Introduction to the Elliptical Trigonometry in Euclidian 2D-space with simulation of four elliptical trigonometric functions *Jes, Jes-x, Mar* and *Rit*

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Abstract: - Trigonometry is a branch of mathematics that deals with relations between sides and angles of triangles. The trigonometric functions are very important in technical subjects like Astronomy, Relativity, science, engineering, architecture, and even medicine. In this paper, the elliptical trigonometry is introduced in order to be in the future a part of the General Trigonometry topic.
The concept of the Elliptical Trigonometry is completely different from the traditional trigonometry in which the study of angles is not the relation between sides of a right triangle that describes a circle as the previous one, but the idea here is to use the relation between angles and sides of an ellipse form with the internal and external circles formed by the intersection of the ellipse form and the positive parts of x'ox and y'oy axis in the Euclidian 2D space and their projections. This new concept of relations will open a huge gate in the mathematical domain and it can resolve many complicated problems that are difficult or almost impossible to solve with the traditional trigonometry, and it can describe a huge number of multi form periodic signals.

Key-words: - Mathematics, geometry, trigonometry, angular function, multi form signal, power electronics.

1 Introduction

The traditional trigonometry is a branch of the mathematics that deals with relationships between sides and angles of a right triangle [3-5]. Six principal functions (e.g. cosine, sine, tangent …) are used to produce signals that have an enormous variety of applications in all scientific domains [6-9]. It can be considered as the basis and foundation of many domains as electronics, signal theory, astronomy, navigation, propagation of signals and many others… [10-13]. Now it is time to introduce a new concept of trigonometry in the Euclidean 2D-space to replace the previous one, and open new gates and new challenges by the reconstruction of the science [14-16].

In this paper, the new concept of the elliptical trigonometry is introduced and few examples are shown and discussed briefly. Figures are drawn and simulated using AutoCAD and Matlab.

In the second section, the angular functions are defined, these functions have enormous applications in all domains, and it can be considered as the basis of this trigonometry [1-2]. The definition of the Elliptical trigonometry is presented and discussed briefly in the third section. In the fourth section, a survey on the Elliptical Trigonometric functions is discussed and four different functions are simulated with brief examples. A survey on the application of the elliptical trigonometry in engineering domain is presented in the section 5. Finally, a conclusion about the elliptical trigonometry is presented in the section 6.

2 The angular functions

Angular functions are new mathematical functions that produce a rectangular signal, in which period is function of angles, it is also introduced by the author [20]. Similar to trigonometric functions, the angular functions have the same properties as the precedent, but the difference is that a rectangular signal is obtained instead of a sinusoidal signal [14-16] and moreover, one can change the width of each positive and negative alternate in the same period. This is not the case of any other trigonometric function. In other hand, one can change the frequency, the amplitude and the width of any period of the signal by using the general form of the angular function.

In this section three types of angular functions are presented, they are used in this trigonometry; of course there are more than three types, but in this paper the study is limited to three functions.

2.1 Angular function $ang_x(x)$

The expression of the angular function related to the (ox) axis is defined, for $K \in \mathbb{Z}$, as:

$$ang_x(\beta(x+\gamma)) = \begin{cases} +1 \ \ for \ \ (4K-1)\frac{\pi}{2\beta} - \gamma \le x \le (4K+1)\frac{\pi}{2\beta} - \gamma \\ -1 \ \ for \ (4K+1)\frac{\pi}{2\beta} - \gamma < x < (4K+3)\frac{\pi}{2\beta} - \gamma \end{cases} \quad (1)$$

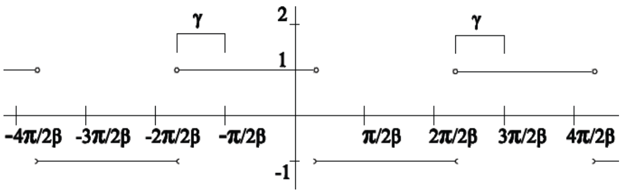


Fig. 1: The $ang_x(\beta(x+\gamma))$ waveform.

For $\beta = 1$ and $\gamma = 0$, the expression of the angular function becomes:

$$ang_x(x) = \begin{cases} +1 \ \ for \ \ \cos(x) \ge 0 \\ -1 \ \ for \ \ \cos(x) < 0 \end{cases}$$

2.2 Angular function $ang_y(x)$

The expression of the angular function related to the (oy) axis is defined, for $K \in \mathbb{Z}$, as:

$$ang_y(\beta(x+\gamma)) = \begin{cases} +1 \ \ for \ \ 2K\pi/\beta - \gamma \le x \le (2K+1)\pi/\beta - \gamma \\ -1 \ \ for \ \ (2K+1)\pi/\beta - \gamma < x < (2K+2)\pi/\beta - \gamma \end{cases} \quad (2)$$

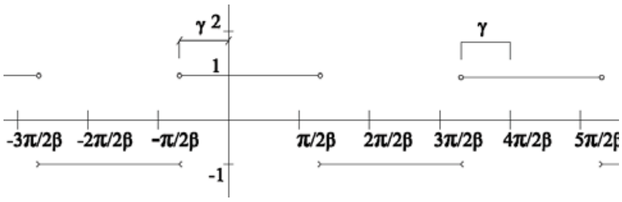


Fig. 2: The $ang_y(\beta(x+\gamma))$ waveform.

For $\beta = 1$ and $\gamma = 0$, the expression of the angular function becomes:

$$ang_y(x) = \begin{cases} +1 \ \ for \ \ \sin(x) \ge 0 \\ -1 \ \ for \ \ \sin(x) < 0 \end{cases}$$

2.3 Angular function $ang_\alpha(x)$

α (called firing angle) represents the angle width of the positive part of the function in a period. In this case, we can vary the width of the positive and the negative part by varying only α. The firing angle must be positive.

$$ang_\alpha(\beta(x+\gamma)) = \begin{cases} +1 \ \ for \\ (2K\pi - \alpha)/\beta - \gamma \le x \le (2K\pi + \alpha)/\beta - \gamma \\ \\ -1 \ \ for \\ (2K\pi + \alpha)/\beta - \gamma < x < (2(K+1)\pi - \alpha)/\beta - \gamma \end{cases} \quad (3)$$

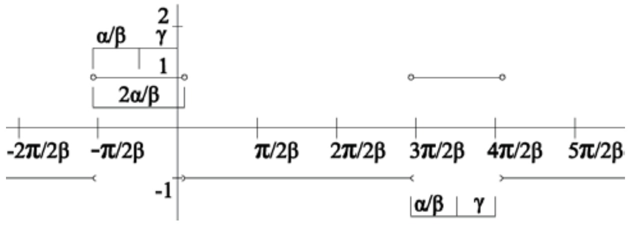


Fig. 3: The $ang_\alpha(\beta(x+\gamma))$ waveform.

3 Definition of the Elliptical Trigonometry

3.1 The Elliptical Trigonometry unit

The Elliptical Trigonometry unit is an ellipse with a center O (x = 0, y = 0) and has the equation form:

$$(x/a)^2 + (y/b)^2 = 1 \quad (4)$$

With:

'a' is the radius of the ellipse on the (x'ox) axis,

'b' is the radius of the ellipse on the (y'oy) axis.

