya}(u)$. Therefore, y satisfies the second condition for hyper independence of $X - \{x\}$. Therefore, $X - \{x\}$ is hyper independent set.

Conversely, suppose for some $x \in X$, $X - \{x\}$ is hyper independent. Then for any $y \in Y$, $N_{Xa}(y)$ is not contained in $X - \{x\}$ or if $N_{Xa}(y) \subseteq X - \{x\}$ then there exists $u \in N_{Xa}(y)$ such that $N_{Ya}(u)$ is not contained in $X - \{x\}$. That is, for any $y \in Y$, either y is adjacent to x or there exists $u \in X$ such that y is adjacent to u and u is $Xa-$adjacent to x. That is, $\gamma_{ha}(G) = 1$. □

3.4 Inequalities involving dominating parameters and irredundant parameter

The vertex set X is partitioned as follows. Let $S_0 \subseteq X$. Let S_1 be the set of vertices in $X - S_0$, $Xa-$dominated by S_0. i.e., $S_1 = N_{Xa}[S_0] - S_0$. Let $S_2 = X - S_0 - S_1$. Thus any subset $S_0 \subseteq X$ defines a partition of X into three sets $\{S_0, S_1, S_2\}$.

In what follows S_1 is a semigraph without isolates.

Theorem 52 *Let $\{S_0, S_1, S_2\}$ be a partition of X of a connected nontrivial semigraph S defined by S_0. If S_0 is a hyper dominating set, then S_2 is a $Xa-$hyper independent set of S.*

Proof: Let S_0 be a hyper dominating set. Let us assume that S_2 is not a $Xa-$hyper independent set. Then there exists $y \in Y$ such that $N_{Xa}(y) \subseteq S_2$.

Therefore, $y \in Y$ is not hyper dominated by any element of S_0, a contradiction. Hence, S_2 is a $Xa-$hyper independent set. □

Theorem 53 *Let $\{S_0, S_1, S_2\}$ be a partition of X of a connected nontrivial bipartite semigraph S defined by S_0. If S_0 is a minimal hyper dominating set, S_1 is a $Xa-$dominating set of S.*

Proof: Let S_0 be minimal hyper dominating set of S. Suppose S_1 is not a $Xa-$dominating set of S. Then there exists a vertex $v \in X - S_1$, not $Xa-$adjacent to any vertex in S_1. Since, $v \notin S_1$ either $v \in S_0$ or $v \in S_2$.
Case 1: Assume $v \in S_0$. Since, S_0 is minimal hyper dominating set, S_0 is maximal hyper irredundent set. Therefore, v has a private hyper neighbour $y \in Y$. Equivalently, $y \in N_a(N_{Ya}(v))$ or $y \in N_a(v)$. In either case v is $Xa-$adjacent to a vertex of S_1, a contradiction.
Case 2: Let $v \in S_2$. By the above theorem, S_2 is $Xa-$hyper independent set of S. Since v is not a isolate of S, there exists $y \in Y$ such that v is $Ya-$ adjacent to y. Since, S_2 is hyper independent, $N_{Xa}(y)$ is not contained in S_2. Therefore, there exists $x \in N_{Xa}(y)$ such that $x \in S_0 \cup S_1$. If $x \in S_0$, then $v \in S_1$, a contradiction. Therefore, S_1 is a $Xa-$dominating set. □

Theorem 54 *In a bipartite semigraph S, $\Gamma_{ha}(S) + \gamma_{Xa}(S) \leq p$.*

Proof: Let D be a $\Gamma_{ha}-$set of S. Then, $X - D$ is a $Xa-$dominating set. Therefore, $\gamma_{Xa}(S) \leq |X - D|$. Hence, $\Gamma_{ha}(S) + \gamma_{Xa}(S) \leq p$. □

Theorem 55 *In a bipartite semigraph S, $\Gamma_{Xa}(S) + \gamma_{ha}(S) \leq p$.*

Proof: Let D be a $\Gamma_{Xa}-$set of S. Then, D is a minimal $Xa-$dominating set of S. Therefore, $X - D$ is a $Xa-$dominating set of S. Hence, $X - D$ is a hyper dominating set of S. Therefore, $\gamma_{ha}(S) \leq |X = D|$. Hence, $\Gamma_{Xa}(S) + \gamma_{ha}(S) \leq p$. □

The upper bound for the sum of domination number and irredundance number was found in [8]. The upper bound for the sum of upper hyper irredundance number and $Xa-$domination number, upper $Xa-$domination number and hyper irredundance number are found.

