The Nonexistence of the Solution for Quasilinear Parabolic Equation Related to the P-Laplacian

HUASUI ZHAN
Jimei University
School of Sciences
Xiamen
P.R.CHINA
huashuizhan@163.com, hszhan@jmu.edu.cn

Abstract: Consider the following Cauchy problem

$$u_t = div(|\nabla u^m|^{p-2}\nabla u^m) - u^q, (x,t) \in S_T = R^N \times (0,T),$$
$$u(x,0) = \delta(x), \quad x \in R^N,$$

where $1 < p < 2$, and $\delta(x)$ is the Dirac measure centered at the origin. If $m(p-1) + \frac{p}{N} \leq 1$ and $q > 0$, it can be proved that there is not solution for the above narrated problem.

Key–Words: Quasilinear parabolic equation, Cauchy problem, Nonexistence, Dirac measure

1 Introduction

The paper is interested in the following equation

$$u_t = div(|\nabla u^m|^{p-2}\nabla u^m) - u^q, \tag{1}$$

where $(x,t) \in S_T = R^N \times (0,T)$, with the following initial condition

$$u(x,0) = \mu, \quad x \in R^N, \tag{2}$$

where μ is a nonnegative σ-finite measure in R^N. In what follows, $B_R(x_0) = \{x : |x - x_0| < R\}$, and if $x_0 = 0$, simply denote it as B_R.

Definition 1 *A measurable function u is said to be a weak solution of problem (1)-(2), if u satisfies the following conditions*

$$u \in C(0,T; L^1_{loc}(R^N)) \tag{3}$$

$$u^m \in L^p(0,T; W^{1,p}_{loc}(R^N)) \tag{4}$$

$$\nabla u^m \in L^\infty_{loc}(S_T), \tag{5}$$

$$\int_{R^N} u(x,t)\varphi(x,t)dx + \iint_{S_T}(-u\varphi_t + |\nabla u^m|^{p-2}\nabla u^m \nabla\varphi)dxdt + \int\int_{S_T} u^q\varphi dxdt = \int_{R^N}\varphi(x,0)d\mu \tag{6}$$

where $\varphi \in C^1(\bar{S}_T)$ and $\varphi = 0$ if $|x|$ is large enough.

The problem arises in the fields of mechanics, physics and biology, including the non-Newtonian fluids, the gas flow in porous media, the spread of biological population, etc.

If the initial value $u(x,0) = u_0(x)$ is appropriately smooth, there are many papers in devoting to the solvability of the Cauchy problem of (1), one can refer to Wu-Zhao [1], Gmira [2], Yang-Zhao [3], Zhao [4, 5, 6], Zhao-Yuan [7], Dibenedetto-Friedman [8], Li-Xia [9], Dibenedetto-Herrero [10], Benilan-Crandall-Pierre [11], Zhao-Xu [12], Fan [13] and the references therein for details.

For example, when $p = 2$, we have the following basic results

(i) If $q > m > 1$, $u_0 \in L^1_{loc}(R^N)$, or $1 < m < q < m + \frac{2}{N}$, $u(x,0) = \mu$ is a nonnegative σ-finite measure, then the Cauchy problem of (1) has a global solution.

(ii) If $q \geq m + \frac{2}{N}$, $u(x,0) = \mu$ is a nonnegative σ-finite measure, then the Cauchy problem of (1) has not any solution.

This fact means that, in the case of $p = 2$, in order that the Cauchy problem of (1) has a solution , $q < m + \frac{2}{N}$ not only acts as a sufficient condition, but also acts as a necessary condition. By the way, it is well-known that, in order that the Cauchy problem of the equation

$$u_t = \triangle u^m, u(x,0) = u_0(x) \tag{7}$$

has a solution, one should pose some restrictions on the growth order of the initial value. However, if $q > m > 1$, the existence of the solution for the Cauchy problem of equation (1) has not any restrictions on the growth order of the initial value.

(iii) If $q = m$,

$$\int_{R^N} \exp\{-\sqrt{1+|x|^2}\}u_0(x)dx < \infty, \tag{8}$$

or $1 < q < m$, $u_0 \in L^1_{loc}(R^N)$,

$$u_0(x) \le c_1(c_2 + |x|^2)^{\frac{1}{m-q}},\ a.e.\ R^N, \tag{9}$$

where c_1, c_2 are constants,

$$c_1 < \left(\frac{(m-q)^2}{2Nm(m-q)+4mp}\right)^{\frac{1}{m-q}},$$

then the Cauchy problem of (1) has a global solution. The condition (9) restricts the growth order of the initial value u_0 is less than $|x|^{\frac{2}{m-q}}$, this restriction is weaker than that of the equation (7), which restricts initial value u_0 satisfying that: for some $r > 0$,

$$\sup_{\rho\ge r} \rho^{N-\frac{2}{m-1}} \int_{B_\rho} u_0(x)dx < \infty,$$

which roughly means that the growth order of the initial value u_0 should be less than $\frac{|x|^2}{m-1}$. Moreover, Zhao-Li [14] showed that the condition (9) is almost extremely, in fact, when $1 < q < m$, if $u_0 \in L^1_{loc}(R^N)$, and there is constant $\alpha > \frac{2}{m-q}$, such that

$$\lim_{|x|\to\infty} \inf \frac{u_0(x)}{|x|^\alpha} > 0,$$

then the Cauchy problem (7) has not any weak solution.

(iv) If $0 < q < m + \frac{2}{N}$, $m > (1-\frac{2}{N})_+$, the Cauchy problem (1)-(2) has a very singular solution $U(x,t)$, which satisfies that

$$U \in C(\overline{S}_T \setminus 0), U(x,0) = 0, \forall x \in R^N,$$

$$\lim_{t\to 0} \int_{B_r} U(x,t)dx = +\infty, \forall r > 0.$$

For another example, when $m = 1$, we have the following basic results.