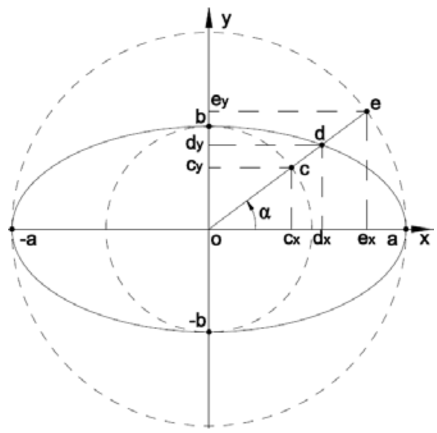


Fig. 4: The elliptical trigonometry unit.

It is essential to note that 'a' and 'b' must be positive. In this paper, 'a' is fixed to 1. One is interested to vary only a single parameter which is 'b'.

3.2 Intersections and projections of different elements of the Elliptical Trigonometry on the relative axes

From the intersections of the ellipse with the positive parts of the axes (ox) and (oy), define respectively two circles of radii $[oa]$ and $[ob]$. These radii can be variable or constant according to the form of the ellipse.

The points of the intersection of the half-line $[od)$ (figure 4) with the internal and external circles and with the rectangle and their projections on the axes (ox) and (oy) can be described by many functions that have an extremely importance in creating plenty of signals and forms that are very difficult to be created in the traditional trigonometry.

Definition of the letters in the Figure 4:

a: Is the intersection of the ellipse with the positive part of the axe (ox) that gives the relative circle of radius "a". It can be variable.

b: Is the intersection of the ellipse with the positive part of the axe (oy) that gives the relative circle of radius "b". It can be variable.

c: Is the intersection of the half-line $[od)$ with the circle of radius b.

d: Is the intersection of the half-line $[od)$ with the ellipse.

e: Is the intersection of the half-line $[od)$ with the circle of radius a.

c_x: Is the projection of the point c on the ox axis.

d_x: Is the projection of the point d on the ox axis.

e_x: Is the projection of the point e on the ox axis.

c_y: Is the projection of the point c on the oy axis.

d_y: Is the projection of the point d on the oy axis.

e_y: Is the projection of the point e on the oy axis.

α: Is the angle between the (ox) axis and the half-line $[od)$.

o: Is the center (0, 0).

3.3 Definition of the Elliptical Trigonometric functions $\boldsymbol{Efun(\alpha)}$

The traditional trigonometry contains only 6 principal functions: Cosine, Sine, Tangent, Cosec, Sec, Cotan [6], [16], [17]. But in the Elliptical Trigonometry, there are 32 principal functions and each function has its own characteristics. These functions give a new vision of the world and will be used in all scientific domains and make a new challenge in the reconstruction of the science especially when working on the economical side of the power of electrical circuits, the electrical transmission, the signal theory and many other domains [15],[18].

The functions $Cjes(\alpha), Cmar(\alpha), Cter(\alpha)$ and $Cjes_y(\alpha)$, which are respectively equivalent to cosine, sine, tangent and cotangent. These functions are particular cases of the "*Circular Trigonometry*". The names of the cosine, sine, tangent and cotangent are replaced respectively by *Circular Jes*, *Circular Mar*, *Circular Ter* and *Circular Jes-y*.

$Cjes(\alpha) \Leftrightarrow cos(\alpha)$; $Cmar(\alpha) \Leftrightarrow sin(\alpha)$

$Cter(\alpha) \Leftrightarrow tan(\alpha)$; $Cjes_y(\alpha) \Leftrightarrow cotan(\alpha)$.

The Elliptical Trigonometric functions are denoted using the following abbreviation "$Efun(\alpha)$":

-the first letter "E" is related to the Elliptical trigonometry.

-the word "$fun(\alpha)$" represents the specific function name that is defined hereafter: (refer to Figure 4).

• **Elliptical *Jes* functions:**

El. Jes: $Ejes(\alpha) = \frac{od_x}{oa} = \frac{od_x}{oe}$ (5)

El. Jes-x: $Ejes_x(\alpha) = \frac{od_x}{oe_x} = \frac{Ejes(\alpha)}{Cjes(\alpha)}$ (6)

El. Jes-y: $Ejes_y(\alpha) = \frac{od_x}{oe_y} = \frac{Ejes(\alpha)}{Cmar(\alpha)}$ (7)

• **Elliptical *Mar* functions:**

El. Mar: $Emar(\alpha) = \frac{od_y}{ob} = \frac{od_y}{oc}$ (8)

El. Mar-x: $Emar_x(\alpha) = \frac{od_y}{oc_x} = \frac{Emar(\alpha)}{Cjes(\alpha)}$ (9)

El. Mar-y: $Emar_y(\alpha) = \frac{od_y}{oc_y} = \frac{Emar(\alpha)}{Cmar(\alpha)}$ (10)

• **Elliptical *Ter* functions:**

El. Ter: $Eter(\alpha) = \frac{Emar(\alpha)}{Ejes(\alpha)}$ (11)

El. Ter-x:

$Eter_x(\alpha) = \frac{Emar_x(\alpha)}{Ejes_y(\alpha)} = Eter(\alpha) \cdot Cter(\alpha)$ (12)

El. Ter-y: $Eter_y(\alpha) = \frac{Emar_y(\alpha)}{Ejes_x(\alpha)} = \frac{Eter(\alpha)}{Cter(\alpha)}$ (13)

• **Elliptical *Rit* functions:**

El. Rit: $Erit(\alpha) = \frac{od_x}{ob} = \frac{od_x}{oc} = \frac{Emar(\alpha)}{Cter(\alpha)}$ (14)

El. Rit-y: $Erit_y(\alpha) = \frac{od_x}{oc_y} = \frac{Erit(\alpha)}{Cmar(\alpha)}$ (15)

• **Elliptical *Raf* functions:**

El. Raf: $Eraf(\alpha) = \frac{od_y}{oa} = Cter(\alpha).Ejes(\alpha)$ (16)

El. Raf-x: $Eraf_x(\alpha) = \frac{od_y}{oe_x} = \frac{Eraf(\alpha)}{Cjes(\alpha)}$ (17)

• **Elliptical *Ber* functions:**

El. Ber: $Eber(\alpha) = \frac{Eraf(\alpha)}{Erit(\alpha)}$ (18)

El. Ber-x:

$Eber_x(\alpha) = \frac{Eraf_x(\alpha)}{Erit_y(\alpha)} = Eber(\alpha) \cdot Cter(\alpha)$ (19)

El. Ber-y: $Eber_y(\alpha) = \frac{Eraf_y(\alpha)}{Erit_x(\alpha)} = \frac{Eber(\alpha)}{Cter(\alpha)}$ (20)

3.4 The reciprocal of the Elliptical Trigonometric function

$Efun^{-1}(\alpha)$ is defined as the inverse function of $Efun(\alpha)$. $(Efun^{-1}(\alpha) = 1/Efun(\alpha))$. In this way the reduced number of functions is equal to 32 principal functions.

E.g.: $Ejes^{-1}(\alpha) = \frac{1}{Ejes(\alpha)}$

3.5 Definition of the Absolute Elliptical Trigonometric functions $\bar{E}fun(\alpha)$

The *Absolute Elliptical Trigonometry* is introduced to create the absolute value of a function by varying only one parameter without using the absolute value "| |". The advantage is that we can change and control the sign of an Elliptical Trigonometric function without using the absolute value in an expression. Some functions are treated to get an idea about the importance of this new definition. To obtain the Absolute Elliptical Trigonometry for a specified function (e.g.: $jes(\alpha)$) we must multiply it by the corresponding Angular Function (e.g.: $\left(ang_x(\alpha)\right)^i$) in a way to obtain the original function if i is even, and to obtain the absolute value of the function if i is odd (e.g.: $|Ejes(\alpha)|$).

