The Progress for Stability of Essential and Critical Spectra of Perturbed C_0-semigroups and its Applications to Models of Transport Theory

Mengmeng Xu
Zhengzhou University
Department of Mathematics
Zhengzhou 450001
P. R. China
xupng873@126.com

Junguo Jia
Zhengzhou University
Department of Mathematics
Zhengzhou 450001
P. R. China
junguojia2@zzu.edu.cn

Peng Zhao
Zhengzhou University
Department of Mathematics
Zhengzhou 450001
P. R. China
arpeng1000@126.com

Abstract: - In this paper, we make an analysis for the stable results published recent years about the essential and critical spectra of perturbed C_0-semigroups. We also show how these results are applied to models of transport theory. At last we provide and discuss some concerning problems for further study.

Key-words: - Perturbed C_0-Semigroup, Essential and Critical Spectrum, Transport Theory, Compactness, Norm Continuity, Stability

1 Introduction

Let X be a Banach space and let $T : D(T) \subset X \to X$ be the generator of a C_0-semigroup $(U(t))_{t\geq 0}$ on X. Consider the abstract Cauchy problem

$$\begin{cases} \frac{\mathrm{d}\varphi}{\mathrm{d}t}(t) = (T+K)\varphi(t), \ \ t>0, \\ \varphi(0) = \varphi_0. \end{cases} \tag{1}$$

Here $\varphi_0 \in X$ and K is a T-bounded operator on X (i.e. $D(T) \subset D(K)$, K is bounded on $D(T)$, where $D(T)$ is equipped with the graph norm) such that

$$\int_0^h \|KU(s)x\| ds \leq q(h)\|x\|, \ \ x \in X, h \geq 0$$

where $q : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ = (0,\infty)$) satisfies $\lim_{t\downarrow 0} q(t) = 0$. By Miyadera-Voigt perturbation theorem, $A := T + K$ with domain $D(A) = D(T)$ generates a C_0-semigroup $(V(t))_{t\geq 0}$ given by the Dyson-Phillips expansion

$$V(t) = \sum_{j=0}^{\infty} U_j(t) \tag{2}$$

where $U_0(t) = U(t)$,

$$U_j(t) = \int_0^t U_{j-1}(t-s)KU(s)\mathrm{d}s, (j \geq 1).$$

Thus the Cauchy problem (1) has a unique classical solution given by $\varphi(t) = V(t)\varphi_0$ if $\varphi_0 \in D(A)$. In order to get more information on the solution, in particular, its asymptotic behavior of large time, we first recall some results and progress of the concerning spectrum theory of $T + K$ or $(V(t))_{t\geq 0}$, since the spectrum of A or the semigroup $V(t)$ plays a central role.

Let $R_k(t)$ be the k-th order remainder term of the Dyson-Phillips expansion series

$$R_k(t) := \sum_{j=k}^{\infty} U_j(t). \tag{3}$$

It is known (e.g., see [1, 2, 3]) that the compactness of $R_k(t)$ for all $t \geq 0$ implies the stability of the essential spectrum of the semigroups, i.e. $\sigma_{\mathrm{ess}}(V(t)) = \sigma_{\mathrm{ess}}(U(t))$ if $k = 1$ and the stability of the essential growth type, i.e. $\omega_{\mathrm{ess}}(V(t)) = \omega_{\mathrm{ess}}(U(t))$ if for some $k > 1$. On the other hand, the norm continuity of $0 \leq t \mapsto R_k(t)$ ensures the stability of the critical spectrum, i.e. $\sigma_{\mathrm{crit}}(V(t)) = \sigma_{\mathrm{crit}}(U(t))$, if $k = 1$ and the stability of the critical growth type, i.e. $\omega_{\mathrm{crit}}(V(t)) = \omega_{\mathrm{crit}}(U(t))$, if for some $k > 1$. The concepts of essential and critical spectra will be given in section 2.

In the last decade, many authors researched the compactness and the norm continuity of the remainder term (3) and applied these results to models in transport theory.

Consider the transport equation (also called the linear Boltzmann equation)

$$\begin{aligned}\frac{\partial\varphi}{\partial t}(x,v,t) = & -v\cdot\nabla_x\varphi(x,v,t)-\sigma(x,v)\varphi(x,v,t)\\ & +\int_V k(x,v,v')\varphi(x,v',t)\mathrm{d}\mu(v')\end{aligned} \tag{4}$$

with no-reentry boundary condition

$$\varphi|_{\Gamma_-}(x,v,t)=0,\ \ t>0 \tag{5}$$

where $(x,v)\in\Omega\times V$, Ω is a convex open subset of $\mathbb{R}^N$ $(N\geq 1)$ with smooth boundary $\partial\Omega$ and V (the velocity space) is a support of a positive Radon measure μ on $\mathbb{R}^N$. The function $\varphi(x,v)$ represents the number (or probability) density of gas particles having the position x and the velocity v. The subset Γ_- of the boundary set is defined by

$$\Gamma_-=\{(x,v)\in\partial\Omega\times V;\ v\cdot\eta(x)<0\}$$

where $\eta(x)$ is the unit outward normal at $x\in\partial\Omega$. The collision frequency $\sigma(\cdot,\cdot)\in L^\infty(\Omega\times V)$ is a non-negative function while $k(\cdot,\cdot,\cdot)$ is the scattering kernel. The unbounded linear operator, called the streaming operator defined by

$$T:\varphi\in D(T)\to -v\cdot\frac{\partial\varphi}{\partial x}-\sigma\varphi$$

with domain for $1\leq p<\infty$

$$D(T)=\left\{\varphi\in L^p(\Omega\times V)\ \Big|\ \begin{array}{l} v\cdot\frac{\partial\varphi}{\partial x}\in L^p(\Omega\times V)\\ \varphi|_{\Gamma_-}=0,\end{array}\right\},$$

generates the so-called streaming C_0-semigroup

$$U(t)\varphi=\begin{cases} e^{-\int_0^t\sigma(x-sv,v)ds}\varphi(x-tv,v), t\leq t_-(x,v)\\ 0,\quad t>t_-(x,v)\end{cases} \tag{6}$$

where

$$\begin{aligned}t_-(x,v) &= \sup\{t>0; x-sv\in\Omega,\forall 0<s<t\}\\ &=\inf\{s>0;x-sv\notin\Omega\},\end{aligned} \tag{7}$$