(i)If $q > p-1$, $u_0 \in L^1_{loc}(R^N)$, or $p-1 < q < p-1+\frac{p}{N}$, $u(x,0) = \mu$ is a nonnegative σ finite measure, then the Cauchy problem of (1) has a global solution.

(ii) If $q \ge p-1+\frac{p}{N}$, $u(x,0) = \mu$ is a nonnegative σ measure, then the Cauchy problem of (1) has not any global solution; this fact means that, in order that the Cauchy problem of (1) in this case has a solution , the condition $q < p-1+\frac{p}{N}$ not only acts as a sufficient condition, but also acts as a necessary condition. By the way, it is well-known that, in order that the Cauchy problem of the equation

$$u_t = div(|\nabla u|^{p-2}\nabla u), u(x,0) = u_0(x), \tag{10}$$

has a solution, one should pose some restrictions on the growth order of the initial value. However, if $q > p-1$, the existence of the solution for the Cauchy problem of equation (1) has not any restrictions on the growth order of the initial value.

(iii) If $q = p-1$, $u_0 \in L^{1+\alpha}_{loc}(R^N)$,

$$\int_{R^N} \exp\{-c\sqrt{1+|x|^2}\}u_0^{1+\alpha}(x)dx < \infty,$$

where the constants $\alpha > 0$, $c < p(p-1)^{\frac{1-p}{p}}\alpha^{\frac{p-1}{p}}$, or $1 < q < p-1$, $u_0 \in L^1_{loc}(R^N)$,

$$u_0(x) \le c_1(c_2+|x|^2)^{\frac{p}{2(p-1-q)}},\ a.e.\ R^N, \tag{11}$$

where c_1, c_2 are constants,

$$c_1 < \left(\frac{(p-1-q)^p}{p^{p-1}(pq+N(p-1-q))}\right)^{\frac{1}{p-1-q}},$$

then the Cauchy problem of (1) has a global solution. The condition (11) restricts the growth order of the initial value u_0 is less than $|x|^{\frac{p}{p-1-q}}$, this restriction is weaker than that of the equation (10), which restricts initial value u_0 has the growth order less than $\frac{|x|^p}{p-1}$. Moreover, the condition (11) is almost extremely, in fact, when $1 < q < p-1$, if $u_0 \in L^1_{loc}(R^N)$, and there are constants $\alpha > \frac{p}{p-1-q}$, $B > 0$, such that

$$\lim_{|x|\to\infty} \frac{u_0(x)}{|x|^\alpha} = B,$$

then the Cauchy problem (10) has not any weak solution.

(iv) If $\max\{1, p-1\} < q < p-1+\frac{p}{N}$, $p > \frac{2N}{N+1}$, the Cauchy problem (1)-(2) has a very singular solution $U(x,t)$ too.

Recently, the author has been studying the solvability of the equation in [15]. By discussing the existence of the self-similar solution, the author [15] had got the singular solution of the following more general equation

$$u_t = div(|Du^m|^{p-2}Du^m) - u^{q_1}|Du^m|^{p_1}, \tag{12}$$

provided that $p > 2, m > 1$, and

$$p > p_1, q_1 + p_1 m > m(p-1) > 1. \tag{13}$$

While [16-17] had discussed the large time behavior of the solution of the Cauchy problem of (12). There are so many papers such as [18-40] which studied the posedness of the solutions, the HÖlder continuity, the large time behaviors of the solutions, and other related results of the special cases of (12) or (1), so it is impossible to point them one by one here.

2 The main result of the paper

Especially, we quote the following important propositions obtained in [13].

Proposition 2 *Let $N \geq 1$. Suppose that $u(x,0) = \mu$ is a nonnegative σ finite measure. If m,p,q,N satisfy the conditions that $m(p-1)+\frac{p}{N} > 1$, and $0 < q < m(p-1)+\frac{p}{N}$, then there exists a generalized solution to the Cauchy problem of (1).*

Proposition 3 *Let $N \geq 1$. If m,p,q,N satisfy the conditions that either $m(p-1)+\frac{p}{N} < 1, q > 0$, or $1 < m(p-1)+\frac{p}{N} < q$, then the Cauchy problem (1)-(2) has not a solution.*

In this paper, we will discuss Cauchy problem (1)-(2) when $m(p-1) < 1$. We will prove the following

Theorem 4 *Suppose $2 > p > 1$, $m > 1$, $q > p-1$ and*

$$2-p+\frac{p}{N} < m(p-1)+\frac{p}{N} \leq 1, \qquad (14)$$

then there is not nonnegative solution for Cauchy problem (1) with the following initial value

$$u(x,0) = \delta(x), \qquad (15)$$

where δ is the Dirac measure centered at the origin.

Compared to the above Proposition 3, which is one of the main results of [13], it can be found that our Theorem 4 improves the result of Proposition 3 in the case of that $m(p-1)+\frac{p}{N} < 1, q > 0$. This improvement is obtained by quoting the above definition of the weak solution (Definition 1, which is equivalent to the corresponding weak solution defined in [13]), and by choosing suitable testing functions to make more meticulous estimates. However, the method we used is different from that of 13], and we use some ideas in [3], in which $m = 1$, $p \leq \frac{2N}{N+1}$ $q \geq 0$. However, compared to [3], to get Theorem 4, the conditions $m > 1, q > p-1$ are necessary in our proof. By the way, the condition $2-p+\frac{p}{N} < m(p-1)+\frac{p}{N}$ implies $m(p-1) > 2-p$, we conjecture that this condition may be weaken to that $m(p-1) > 0$.