If the function doesn't have a negative part (not alternative), we multiply it by $\left(ang_x(\beta(\alpha-\gamma)\right)^i$ to obtain an alternating signal which form depends on the value of the frequency "β" and the translation value "γ". By varying the last parameters, one can get a multi form signals.

• $\bar{E}jes_i(\alpha) = \left(ang_x(\alpha)\right)^i \cdot Ejes(\alpha)$ (21)

$$= \begin{cases} \left(ang_x(\alpha)\right)^1 \cdot Ejes(\alpha) = |Ejes(\alpha)| & if\ i=1 \\ \left(ang_x(\alpha)\right)^2 \cdot Ejes(\alpha) = Ejes(\alpha) & if\ i=2 \end{cases}$$

• $\bar{E}jes_{i,x}(\alpha) = \left(ang_x(\alpha-\gamma)\right)^i \cdot Ejes_x(\alpha)$ (22)

$$= \begin{cases} ang_x(\alpha-\gamma) \cdot Ejes_x(\alpha) & if\ i=1 \\ Ejes_x(\alpha) & if\ i=2 \end{cases}$$

• $\bar{E}jes_{i,y}(\alpha) = \left(ang_y(2\alpha)\right)^i \cdot Ejes_y(\alpha)$ (23)

$$= \begin{cases} ang_y(2\alpha) \cdot Ejes_y(\alpha) = |Ejes_y(\alpha)| & if\ i=1 \\ Ejes_y(\alpha) & if\ i=2 \end{cases}$$

• $\bar{E}mar_i(\alpha) = \left(ang_y(\alpha)\right)^i \cdot Emar(\alpha)$ (24)

• $\bar{E}mar_{i,x}(\alpha) = \left(ang_y(2\alpha)\right)^i \cdot Emar_x(\alpha)$ (25)

• $\bar{E}mar_{i,y}(\alpha) = \left(ang_x(\alpha-\gamma)\right)^i \cdot Emar_y(\alpha)$ (26)

• $\bar{E}rit_i(\alpha) = \left(ang_x(\alpha)\right)^i \cdot Erit(\alpha)$ (27)

And so on…

4 A survey on the Elliptical Trigonometric functions

As previous sections, a brief study on the Elliptical Trigonometry is given. Four functions of 32 are treated with examples to show multi form signals made using the characteristic of this trigonometry.

Elliptic cosine and Elliptic sine that appear in the previous articles [1] and [2], are particular cases of the Elliptic *Jes* and Elliptic *Mar* respectively.

For this study the following conditions are taken:

- $a = 1$
- $b > 0$ the height of the rectangle from the center.

4.1 Determination of the Elliptic *Jes* function

The Elliptical form in the figure 4 is written as the equation (4). Thus, given (5), the Elliptical *Jes* function can be determined using following method. In fact:

$Cter(\alpha) = \frac{y}{x} = \frac{oc_y}{oc_x}$, it is significant to replace the equation $y = Cter(\alpha).x$ in that defined in (4).

$$\left(\frac{x}{a}\right)^2 + \left(Cter(\alpha)\frac{x}{b}\right)^2 = \left(\frac{x}{a}\right)^2\left(1 + \left(Cter(\alpha)\frac{x}{b}\right)^2\right) = 1 \Rightarrow$$

$$Ejes_b(\alpha) = \frac{\pm 1}{\sqrt{1+\left(\frac{a}{b}\text{Cter}(\alpha)\right)^2}}$$

Therefore:

•$Ejes_b(\alpha) = \frac{+1}{\sqrt{1+\left(\frac{a}{b}\text{Cter}(\alpha)\right)^2}}$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$; $\frac{x}{a} \ge 0$

•$Ejes_b(\alpha) = \frac{-1}{\sqrt{1+\left(\frac{a}{b}\text{Cter}(\alpha)\right)^2}}$ for $\frac{\pi}{2} < x < \frac{3\pi}{2}$; $\frac{x}{a} < 0$

Thus the expression of the Elliptic *Jes* can be unified by using the angular function expression (1), therefore the expression becomes:

$$Ejes_b(\alpha) = \frac{ang_x(\alpha)}{\sqrt{1+\left(\frac{a}{b}\text{Cter }(\alpha)\right)^2}} \Rightarrow$$

$$Ejes_b(x) = \frac{ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter }(x)\right)^2}} \tag{28}$$

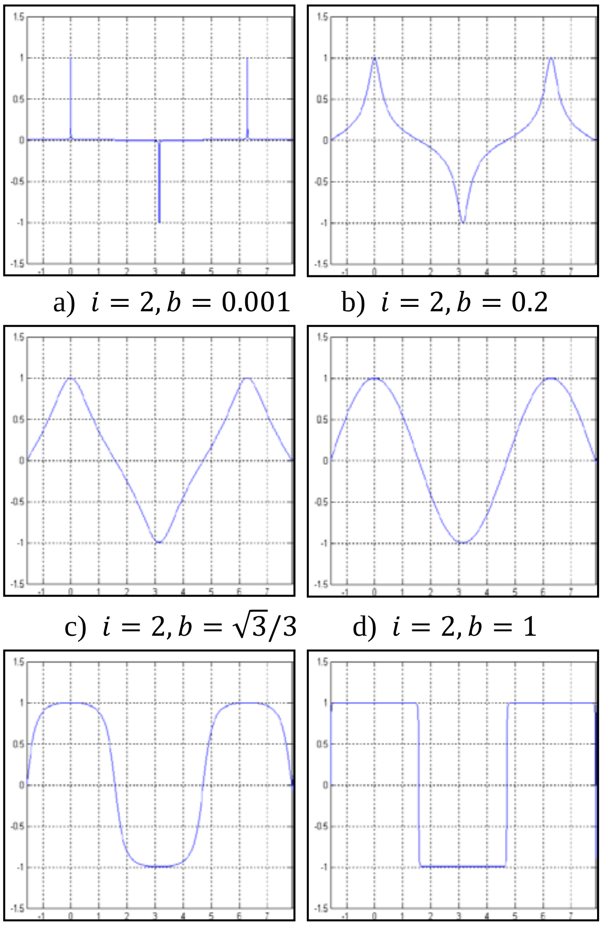
• Expression of the Absolute Elliptic *Jes*:

$$\bar{E}jes_{i,b}(x) = \frac{ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter }(x)\right)^2}} \cdot \left(ang_x(x)\right)^i \tag{29}$$

The Absolute Elliptic *Jes* is a powerful function that can produce more than 14 different signals by varying only two parameters i and b. Similar to the cosine function in the traditional trigonometry, the Absolute Elliptical *Jes* is more general than the precedent.

• Multi form signals made by $\bar{E}jes_{i,b}(x)$:

Figures 5 and 6 represent multi form signals obtained by varying two parameters (i and b). For the figures 5.a to 5.f the value of $i = 2$, for the figures 6.a to 6.f the value of $i = 1$.