Remark 56 *By $ir_{ha}-$set and $IR_{ha}-$set, we mean minimum cardinality and maximum cardinality of a maximal hyper irredundant set.*

Theorem 57 *Let S be a bipartite semigraph with $N_{Ya}(x) \neq \phi$ for every $x \in X$. Then $IR_{ha}(S) + \gamma_{Xa}(S) \leq |X|$.*

Proof: Let D be a $IR_{ha}-$set of S. Then, D is a maximal irredundant set. Therefore, D is hyper irredundant set. That is every $x \in D$ has a private neighbour $y \in Y$. Then x is $Ya-$adjacent to y or $y \in N_a(N_{Ya}(x))$ and for all vertices $x_1 \in D - \{x\}$, x_1 is not adjacent to y and $y \notin N_a(N_{Ya}(x))$.
Case (i): x is $Ya-$adjacent with y.

Since, $N_{Ya}(x) \neq \phi$, x has $Xa-$neighbours. Let z be any $Xa-$neighbour of x. Suppose, $x \in D$. Then z is not $Xa-$adjacent to y and $y \notin N_a(N_{Ya}(z))$. But $y \in N_a(N_{Ya}(x))$, since x is $Xa-$neighbour of z, a contradiction. Therefore, any $Xa-$neighbour of x is in $X - D$.
Case (ii): $y \in N_a(N_{Ya}(x))$.

Vertices in $N_a(y)$ are in $X - D$. Then $N_a(y) \subseteq X - D$. Otherwise, we get a contradiction to $y \in Y$ is a private hyper neighbour of $x \in D$. Hence, for every $x \in D$ there exists $x_1 \in X - D$ such that x and x_1 are $Xa-$adjacent. That is, $X - D$ is a $Xa-$dominating set. Therefore, $\gamma_{Xa}(S) \leq |X - D|$. Hence, $IR_{ha}(S) + \gamma_{Xa}(S) \leq |X|$. □

Theorem 58 *Let S be a bipartite semigraph with $N_{Ya}(x) \neq \phi$ for every $x \in X$. Then, $IR_{Xa}(S) + \gamma_{ha}(S) \leq |X|$.*

Proof: Let D be a $IR_{Xa}-$set of S. Every element $x \in D$ has a private $Xa-$neighbour. Consider the set $X - D$. Since $X - D$is a $Xa-$dominating set, elements of Y are either $Ya-$adjacent to $X - D$ or $Ya-$adjacent to vertices which are $Xa-$adjacent to elements of $X - D$. Therefore, $X - D$ is a hyper dominating set. Therefore, $\gamma_{ha} \leq |X - D|$. Hence, $IR_{Xa}(S) + \gamma_{ha}(S) \leq |X|$. □

Theorem 59 *In a bipartite semigraph S, $\Gamma_{ha}(S) \leq \Gamma_{Xa}(S)$.*

Proof: Let D be a $\Gamma_{ha}-$set. Then D is a minimal hyper dominating set of maximum cardinality. Let $D = \{u_1, u_2, u_3, \cdots, u_d\}$. Then every $u_i \in D$ is a $Ya-$isolate of $\langle D\rangle$ or there exists $y \in Y$ such that y is private hyper neighbour of u.

Case A: Every $u_i \in D$ is $Ya-$isolate of $\langle D\rangle$.

For $i \neq j$, $N_{Ya}(u_i) \neq N_{Ya}(u_j)$. Otherwise, $i \neq j$, $N_{Ya}(u_i) = N_{Ya}(u_j)$ then $D-\{u_i\}$ or $D-\{u_j\}$ is hyper dominating set, a contradiction to the fact that D is minimal hyper dominating set. Hence every vertex in D has a unique $Ya-$private hyper neighbour in S. Therefore, every minimal $Xa-$dominating set must contain at least $|D|$ elements. Hence, $\Gamma_{Xa}(S) \geq$

$|D|$ elements. Hence, we have the inequality $\Gamma_{Xa} \geq |D| = \Gamma_{ha}(S)$.

Case B: There exists $y \in Y$ such that y is private hyper neighbour of u.

Sub case(i): $y \in N_a(u)$.

Since y is private hyper neighbour $N(y) \subseteq (X - D) \cup \{u\}$. No other vertex in $D - \{u\}$ is $Xa-$adjacent to $N_a(y)$, for otherwise, we get a contradiction to y is private hyper neighbour of u.

Sub case(ii): $y \in N_a(N_{Ya}(u))$.

Clearly $N_a(y) \in X - D$. Otherwise, we get a contradiction to y is private hyper neighbour of u and no other vertex in $D - \{u\}$ is $Xa-$adjacent to vertices in $N_a(y)$.

In the above two cases, any minimal $Xa-$dominating set must contain at least $|D|$ elements. Hence, $\Gamma_{Xa}(S) \geq |D| = \Gamma_{ha}(S)$.

4 Conclusion

The concept of hyper domination in bipartite semigraph was defined. The dominating chain involving hyperdomination in bipartite semigraph are proved. Some inequalities involving domination parameters and irredundant parameters is proved. Gallai type theorem involving hyperdomination number and hyper independence number are also proved.

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