3 An important lemma and its proof

Lemma 5 *Suppose $1 < p < 2$, $m(p-1) < 1$, $q > 0$, then the nonnegative solution u of Cauchy problem (1)-(2) satisfies*

$$\sup_{0<\tau<t}\int_{B_R} u(x,\tau)dxd\tau$$

$$\leq c + ct^{\frac{1}{1-mp+m}}R^{N-\frac{p}{1-mp+m}}. \qquad (16)$$

$$\int_0^T\int_{B_R}|\nabla u^m|^{p-1}dxdt$$

$$\leq cR^{1-N[1-m(p-1)]-p}\left(\int_0^T\int_{B_{2R}}t^{\frac{1}{2}}udxdt\right)^{m(p-1)}$$

$$+cR^{\frac{N(m-mp+1)}{p}}\left(\int_0^T\int_{B_{2R}}t^{\frac{1}{2}}udxdt\right)^{\frac{(m+1)(p-1)}{p}}. \qquad (17)$$

$$\int_0^T\int_{B_R}u^q dxdt \leq c(R,T). \qquad (18)$$

Proof Let ξ be the cut function on B_{2R}, satisfying that $\xi = 1$ on B_{l2R}, and $|\nabla\xi| \leq (1-l)^{-1}R^{-1}$, $l \in [\frac{1}{2}, 1)$. For any $t > s > 0$, let ξ^p be as a testing function. We have

$$\int_{B_{2lR}}u(x,t)\xi dx$$

$$\leq \int_{B_{2R}}u(x,t)\xi dx + \int_s^t\int_{B_{2R}}\xi^p u^q dxd\tau$$

$$= \int_{B_{2R}}u(x,s)\xi dx - \int_s^t\int_{B_{2R}}|\nabla u^m|^{p-2}\nabla u^m\cdot\nabla\xi dxd\tau$$

$$\leq \int_{B_{2R}}u(x,s)dx + \frac{c}{(1-l)R}\int_s^t\int_{B_{2R}}|\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$

$$(19)$$

The calculations as follows are formal on that they require u to be strictly positive. They can be make rigorous by replacing u with $u+\varepsilon$ and letting $\varepsilon \to 0$. By Hölder inequality, we have

$$\int_s^t\int_{B_{2R}}|\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$

$$= \int_s^t\int_{B_{2R}}\xi^{p-1}|\nabla u^m|^{p-1}(\tau-s)^{\beta}u^{m\alpha}u^{-m\alpha}(\tau-s)^{-\beta}dxd\tau$$

$$\leq \left(\int_s^t\int_{B_{2R}}\xi^p(\tau-s)^{\frac{\beta p}{p-1}}|\nabla u^m|^p u^{\frac{-m\alpha p}{p-1}}dxd\tau\right)^{\frac{p-1}{p}}$$

$$\times\left(\int_s^t\int_{B_{2R}}(\tau-s)^{-p\beta}u^{m\alpha p}dxd\tau\right)^{\frac{1}{p}}, \qquad (20)$$

where α, β are constants to be chosen later.

In Definition 1, we choose testing function

$$\varphi = \xi^p (\tau - s)^{\frac{\beta p}{p-1}} u^{m(1-\frac{\alpha p}{p-1})} \eta_h(\tau - s),$$

where

$$\eta_h(t) \in C^1(R),\ \eta_h \geq 0,$$

and when $t > s + h$,

$$\eta_h(t) = 1,$$

when $t < s$,

$$\eta_h(t) = 0,$$

and

$$\lim_{h\to 0} \eta_h'(t)t = 0, \tag{21}$$

then, from the definition of the weak solution, we have

$$\int_s^t \int_{R^N} u_\tau(x,\tau)\varphi(x,\tau)dxd\tau$$

$$= \int_s^t \int_{B_{2R}} u_\tau(x,\tau)\xi^p(\tau-s)^{\frac{\beta p}{p-1}} u^{m(1-\frac{\alpha p}{p-1})}\eta_h(\tau-s)dxd\tau$$

$$= \frac{p-1}{m[(1-\beta)p-1]+p-1}$$

$$\times \int_s^t \int_{B_{2R}} \xi^p (\tau - s)^{\frac{\beta p}{p-1}} u_\tau^{m(1-\frac{\alpha p}{p-1})+1} \eta_h(\tau - s)dxd\tau$$

$$= \frac{p-1}{m[(1-\beta)p-1]+p-1}$$

$$\times\{\int_{B_{2R}} \xi^p (\tau - s)^{\frac{\beta p}{p-1}} u^{m(1-\frac{\alpha p}{p-1})+1} \eta_h(t - s)dx$$

$$-\frac{\beta p}{p-1}(\tau - s)^{\frac{\beta p}{p-1}-1}$$

$$\times[\int_s^t \int_{B_{2R}} \xi^p u^{m(1-\frac{\alpha p}{p-1})+1} \eta_h(\tau - s)dxd\tau$$

$$-\int_s^t \int_{B_{2R}} \xi^p u^{m(1-\frac{\alpha p}{p-1})+1} (\tau - s)\eta_h'(\tau - s)dxd\tau]\},$$

and

$$\int_s^t \int_{B_{2R}} div(|\nabla u^m|^{p-2}\nabla u^m)\varphi(x,\tau)dxd\tau$$

$$= \int_s^t \int_{B_{2R}} div(|\nabla u^m|^{p-2}\nabla u^m)$$

$$\times\xi^p(\tau - s)^{\frac{\beta p}{p-1}} u^{m(1-\frac{\alpha p}{p-1})}\eta_h(\tau - s)dxd\tau$$

$$= -p\int_s^t \int_{B_{2R}} (\tau - s)^{\frac{\beta p}{p-1}}\eta_h(\tau - s)\xi^{p-1}u^{m(1-\frac{\alpha p}{p-1})}$$