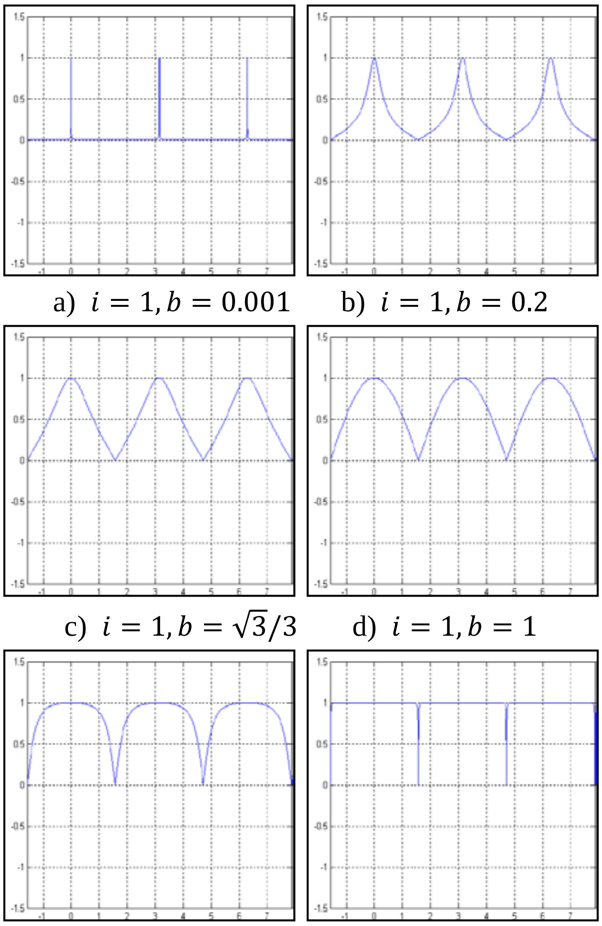


a) $i = 2, b = 0.001$ b) $i = 2, b = 0.2$

c) $i = 2, b = \sqrt{3}/3$ d) $i = 2, b = 1$

e) $i = 2, b = 3$ f) $i = 2, b = 90$

Fig. 5: multi form signals of the function $\bar{E}jes_{i,b}(x)$ for $i = 2$ and for different values of $b > 0$.



a) $i = 1, b = 0.001$ b) $i = 1, b = 0.2$

c) $i = 1, b = \sqrt{3}/3$ d) $i = 1, b = 1$

e) $i = 1, b = 3$ f) $i = 1, b = 90$

Fig. 6: multi form signals of the function $\bar{E}jes_{i,b}(x)$ for $i = 1$ and for different values of $b > 0$.

Important signals obtained using this function: Impulse train with positive and negative part, elliptic deflated, quasi-triangular, sinusoidal, elliptical swollen, square signal, rectangular signal, impulse train (positive part only), rectified elliptic deflated, saw signal, rectified elliptical swollen, continuous signal…

These types of signals are widely used in power electronics, electrical generator and in transmission of analog signals [18].

• Example using the $Ejes_b(x)$:

In this part, an example is treated in a goal to form many shapes obtained by varying only one parameter. For this, consider the following equation:

$$y^2 = \left(k.rect_T(x - x_0).Ejes_b(\alpha x)\right)^2 \tag{30}$$

With k is the amplitude of the signal,
'x' is a variable parameter,
α is the frequency of the $Ejes_b(\alpha x)$,
k, α and T are positive values(> 0).

In fact, the rectangular function $rect_T(x - x_0)$ which is illustrated in figure 7 is defined as

$$rect_T(x - x_0) = \begin{cases} 1 \;\; for \;\; x_0 - \frac{T}{2} \leq x \leq x_0 + \frac{T}{2} \\ 0 \qquad otherwise \end{cases} \quad (31)$$

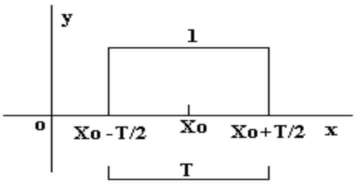


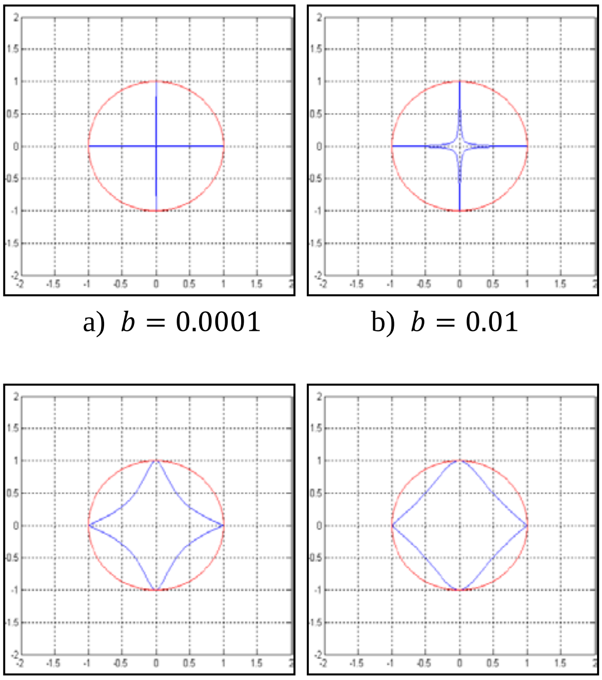
Fig. 7: $rect_T(x - x_0)$ wave form

Then $y = \pm\left(k.rect_T(x - x_0).Ejes_b(\alpha x)\right)$ (32)

The period of $Ejes_b(\alpha x)$ is equal to $T_1 = 2\pi/\alpha$

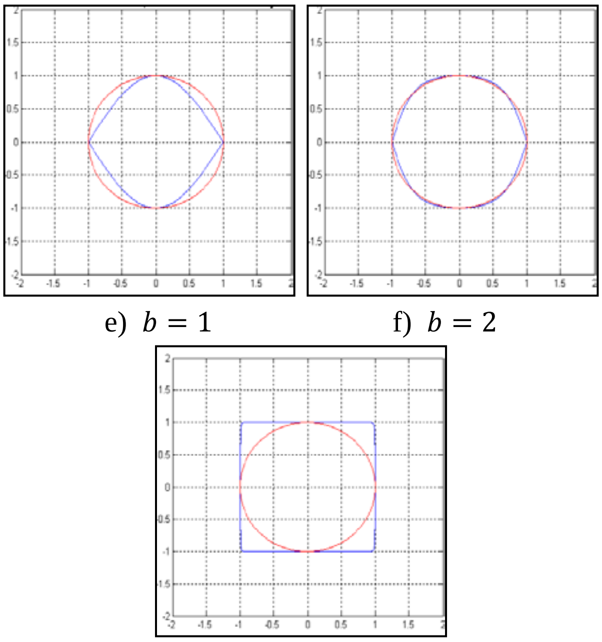
For a particular case, let's take $T = 2; \alpha = \pi/2$; $x_0 = 0$ and $k = 1$.

By varying the parameter b in (32), the following remarkable shapes are illustrated in figures 8.a to 8.g.



a) $b = 0.0001$ b) $b = 0.01$

c) $b = 0.3$ d) $b = \sqrt{3}/3$



e) $b = 1$ f) $b = 2$

g) $b = 120$

Fig. 8: different shapes of the function (32) for different values of b (presented in blue color).

Important shapes obtained respectively using this expression:

Plus form, cross form, star form, Quasi-rhombus, eye form, quasi-circular, square. More signals can be obtained by varying more than one parameter.