$$\times|\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\xi dxd\tau$$

$$+(\frac{\alpha p}{p-1} - 1)\int_s^t \int_{B_{2R}} \eta_h(\tau - s)(\tau - s)^{\frac{\beta p}{p-1}}\xi^p u^{-\frac{m\alpha p}{p-1}}$$

$$\times|\nabla u^m|^p dxd\tau$$

$$\geq -\int_s^t \int_{B_{2R}} (\tau - s)^{\frac{\beta p}{p-1}}\eta_h(\tau - s)u^{-\frac{m\alpha p}{p-1}}$$

$$\times[\varepsilon\xi^p|\nabla u^m|^p + c(\varepsilon)u^{mp}|\nabla\xi|^p]dxd\tau$$

$$+(\frac{\alpha p}{p-1} - 1)\int_s^t \int_{B_{2R}} \eta_h(\tau - s)(\tau - s)^{\frac{\beta p}{p-1}}\xi^p u^{-\frac{m\alpha p}{p-1}}$$

$$\times|\nabla u^m|^p dxd\tau$$

and

$$\int_s^t \int_{B_{2R}} u^q\varphi(x,\tau)dxd\tau$$

$$= \int_s^t \int_{B_{2R}} \xi^p(\tau - s)^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})}\eta_h(\tau - s)dxd\tau.$$

If we let $h \to 0$. By (21), choosing α such that

$$\frac{\alpha p}{p-1} - 1 > 0,$$

the readers will find that in the following discussion, when we choose the constant α, it always satisfies this inequality. Now, we have

$$\int_s^t \int_{B_{2R}} \xi^p u^{-\frac{\alpha pm}{p-1}}(\tau - s)^{\frac{\beta p}{p-1}}|\nabla u^m|^p dxd\tau$$

$$\leq c\int_s^t \int_{B_{2R}} u^{m(1-\frac{\alpha p}{p-1})+1}(\tau - s)^{\frac{\beta p}{p-1}}dxd\tau$$

$$+\frac{c}{(1-l)^pR^p}\int_s^t \int_{B_{2R}} (\tau - s)^{\frac{\beta p}{p-1}}u^{-\frac{\alpha pm}{p-1}+pm}dxd\tau. \tag{22}$$

Substituting (22) in (20),

$$\int_s^t \int_{B_{2R}} |\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$

$$\leq c\{[(1-l)R]^{-p}\int_s^t \int_{B_{2R}} (\tau - s)^{\frac{\beta p}{p-1}}u^{mp-\frac{\alpha pm}{p-1}}dxd\tau$$

$$+\int_s^t \int_{B_{2R}} (\tau - s)^{\frac{\beta p}{p-1}-1}u^{m(1-\frac{\alpha p}{p-1})+1}dxd\tau\}^{\frac{p-1}{p}}$$

$$\times\left(\int_s^t \int_{B_{2R}} (\tau - s)^{-p\beta}u^{\alpha pm}dxd\tau\right)^{\frac{1}{p}}. \tag{23}$$

We consider the following two cases.

(1) If $\frac{1}{mp} < p - 1$, we choose $\alpha = \frac{1}{mp}, \beta = \frac{1}{2p}$ in (23). Then

$$\int_s^t \int_{B_{2R}} |\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$

$$\leq c[[(1-l)R]^{-p}\int_s^t\int_{B_{2R}}(\tau-s)^{\frac{1}{2(p-1)}}u^{mp-\frac{1}{p-1}}dxd\tau$$

$$+\int_s^t\int_{B_{2R}}\tau^{\frac{1}{2(p-1)}-1}u^{m[1-\frac{1}{m(p-1)}]+1}dxd\tau]^{\frac{p-1}{p}}$$

$$\times\left(\int_s^t\int_{B_{2R}}(\tau-s)^{-\frac{1}{2}}udxd\tau\right)^{\frac{1}{p}}. \qquad (24)$$

The conditions $1<p<2, m(p-1)<1$ assure that

$$1>m[1-\frac{1}{m(p-1)}]+1>0.$$

By Hö lder inequality,

$$\{\int_s^t\int_{B_{2R}}(\tau-s)^{\frac{1}{2(p-1)}-1}u^{m[1-\frac{1}{m(p-1)}]+1}dxd\tau\}^{\frac{p-1}{p}}$$

$$\leq\{(\int_s^t\int_{B_{2R}}u(\tau-s)^{-\frac{1}{2}}dxd\tau)^{m[1-\frac{1}{m(p-1)}]+1}$$

$$\times(\int_s^t\int_{B_{2R}}(\tau-s)^{\frac{m-1}{2}\frac{p-1}{1-m(p-1)}}dxd\tau)^{\frac{1}{p-1}-m}\}^{\frac{p-1}{p}}$$

$$\leq c\left(\int_s^t\int_{B_{2R}}u(\tau-s)^{-\frac{1}{2}}dxd\tau\right)^{\frac{m(p-1)+p-2}{p}}$$

$$\times R^{\frac{m-mp+1}{p}N}t^{\frac{2-p}{1-m(p-1)}},$$

in which we have used the conditions $p<2$, $m>1$. At the same time, we have

$$\left(\int_s^t\int_{B_{2R}}(\tau-s)^{\frac{1}{2(p-1)}}u^{mp-\frac{1}{p-1}}dxd\tau\right)^{\frac{p-1}{p}}$$

$$\leq\{\int_s^t\int_{B_{2R}}[(\tau-s)^{-\frac{1}{2}(pm-\frac{1}{p-1})}u^{mp-\frac{1}{p-1}}]^{\frac{1}{pm-\frac{1}{p-1}}}$$

$$dxdt\}^{(mp-\frac{1}{p-1})\frac{p-1}{p}}$$

$$\times\{\int_s^t\int_{B_{2R}}[(\tau-s)^{\frac{1}{2(p-1)}+\frac{1}{2}(pm-\frac{1}{p-1})}]^{\frac{1}{\frac{1}{p-1}+1-pm}}$$