4.2 The Elliptic *Jes-x* function

The elliptical form in the figure 4 is written as the equation (4). Thus, given (6), the Elliptical *Jes-x* function can be determined. In fact,

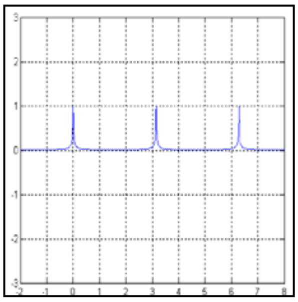
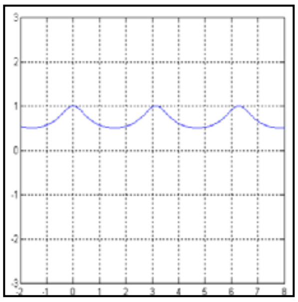
$$Ejes_{x_b}(x) = \frac{Ejes_b(x)}{Cjes(x)} = \frac{ang_x(x)}{Cjes(x) \cdot \sqrt{1 + \left(\frac{a}{b} \text{Cter}\,(x)\right)^2}} \quad (33)$$

• Expression of the Absolute Elliptic *Jes-x*

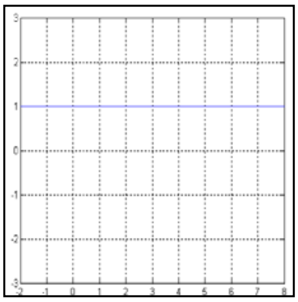
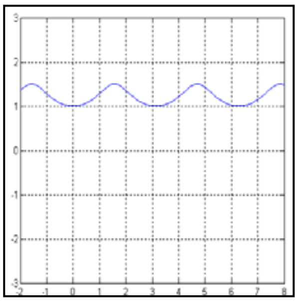
$$\bar{E}jes_{x_b^i}(x) = Ejes_{x_b}(x) \cdot \left(ang_x(x - \gamma)\right)^i \quad (34)$$

• Multi form signals made by $\bar{E}jes_{x_b^i}(x)$:

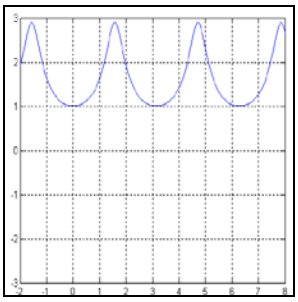
Taking $\gamma = 0$ for this example, figures 9 and 10 represent multi form signals obtained by varying two parameters (i and b). For the figures 9.a to 9.e the value of $i = 2$, for the figures 10.a to 10.f the value of $i = 1$.

a) $i = 2; b = 0.01$ b) $i = 2; b = 0.5$

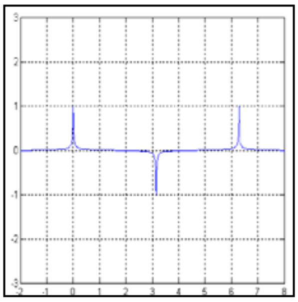
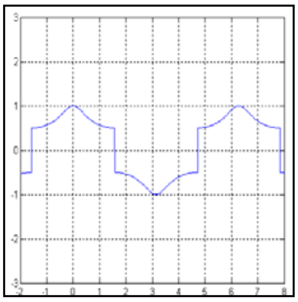



c) $i = 2; b = 1$ d) $i = 2; b = 1.5$

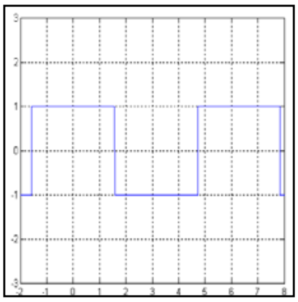
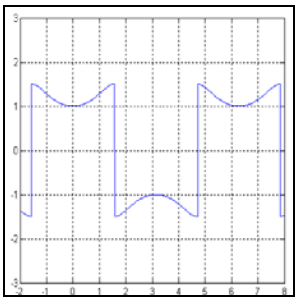


e) $i = 2; b = 2.9$

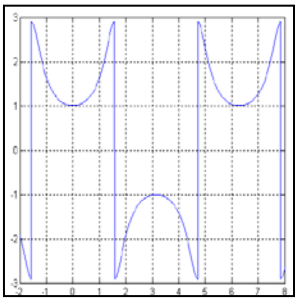
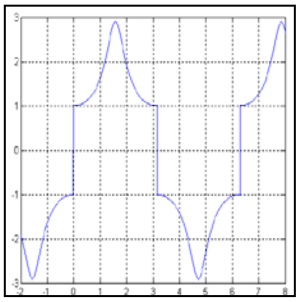
Fig. 9: multi form signals of the function $\bar{E}jes_{x_b^i}(x)$ for $i = 2$ and for different values of $b > 0$.

a) $i = 1; b = 0.01$ b) $i = 1; b = 0.5$

c) $i = 1; b = 1$ d) $i = 1; b = 1.5$

e) $i = 2; b = 2.9$ f) $i = 2; b = 2.9;\ \gamma = \pi/2$

Fig.10: multi form signals of the function $\bar{E}jes_{x_b^i}(x)$ for $i = 1$ and for different values of $b > 0$.

Important signals obtained using this function:
Impulse train with positive part only, sea waves, continuous signal, amplified sea waves, impulse train with positive and negative part, square, saw signal …

• Example using the Elliptic *Jes-x*:

In this part, an example is treated in a goal to form many shapes obtained by varying only one parameter. For this, consider the following equation:

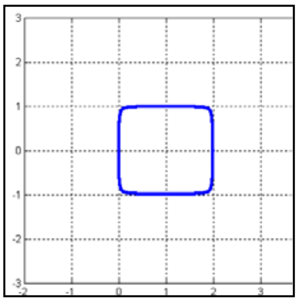
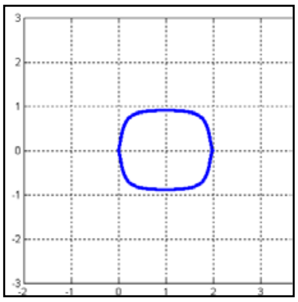
$$y^2 = \left(k.\left(rect_T(x - x_0).Ejes_{x_b}(\alpha x) - 1\right)\right)^2 \quad (35)$$

With k is the amplitude of the signal,
'x' is a variable parameter,
α is the frequency of the $Ejes_{x_b}(\alpha x)$,
k, α and T are positive values(> 0).

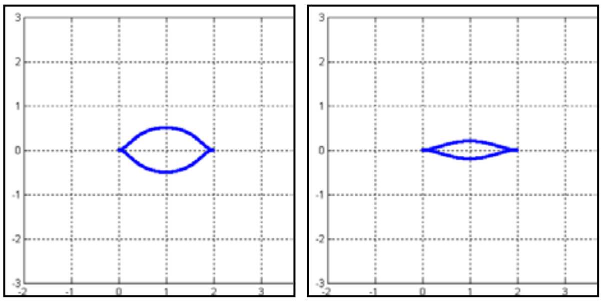
Then,

$$y = \pm\left(k.\left(rect_T(x - x_0).Ejes_{x_b}(\alpha x) - 1\right)\right) \quad (36)$$

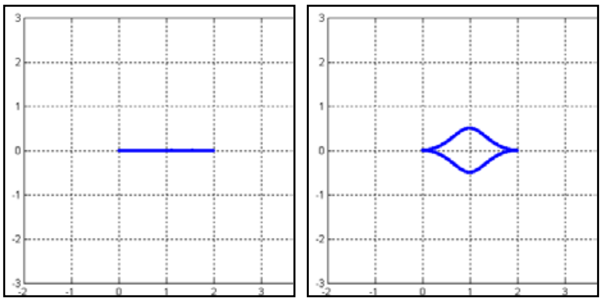
The period of $Ejes_{x_b}(\alpha x)$ is equal to $T_1 = \pi/\alpha$.
For a particular case, let's take $T = 2$; $T_1 = 2$; $\alpha = \pi/2$; $x_0 = 1$ and $k = 1$.
By varying the parameter b in (36), the following remarkable shapes are illustrated in figures 11.a to 11.h.