$$dxd\tau\}^{(\frac{1}{p-1}+1-pm)\frac{p-1}{p}}$$

$$\leq c\left(\int_s^t\int_{B_R}(\tau-s)^{-\frac{1}{2}}udxd\tau\right)^{m(p-1)-\frac{1}{p}}$$

$$\times R^{N(1-mp+m)}t^{\frac{2-m(p-1)}{2}},$$

we have

$$\int_s^t\int_{B_{2R}}|\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$

$$\leq c(1-l)^{1-p}R^{N[1-m(p-1)]}t^{\frac{2-m(p-1)}{2}}$$

$$\times\left(\int_s^t\int_{B_{2R}}(\tau-s)^{-\frac{1}{2}}udxd\tau\right)^{m(p-1)}$$

$$+cR^{\frac{N(m-mp+1)}{p}}t^{\frac{(m+1)[1-m(p-1)]}{2p}}$$

$$\times\left(\int_s^t\int_{B_{2R}}(\tau-s)^{-\frac{1}{2}}udxd\tau\right)^{\frac{(m+1)(p-1)}{p}} \qquad (25)$$

By (19), (25),

$$\sup_{s<\tau<t}\int_{B_{l2R}}u(x,\tau)dx\leq\int_{B_{2R}}u(x,s)dx$$

$$+c(1-l)^{-p}R^{-k}\left(\int_s^t\int_{B_{2R}}u(\tau-s)^{-\frac{1}{2}}dxd\tau\right)^{m(p-1)}$$

$$\times t^{\frac{2-m(p-1)}{2}}$$

$$+c(1-l)^{-1}R^{-\frac{k}{p}}\left(\int_s^t\int_{B_{2R}}u(\tau-s)^{-\frac{1}{2}}dxd\tau\right)^{\frac{(m+1)(p-1)}{p}}$$

$$\times t^{\frac{(m+1)[1-m(p-1)]}{2p}}$$

$$\leq\int_{B_{2R}}u(x,s)dx+c(1-l)^{-p}R^{-k}t$$

$$\times\left(\sup_{s<\tau<t}\int_{B_{2R}}udx\right)^{m(p-1)}$$

$$+c(1-l)^{-1}R^{-\frac{k}{p}}t^{\frac{(m+1)[p-m(p-1)]}{2p}}$$

$$\times\left(\sup_{s<\tau<t}\int_{B_{2R}}udx\right)^{\frac{(m+1)(p-1)}{p}}$$

$$\leq\int_{B_{2R}}u(x,s)dx+c(1-l)^{-\frac{p}{1-m(p-1)}}R^{\frac{-k}{1-m(p-1)}}t^{\frac{1}{1-m(p-1)}}$$

$$+\frac{1}{2}\sup_{0<\tau<t}\int_{B_{2R}}udx. \qquad (26)$$

(2) If $\frac{1}{mp}\geq p-1$, we choose $\alpha=p-1, \beta=\frac{p-1}{2}$ in (23). Then

$$\int_s^t\int_{B_{2R}}|\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$

$$\leq c\{[(1-l)R]^{-p}\int_s^t\int_{B_{2R}}(\tau-s)^{\frac{p}{2}}dxd\tau$$

$$+\int_s^t\int_{B_{2R}}(\tau-s)^{\frac{p-2}{2}}u^{m+1-mp}dxd\tau\}^{\frac{p-1}{p}}$$

$$\times\left(\int_s^t\int_{B_{2R}}u^{mp(p-1)}(\tau-s)^{-\frac{p(p-1)}{2}}dxd\tau\right)^{\frac{1}{p}}.$$

Since

$$\left(\int_s^t\int_{B_{2R}}(\tau-s)^{\frac{p-2}{2}}u^{m+1-mp}dxdt\right)^{\frac{p-1}{p}}$$

$$\leq [\int_s^t \int_{B_{2R}} (u^{m+1-mp}\tau^{-\frac{m+1-mp}{2}})^{\frac{1}{m+1-mp}} dxd\tau]^{\frac{(m+1-mp)(p-1)}{p}}$$

$$\times \left(\int_s^t \int_{B_{2R}} (\tau-s)^{(\frac{p-2}{2}+\frac{m+1-mp}{2})\frac{1}{m(p-1)}} dxd\tau\right)^{\frac{m(p-1)^2}{p}}$$

$$\leq c\left(\int_s^t \int_{B_{2R}} u(\tau-s)^{-\frac{1}{2}} dxd\tau\right)^{\frac{(m+1-mp)(p-1)}{p}} \times t^{\frac{(p-1)^2(m+1)}{2p}} R^{\frac{mN(p-1)^2}{p}},$$

and

$$\left(\int_s^t \int_{B_{2R}} u^{mp(p-1)}(\tau-s)^{-\frac{p(p-1)}{2}} dxd\tau\right)^{\frac{1}{p}}$$

$$\leq c\left(\int_s^t \int_{B_{2R}} u(\tau-s)^{-\frac{1}{2}} dxd\tau\right)^{m(p-1)} \times t^{\frac{mp-mp^2-p^2+p+2}{2p}} R^{\frac{N[1-mp(p-1)]}{p}},$$

we have

$$\int_s^t \int_{B_{2R}} |\nabla u^m|^{p-1} \xi^{p-1} dxd\tau$$

$$\leq c(1-l)^{1-p}\left(\int_s^t \int_{B_{2R}} u(\tau-s)^{-\frac{1}{2}} dxd\tau\right)^{m(p-1)} \times t^{\frac{m+2-mp}{2}} R^{1-k}$$

$$+c\left(\int_s^t \int_{B_{2R}} u(\tau-s)^{-\frac{1}{2}} dxd\tau\right)^{\frac{(p-1)(m+1)}{p}} \times R^{\frac{N[1-m(p-1)]}{p}} t^{\frac{m+3-mp-p}{2p}}, \quad (27)$$

where $k = p + N[m(p-1)-1]$.