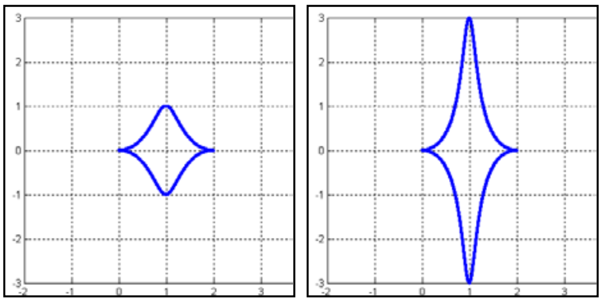
a) $b = 0.01$ b) $b = 0.1$



c) $b = 0.5$ d) $b = 0.8$



e) $b = 1$ f) $b = 1.5$



g) $b = 2$ h) $b = 4$

Fig. 11: different shapes of the function (36) for different values of $b > 0$.

4.3 The Elliptic *Mar* function

The elliptical form in the figure 4 is written as the equation (4). Thus, given (8), the Elliptical *Mar* function can be determined using following method. In fact:

$Cter(\alpha) = \frac{y}{x} = \frac{oc_y}{oc_x} \Rightarrow x = \frac{y}{Cter(\alpha)}$, it is significant to replace the equation $x = \frac{y}{Cter(\alpha)}$ in that defined in (4). Thus, $(\frac{y}{a \cdot Cter(\alpha)})^2 + (y/b)^2 = 1 \Rightarrow$

$$Emar_b(\alpha) = \frac{y}{b} = \frac{\pm\frac{a}{b}\text{Cter}(\alpha)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}(\alpha)\right)^2}}$$

$$\Rightarrow Emar_b(x) = \frac{y}{b} = \frac{\pm\frac{a}{b}\text{Cter}(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}(x)\right)^2}}$$

Therefore:

- $Emar_b(x) = \frac{+\frac{a}{b}\text{Cter}(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}(x)\right)^2}}$ for $0 \le x < \frac{\pi}{2}$
- $Emar_b(x) = \frac{-\frac{a}{b}\text{Cter}(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}(x)\right)^2}}$ for $\frac{\pi}{2} < x \le \pi$
- $Emar_b(x) = \frac{-\frac{a}{b}\text{Cter}(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}(x)\right)^2}}$ for $\pi \le x < 3\frac{\pi}{2}$
- $Emar_b(x) = \frac{+\frac{a}{b}\text{Cter}(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}(x)\right)^2}}$ for $3\frac{\pi}{2} \le x \le 2\pi$

Thus the expression of the elliptic *Mar* can be unified by using the angular function expression (1), therefore the expression becomes:

$$Emar_b(x) = \frac{a}{b}\frac{\text{Cter}(x) ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}(x)\right)^2}} \qquad (37)$$

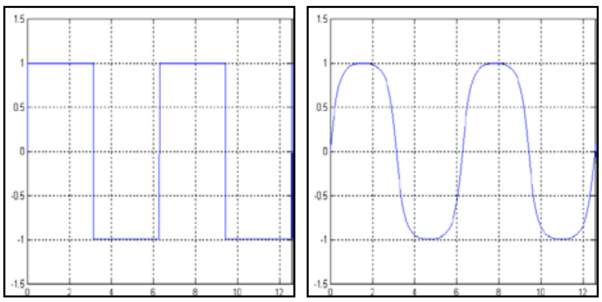
- Expression of the Absolute Elliptic *Mar*:

$$\bar{E}mar_{i,b}(x) = Emar_b(x) \cdot \left(ang_y(x)\right)^i \qquad (38)$$

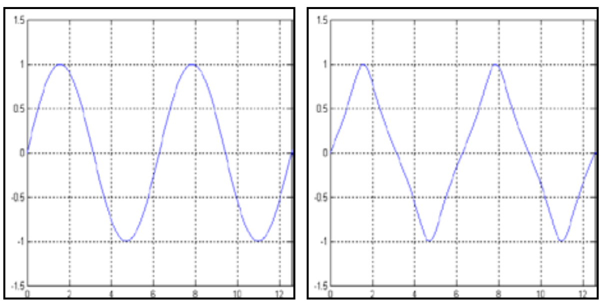
Similar to the Absolute Elliptic *Jes*, the Absolute Elliptic *Mar* is a powerful function that can produce more than 14 different signals by varying only two parameters i and b. Similar to the sine function in the traditional trigonometry, the Absolute Elliptical *Mar* is more general than the precedent.

- Multi form signals made by $\bar{E}mar_{i,b}(x)$:

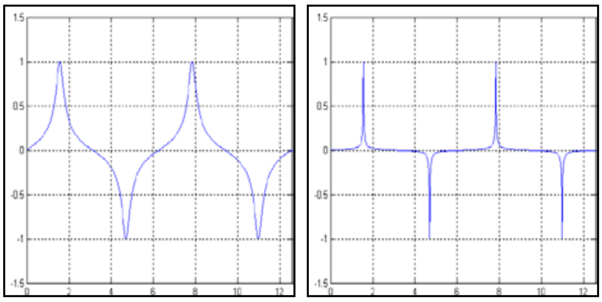
Figures 12 and 13 represent multi form signals obtained by varying two parameters (i and b). For the figures 12.a to 12.f the value of $i = 2$, for the figures 13.a to 13.f the value of $i = 1$.



a) $i = 2; b = 0.001$ b) $i = 2; b = 0.4$

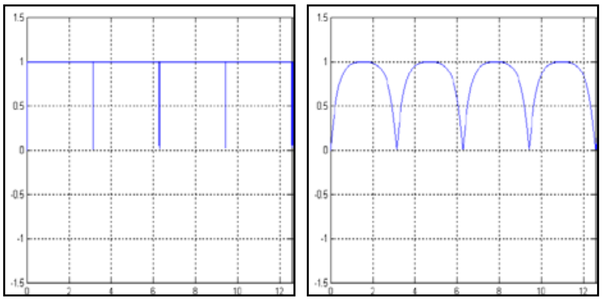


c) $i = 2; b = 1$ d) $i = 2; b = \sqrt{3}$

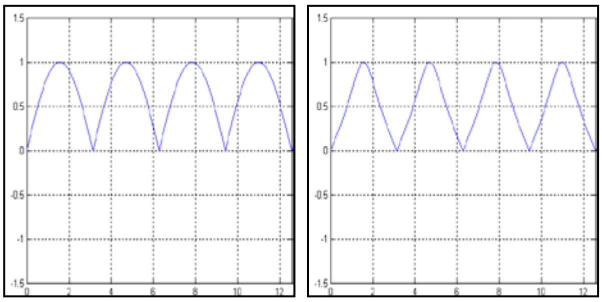


e) $i = 2; b = 6$ f) $i = 2; b = 100$

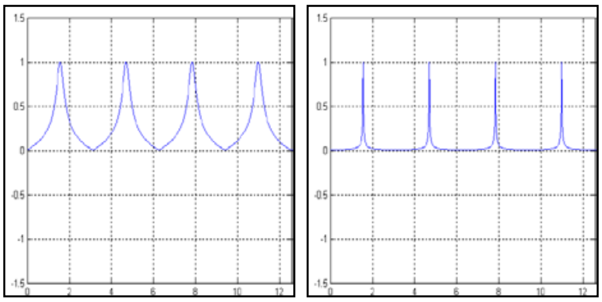
Fig. 12: multi form signals of the function $\bar{E}mar_{i,b}(x)$ for $i = 2$ and for different values of $b > 0$.



a) $i = 1; b = 0.001$ b) $i = 1; b = 0.4$



c) $i = 1; b = 1$ d) $i = 1; b = \sqrt{3}$



e) $i = 1; b = 6$ f) $i = 1; b = 100$

Fig. 13: multi form signals of the function $\bar{E}mar_{i,b}(x)$ for $i = 1$ and for different values of $b > 0$.

4.4 The Elliptic *Rit* function

The elliptical form in the figure 4 is written as the equation (4). Thus, given (14), the Elliptic *Rit* function can be determined. In fact:

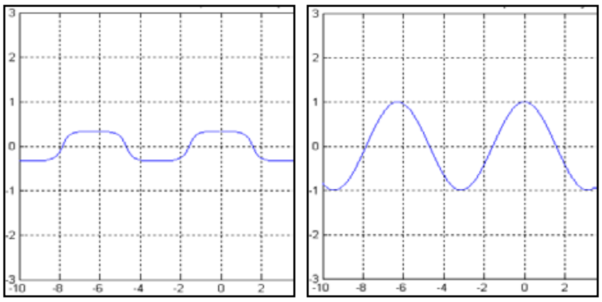
$$Erit_b(x) = \frac{Emar(x)}{\mathrm{Cter}\ (x)} = \frac{a}{b} \cdot \frac{ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\mathrm{Cter}\ (x)\right)^2}} \qquad (39)$$

• Expression of the Absolute Elliptic *Rit*:

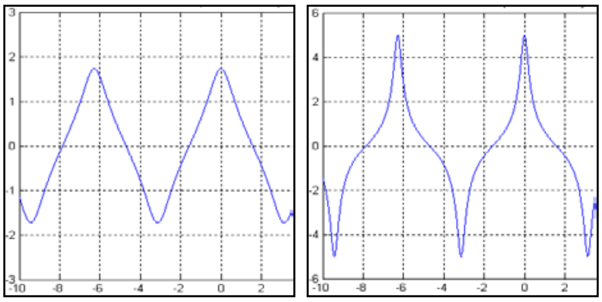
$$\bar{E}rit_{i,b}(x) = Erit_b(x) \cdot \left(ang_x(x)\right)^i \qquad (40)$$

• Multi form signals made by $\bar{E}rit_{i,b}(x)$:

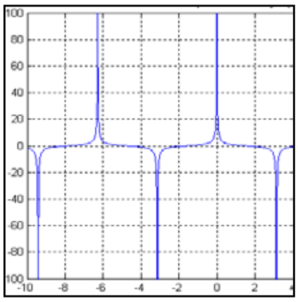
Figures 14 and 15 represent multi form signals obtained by varying two parameters (i and b). For the figures 14.a to 14.e the value of $i = 2$, for the figures 15.a to 15.e the value of $i = 1$.



a) $i = 2; b = 3$ b) $i = 2; b = 1$

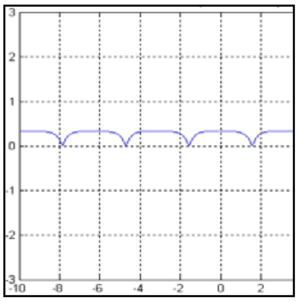


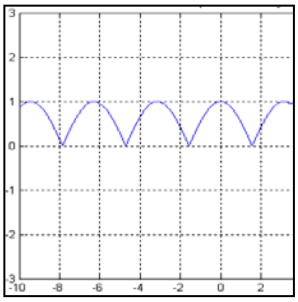
c) $i = 2; b = \sqrt{3}/3$ d) $i = 2; b = 0.2$

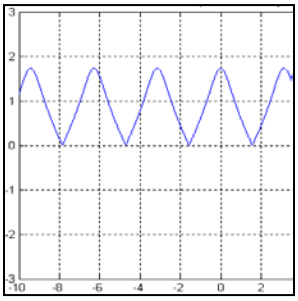


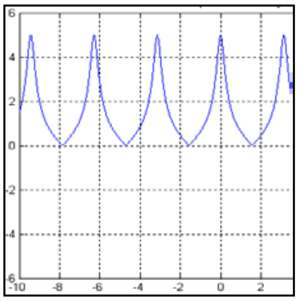
e) $i = 2; b = 0.001$

Fig. 14: multi form signals of the function $\bar{E}rit_{i,b}(x)$ for $i = 2$ and for different values of $b > 0$.


a) $i = 2; b = 3$


b) $i = 2; b = 1$


c) $i = 2; b = \sqrt{3}/3$


d) $i = 2; b = 0.2$

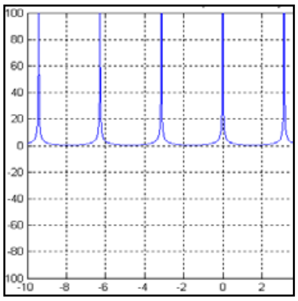

e) $i = 2; b = 0.001$

Fig. 15: multi form signals of the function $\bar{E}rit_{i,b}(x)$ for $i = 1$ and for different values of $b > 0$.

Important signals obtained using this function: Elliptical swollen compressed, sinusoidal, quasi-triangular, elliptical deflated amplified, impulse train (positive and negative part) with controlled amplitude, impulse train (positive part only) with controlled amplitude, saw signal, null signal, rectified sinusoidal …

• **Example using the Elliptical *Rit*:**

In this part, an example is treated in a goal to form many shapes obtained by varying only one parameter. For this, consider the following equation:

$$y^2 = \left(k.rect_T(x - x_0).Erit_b(\alpha x)\right)^2 \qquad (41)$$

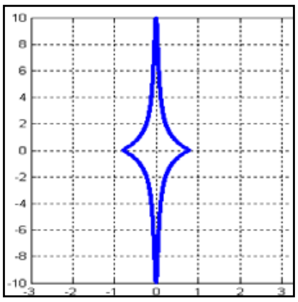
With k is the amplitude of the signal,
'x' is a variable parameter,
α is the frequency of the $Ejes_{x_b}(\alpha x)$,
k, α and T are positive values(> 0).