By (23), (27),

$$\sup_{s<\tau<t} \int_{B_{l2R}} u(x,\tau)dx \leq \int_{B_{2R}} u(x,s)dx$$

$$+c(1-l)^{-p}R^{-k}\left(\int_s^t \int_{B_{2R}} u(\tau-s)^{-\frac{1}{2}} dxd\tau\right)^{m(p-1)} \times t^{\frac{2-m(p-1)}{2}}$$

$$+c(1-l)^{-1}R^{-\frac{k}{p}}\left(\int_s^t \int_{B_{2R}} u(\tau-s)^{-\frac{1}{2}} dxd\tau\right)^{\frac{(m+1)(p-1)}{p}} \times t^{\frac{m+3-mp-p}{2p}}$$

$$\leq \int_{B_{2R}} u(x,s)dx + c(1-l)^{-p}R^{-k}t\left(\sup_{s<\tau<t}\int_{B_{2R}} udx\right)^{m(p-1)}$$

$$+c(1-l)^{-1}R^{-\frac{k}{p}}t^{\frac{2-p(m+1)}{p}}\left(\sup_{s<\tau<t}\int_{B_{2R}} udx\right)^{\frac{(m+1)(p-1)}{p}}$$

$$\leq \int_{B_{2R}} u(x,s)dx + c(1-l)^{-\frac{p}{1-m(p-1)}} R^{\frac{-k}{1-m(p-1)}} t^{\frac{1}{1-m(p-1)}} + \frac{1}{2}\sup_{s<\tau<t}\int_{B_{2R}} udx. \quad (28)$$

From (26), (28), according to [14, Lemma 3.1],

$$\sup_{s<\tau<t} \int_{B_{l2R}} u(x,\tau)dx \leq \int_{B_{2R}} u(x,s)dx + cR^{-\frac{k}{1-m(p-1)}} t^{\frac{1}{1-mp+m}}.$$

Let $s \to 0$. Then (16) is true. By (16), (26) and (27), we get (17). Substituting (17) into (19), by (16), we get (18). Thus the lemma is proved.

4 The proof of Theorem 4

Lemma 6 *If $m(p-1) + \frac{p}{N} \leq 1$, then the solution of the Cauchy problem (1)-(15) satisfies*

(1) For any given $R > 0$,

$$\int_0^T \int \frac{u^{m(\alpha-1)}}{(1+u^{m\alpha})^2} |\nabla u^m| dxdt \leq c(\alpha), \quad (29)$$

and

(2)

$$\int_0^T \int_{B_R} u^{m(p-1)+\frac{p}{N}-\alpha} dxdt < c(\alpha), \quad (30)$$

(3)

$$\int_0^T \int_{B_R} u^{q_1} dxdt < \infty. \quad (31)$$

where $0 < \alpha < p-1 < 1$ and $q_1 = \max\{1, q\}$.

Proof (1) By Definition 1.1, for any $\psi(x) \in C_0^\infty(R^N)$, $\varepsilon \in (0,T)$, we have

$$\int_{R^N} \int_0^{u(x,T)} \frac{s^{m\alpha}}{1+s^{m\alpha}} ds\psi(x)^p dx + \int_\varepsilon^T \int_{R^N} \frac{\alpha u^{m(\alpha-1)}}{(1+u^{m\alpha})^2} |\nabla u^m|^p \psi^p dxdt$$

$$= -p\int_\varepsilon^T \int_{R^N} \frac{u^{m\alpha}}{1+u^{m\alpha}} |\nabla u^m|^{p-2} \nabla u^m \cdot \nabla\psi\psi^{p-1} dxdt + \int_{R^N} \int_0^{u(x,\varepsilon)} \frac{s^{m\alpha}}{1+s^{m\alpha}} ds\psi(x)^p dx$$

$$-\int_\varepsilon^T\int_{R^N}\frac{u^{m\alpha+q}}{1+u^{m\alpha}}\psi^p dxdt$$
$$\le p\int_\varepsilon^T\int_{R^N}\frac{s^{m\alpha}}{1+s^{m\alpha}}ds\psi^p(x)dxdt$$
$$+\int_{R^N}\int_0^{u(x,\varepsilon)}\frac{s^{m\alpha}}{1+s^{m\alpha}}ds\psi(x)^p dx. \tag{32}$$

Noticing that

$$\int_0^T\int_{R^n}\frac{u^{m\alpha}}{1+u^{m\alpha}}|\nabla u^m|^{p-1}|\nabla\psi|\psi^{p-1}dxdt$$
$$\le\eta\int_0^T\int_{R^n}\frac{u^{m(\alpha-1)}}{(1+u^{m\alpha})^2}|\nabla u^m|^p\psi^p dxdt$$
$$+c(\eta)\int_0^T\int_{R^n}u^{m(1+\alpha)(p-1)}|\nabla\psi|^p dxdt$$

and when $\varepsilon\to 0$,

$$\int_{R^N}\int_0^{u(x,\varepsilon)}\frac{s^{m\alpha}}{1+s^{m\alpha}}ds\psi^p(x)dx$$
$$\le\int_{R^N}u(x,\varepsilon)\psi^p(x)dx\to\int_{R^N}\psi^p(x)d\mu. \tag{33}$$