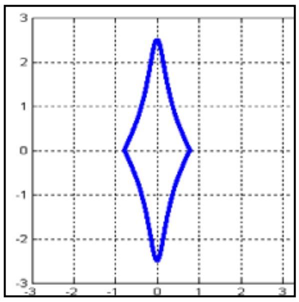
$$\Rightarrow y = \pm\left(k.rect_T(x - x_0).Erit_b(\alpha x)\right) \qquad (42)$$

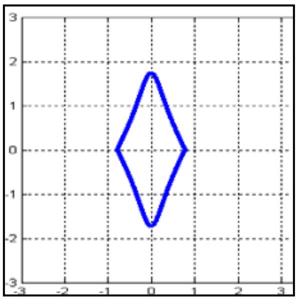
The period of $Erit_b(\alpha x)$ is equal to $T_1 = 2\pi/\alpha$

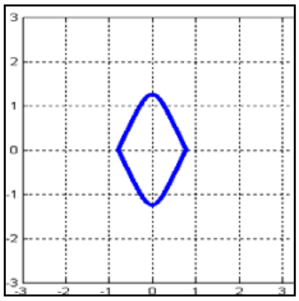
For a particular, let's take $T_1 = \pi; T = \frac{T_1}{2}; \ \alpha = 2;$ $x_0 = 0$ and $k = 1$.

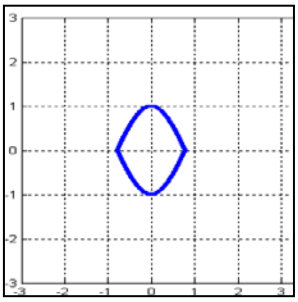
By varying the parameter b in (42), the following remarkable shapes are illustrated in figures 16.a to 16.h.

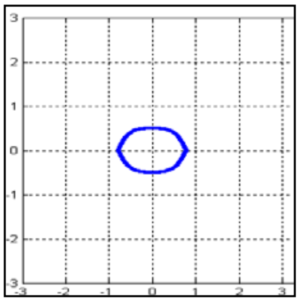

a) $b = 0.1$

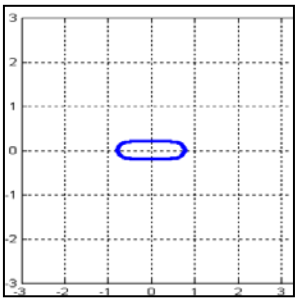

b) $b = 0.4$


c) $b = \sqrt{3}/3$


d) $b = 0.8$


e) $b = 1$


f) $b = 2$


g) $b = 5$

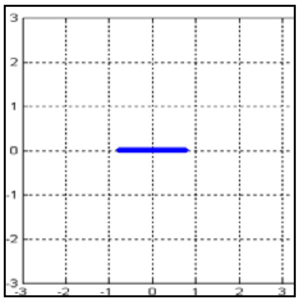

h) $b = 50$

Fig. 16: different shapes of the function (42) for different values of $b > 0$.

4.5 Original formulae of the Elliptical Trigonometry

In this sub-section, a brief review on some remarkable formulae formed using the elliptical trigonometric functions.

• $\left(Ejes_b(x)\right)^2 + \left(Emar_b(x)\right)^2 = 1$ (43)

In fact:

$$\left(Ejes_b(x)\right)^2 + \left(Emar_b(x)\right)^2 =$$

$$\left(\frac{ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2}}\right)^2 + \left(\frac{a}{b}\frac{\mathrm{Cter}\,(x)ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2}}\right)^2 =$$

$$\frac{1}{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2} + \left(\frac{a}{b}\right)^2\frac{\mathrm{Cter}\,(x)^2}{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2} =$$

$$\frac{1+\left(\frac{a}{b}\right)^2\mathrm{Cter}\,(x)^2}{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2} = \frac{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2}{1+\left(\frac{a}{b}\mathrm{Cter}\,(x)\right)^2} = 1$$

• $\frac{1}{Ejes_x^2(x)+Emar_x^2(x)} + \frac{1}{Ejes_y^2(x)+Emar_y^2(x)} = 1$ (44)

In fact:

$$Ejes_x^2(x) + Emar_x^2(x) = \left(\frac{Ejes(x)}{Cjes(x)}\right)^2 + \left(\frac{Emar(x)}{Cjes(x)}\right)^2$$

$$= \frac{1}{\left(Cjes(x)\right)^2} \Rightarrow \frac{1}{Ejes_x^2(x)+Emar_x^2(x)} = \left(Cjes(x)\right)^2$$

and

$$Ejes_y^2(x) + Emar_y^2(x) = \left(\frac{Ejes(x)}{Cmar(x)}\right)^2 + \left(\frac{Emar(x)}{Cmar(x)}\right)^2$$

$$= \frac{1}{\left(Cmar(x)\right)^2} \Rightarrow \frac{1}{Ejes_y^2(x)+Emar_y^2(x)} = \left(Cmar(x)\right)^2$$

Therefore:

$$\frac{1}{Ejes_x^2(x)+Emar_x^2(x)} + \frac{1}{Ejes_y^2(x)+Emar_y^2(x)} =$$

$$Cjes^2(x) + Cmar^2(x) = \cos^2(x) + \sin^2(x) = 1$$

5 Survey on the application of the elliptical trigonometry in engineering domain

As we saw in the previous sections, the main goal of the Elliptical Trigonometry is to produce a huge number of multi form signals using a single function and by varying some parameters of this function, for this reason, one can imagine the importance of this trigonometry in all domains especially in telecommunication, signal theory, electrical and electronic engineering.

Particularly in electrical engineering: motor drives, robotics, and other electronic applications need many controlled circuits that produce different type of signals. In a goal to increase the efficiency, by using this trigonometry, one can develop a circuit that produces multiple forms of signals that respond to requirements [2].

The functions of this trigonometry are easily programmed and simulated with softwares as Matlab and Labview, and many circuits can be formed to describe these functions.

For a particular case, the Elliptic *Mar* function can be obtained as an output signal by using a simple circuit as shown in the figure 17.

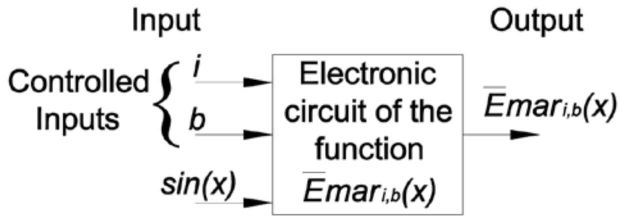


Fig. 17: Electronic circuit with its inputs and output.

Thus, this simple circuit can produce more than 14 different signals by varying only the two parameters (i and b). These output signals have an extreme importance in the power electronics in which the regulation of electronic systems will become simpler and more efficient. In another hand circuits will be more reduced and costless.

6 Conclusion

In this paper, an original study in trigonometry is introduced. The elliptical unit and its trigonometric functions are presented and analyzed. In fact the proposed Elliptical Trigonometry is a new form of trigonometry that permits to produce multiple forms of signals by varying some parameters; it can be used in numerous scientific domains and particularly in mathematics and in engineering. For the case treated in this paper, 32 elliptical trigonometric functions are defined; only four functions are simulated using software as Matlab with a brief study. For each elliptical function, many periodic signals are produced by varying some parameters.

The elliptical trigonometric functions will be widely used in electronic domain especially in power electronics, in signal theory, in signal processing and many other domains. Thus, several studies will be improved and developed after introducing the new functions of this trigonometry.

Some mathematical expressions and electronic circuits will be replaced by simplified expressions and reduced circuits.
In order to illustrate the importance of this trigonometry, some functions are defined briefly and some examples are treated. Briefly, the Elliptical trigonometry is a particular case of the General trigonometry introduced also by the author [21]. The General Trigonometry is the general case of all possible trigonometries which can be exist, it has a huge importance in all domains and especially in the domains related to the trigonometry topic such as electronics, signal processing, signal theory and many others.

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