Then, if we let $\varepsilon\to 0$ in (32), we have

$$\sup_{0<t<T}\int_{R^N}u(x,t)\psi(x)^p dx$$
$$+\int_0^T\int_{R^N}\frac{u^{m(\alpha-1)}}{(1+u^{m\alpha})^2}|\nabla u^m|^p\psi^p dxdt$$
$$=c\left(1+\int_0^T\int_{R^N}u^{m(1+\alpha)(p-1)}|\nabla\psi|^p dxdt\right). \tag{34}$$

This implies that

$$\sup_{0<t<T}\int_{R^N}u(x,t)\psi(x)^p dx$$
$$\le c\left(1+\int_0^T\int_{R^N}u^{m(1+\alpha)(p-1)}|\nabla\psi|^p dxdt\right). \tag{35}$$

We now choose α small enough such that $m(1+\alpha)(p-1)<1$, by Young inequality, we have

$$\int_0^T\int_{R^N}u^{m(1+\alpha)(p-1)}|\nabla\psi|^p dxdt$$
$$\le\varepsilon\int_0^T\int_{R^N}u\psi^p dxdt$$
$$+c(\varepsilon)\int_0^T\int_{R^N}\left(\frac{|\nabla\psi^p|}{\psi^{m(1+\alpha)(p-1)}}\right)^{\frac{1}{1-m(1+\alpha)(p-1)}}dxdt. \tag{36}$$

Let $X\in C_0^\infty(B_{2R})$, $X\mid_{B_R}=1$, and let

$$\psi=X^h, h\ge\frac{1}{1-m(1+\alpha)(p-1)}.$$

By the above inequalities (35)-(36), we have

$$\int_0^T\int_{R^N}u^{m(1+\alpha)(p-1)}|\nabla\psi|^p dxdt\le c, \tag{37}$$

$$\sup_{0<t<T}\int_{R^N}u(x,t)\psi^p(x)dx\le c. \tag{38}$$

Combining (34)-(38), we get (29), i.e.

$$\int_0^T\int_{B_R}\frac{u^{m(\alpha-1)}}{(1+u^{m\alpha})^2}|\nabla u^m|dxdt\le c(\alpha).$$

Now, let

$$w=u^{\frac{m(p-1-\alpha)}{p}}.$$

By Sobolev inequality,

$$\left(\int_{R^N}\psi^p w^r dx\right)^{\frac{1}{\gamma}}\le c\left(\int_{R^N}|\nabla\psi w|^p dx\right)^{\frac{\theta}{p}}$$
$$\times\left(\int_{B_{2R}}w^{\frac{p}{p-1-\alpha}}dx\right)^{\frac{(1-\theta)(p-1-\alpha)}{p}}, \tag{39}$$

where,

$$\theta=\left(\frac{p-1-\alpha}{p}-\frac{1}{\gamma}\right)\left(\frac{1}{N}-\frac{1}{p}+\frac{p-1-\alpha}{p}\right)^{-1}.$$

For $\gamma=\frac{p(p-1+\frac{p}{N}-\alpha)}{p-1-\alpha}$, by (39), we have

$$\int_0^T\int_{R^N}\psi^p w^r dxdt\le\int_0^T\int_{R^N}|\nabla(\psi w)|^p dxdt$$
$$\times\sup_{0<t<T}\left(\int_{B_{2R}}w^{\frac{p}{p-1-\alpha}}dx\right)^{\frac{(\gamma-\theta)(p-1-\alpha)}{p}}.$$

Hence, by (38), (29), we have

$$\int_0^T\int_{R^N}\psi^p u^{m(p-1)+\frac{p}{N}-\alpha}dxdt$$
$$\le c(\alpha)\left(1+\int_0^T\int_{R^N}|\nabla\psi|^p u^{m(p-1-\alpha)}dxdt\right).$$

At the same time, if we rewrite the first equality of the formula (32) as

$$\int_{R^N}\int_0^{u(x,T)}\frac{s^{m\alpha}}{1+s^{m\alpha}}ds\psi(x)^p dx$$
$$+\int_\varepsilon^T\int_{R^N}\frac{\alpha u^{m(\alpha-1)}}{(1+u^{m\alpha})^2}|\nabla u^m|^p\psi^p dxdt$$

$$+\int_{\varepsilon}^{T}\int_{R^N}\frac{u^{m\alpha+q}}{1+u^{m\alpha}}\psi^p dxdt$$
$$=-p\int_{\varepsilon}^{T}\int_{R^N}\frac{u^{m\alpha}}{1+u^{m\alpha}}\left|\nabla u^m\right|^{p-2}\nabla u^m\cdot\nabla\psi\psi^{p-1}dxdt$$
$$+\int_{R^N}\int_0^{u(x,\varepsilon)}\frac{s^{m\alpha}}{1+s^{m\alpha}}ds\psi(x)^p dx,$$

by (33), letting $\varepsilon\to 0$, and setting $u_1=\max\{u,1\}$, we have

$$\sup_{0<t<T}\int_{R^N}u_1\psi(x)^p dx+\int_0^T\int_{R^N}\psi^p u_1^q dxdt$$
$$+\int_0^T\int_{R^N}\frac{u^{m(\alpha-1)}}{(1+u^{m\alpha})^2}\left|\nabla u^m\right|^p\psi^p dxdt$$
$$\le c\left(1+\int_0^T\int_{R^N}u_1^{m(1+\alpha)(p-1)}\left|\nabla\psi\right|^p dxdt\right),$$

where we choose $q_1=\max\{q,1\}$, $\alpha<\frac{q_1-p+1}{p-1}$, the condition $q>p-1$ assures that $\alpha>0$. This also implies

$$\sup_{0<t<T}\int_{R^N}u_1\psi(x)^p dx$$
$$\le c\left(1+\int_0^T\int_{R^N}u_1^{m(1+\alpha)(p-1)}\left|\nabla\psi\right|^p dxdt\right). \tag{40}$$

Using a similar argument as in the proof of (3.1), we obtain

$$\int_0^T\int_{R^N}u^{q_1}dxdt\le c. \tag{41}$$

Hence, (40) and (41) imply the conclusion (30). Thus, the lemma is proved.

Lemma 7 *If $m(p-1)+\frac{p}{N}\le 1$, $q\ge 0$, then the solution of Cauchy problem (1)-(15) satisfies*

$$\iint_{S_T}(u\xi_t-\left|\nabla u^m\right|^{p-2}\nabla u^m\cdot\nabla\xi-u^q\xi)dxdt=0, \tag{42}$$

where $\xi\in C_0^\infty(R^N\times(-T,T))$.

Proof Let

$$\psi_k(x,t)=\eta_k(x,t)=\eta_k(|x|^2)\xi(x,t),$$

where

$$\xi\in C_0^\infty(R^N\times(-T,T)),\ \eta\in c^\infty(-\infty,+\infty)$$

$\eta(s)=1$ when $s\ge 2$; $\eta(s)=0$, when $s\le 1$.

Let $\eta_k(s)=\eta(ks)$. By the definition of weak solution,

$$\int_0^T\int_{R^N}[u(\xi\eta_k)_t-\left|\nabla u^m\right|^{p-2}\nabla(\xi\eta_k)\nabla u^m-u^q\xi]dxdt=0.$$

By Lemma 6, it is enough to prove that

$$\lim_{k\to\infty}\iint_{S_T}\left|\nabla u^m\right|^{p-2}\nabla u^m\cdot\nabla\eta_k\xi dxdt=0. \tag{43}$$

Denoting $D_k=\{x:k^{-1}<|x|^2<2k^{-1}\}$, clearly mes$D_k\le ck^{\frac{-N}{2}}$. Hence, by Hölder inequality and Lemma 6, we have

$$\left|\iint_{S_T}\left|\nabla u^m\right|^{p-2}\nabla u^m\cdot\nabla\eta_k\xi dxdt\right|$$
$$\le k^{\frac{1}{2}}\int_0^T\int_{D_k}\left|\nabla u^m\right|^{p-1}dxdt$$
$$\le k^{\frac{1}{2}}(\int_0^T\int_{D_k}\frac{u^{m\alpha}}{(1+u^{m\alpha})^2}\left|\nabla u^m\right|^p dxdt)^{\frac{p-1}{p}}$$
$$\times(\int_0^T\int_{D_k}(1+u^{m\alpha})^{2(p-1)}u^{m(p-1)(1-\alpha)}dxdt)^{\frac{1}{p}}$$
$$\le ck^{\frac{1}{2}}(\int_0^T\int_{D_k}u_1^{m(p-1)(1+\alpha)}dxdt)^{\frac{1}{p}}$$
$$\le c_1(\int_0^T\int_{D_k}u_1^{m(p-1)+\frac{p}{N}-\alpha}dxdt)^{\frac{m(p-1)(1+\alpha)}{(m(p-1)+\frac{p}{N}-\alpha)p}}$$
$$\times k^{\frac{1}{2}-\frac{p-N\alpha-\alpha Nm(p-1)}{2p(m(p-1)+\frac{p}{N}-\alpha)}},$$

where $u_1=\max\{u,1\}$. Since $m(p-1)+\frac{p}{N}\le 1$ implies $p\le\frac{(m+1)N}{mN+1}$, we can choose

$$\alpha<\frac{p-p(1-m(p-1))-\frac{p^2}{N}}{N+Nm(p-1)-p},$$

to obtain

$$\frac{1}{2}-\frac{p-N\alpha-\alpha Nm(p-1)}{2p(m(p-1)+\frac{p}{N}-\alpha)}<0.$$

If $p=\frac{(m+1)N}{mN+1}$, we choose ξ in (25) and (27) as $\xi\eta_k$, and notice that $R=k^{\frac{1}{2}}$. Let $s\to 0$ in (25) and (27). Then we also obtain (43) in this case, and so Lemma 7 is got.

The proof Theorem 4 Suppose to the contrary that Cauchy (1)-(15) has a solution. Then by Lemma 7, we have

$$\iint_{S_T}(u\xi_t-\left|\nabla u^m\right|^{p-2}\nabla u^m\cdot\nabla\xi-u^q\xi)dxdt=0, \tag{44}$$

where $\xi\in C_0^\infty(R^N\times(-T,T))$.

Let $\eta_k(t)=1-\int_{-\infty}^{t-\tau-2k}j_h(s)ds$, where

$$j_h\in C_0^1\left(-2h,2h\right),j_h\ge 0,$$

$$\int_R j_h\left(s\right)ds=1,\tau\in\left(0,T\right),\ 2h<T-\tau.$$

Clearly,$\eta_k \in C^\infty(R)$. If $t < \tau + h,\ 0 \le \eta_k \le 1$; if $t < T$, $\lim_{k\to 0} \eta_k(t) = 0$.

For any $\forall\chi \in C_0^\infty(R^N)$, we choose $\xi = \chi(x)\eta_k(t)$ in (44), then

$$-\int_0^T \int_{R^N} j_h(t-\tau-2h)u\chi dxdt$$

$$-\int_0^T \int_{R^N} [|\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\chi\eta_k - u^q\chi(x)\eta_k]dxdt = 0.$$

Let $k \to 0^+$. We have

$$\int_{R^N} u(x,\tau)\chi(x)dx$$

$$= -\int_0^\tau \int_{R^N} |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\chi dxds$$

$$-\int_0^\tau \int_{R^N} u^q\chi(x)dxdt.$$

This implies that for $\forall\chi \in C_0^\infty(R^N)$,

$$\lim_{\tau\to 0} \int_{R^N} u(x,\tau)\chi(x)dx = 0.$$

This contradicts (15).

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