Figure (a) shows that for each fixed $x\in\Omega$ and vectors $v_1,v_2\in V$, the minimum values of the set

$$\{s>0;x-sv_j\notin\Omega\}$$

are s_1 and s_2 respectively, i.e.

$$t_-(x,v_1)=s_1,\quad t_-(x,v_2)=s_2.$$

Moreover, if the collision operator $K:L^p(\Omega\times V)\to L^p(\Omega\times V)$

$$K:\varphi\mapsto\int_V k(x,v,v')\varphi(x,v')\mathrm{d}\mu(v') \tag{8}$$

is bounded on $L^p(\Omega\times V)$, then $T+K$ generates a transport C_0-semigroup $(V(t))_{t\geq 0}$ given by the Dyson-Phillips expansion (2).

For transport equation with no-reentry boundary conditions, many authors had studied the compactness of the remainder term when Ω is bounded, for instance, see the works [4]–[7] and the references therein. However, when Ω is unbounded region of $\mathbb{R}^N$, due to the lack of compactness, it turned out in [8, 9] that the critical spectrum and the norm continuity of the remainder term are the important tools to the spectral analysis of $(V(t))_{t\geq 0}$.

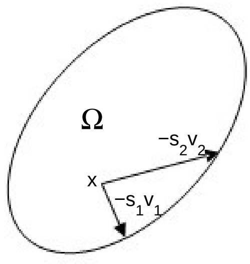

Fig (a)

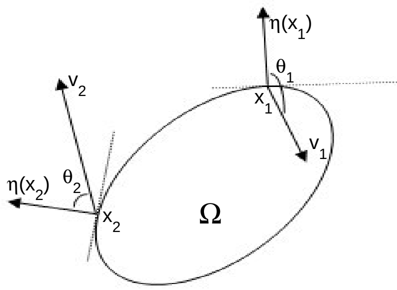

Fig (b)

Now let Ω is a convex open subset of $\mathbb{R}^N$ $(N\geq 1)$ with smooth boundary $\partial\Omega$. In the region $\Omega\times\mathbb{R}^N$, let us consider the transport equation

$$\begin{aligned}\frac{\partial\varphi}{\partial t}(x,v,t) = & -v\cdot\nabla_x\varphi(x,v,t)-\sigma(x,v)\varphi(x,v,t)\\ & +\int_{\mathbb{R}^N} k(x,v,v')\varphi(x,v',t)\mathrm{d}v'\end{aligned} \tag{9}$$

with reentry boundary conditions (including the periodic boundary conditions, reflections boundary conditions and so on)

$$\varphi|_{\Gamma_-}(x,v,t)=H(\varphi|_{\Gamma_+}(x,v,t)),(x,v)\in\Gamma_-, \tag{10}$$

where Γ_- (resp. Γ_+) denotes the incoming (resp. outgoing) part of the boundary of the phase space $\Omega \times V$, i.e.,

$$\Gamma_\pm = \{(x, v) \in \partial\Omega \times \mathbb{R}^N;\ \pm v \cdot \eta(x) > 0\}.$$

From Figure (b) we can see clearly that

$$(x_1, v_1) \in \Gamma_-, \quad (x_2, v_2) \in \Gamma_+.$$

The boundary conditions (10) shows that the incoming flux $\varphi|_{\Gamma_-}$ is related to the outgoing one $\varphi|_{\Gamma_+}$ through a linear bounded operator $H \neq 0$ that we shall assume to be bounded on some suitable trace spaces.

There were a great progress having been made in the last decade in the understanding of the spectral features of one-dimensional models, for examples, see [10]–[21]. In this aspect, the Chinese scholars had done some important works, here we infer to [15, 16] and [17]–[19] and reference therein. In [10]–[13], the compactness of the remainder term has been studied, and the stability of the essential growth type or essential spectrum obtained. While in [14]-[20], some spectral problems of the transport operators and the streaming operator as well as streaming semigroup had been investigated. However, for higher dimension equation, only a few results are available, see [22], the main difficult come from computation of the expression of C_0-semigroup $(U(t))_{t\geq 0}$. Therefore, in the present paper, we summarize the stable results of essential and critical spectra in the perturbed semigroup and discuss them application to some models in transport theory.

The rest is organized as follows: In section 2, we recall the concepts of essential and critical spectra of the semigroup theory and some results of related stability. In section 3 we give some applications of the stability results in the transport theory. In section 4, we provide some unsolved problems for further study.

2 Notations and Preliminaries

Throughout this paper, for Banach spaces X and Y, $\mathcal{B}(X, Y)$ denotes the space of bounded linear operators from X to Y. When $X = Y$, we simply write $\mathcal{B}(X)$. For any linear operator A, $\sigma(A)$, $\rho(A)$ and $r(A)$ denote the spectrum set, the resolvent set and the spectral radius respectively, and the spectral bound $s(A)$ of A is defined by

$$s(A) = \sup\{\mathrm{Re}\lambda; \lambda \in \rho(A)\}.$$

For C_0-semigroup $(U(t))_{t\geq 0}$, $\omega(U)$ denotes the growth type (bound) of $(U(t))_{t\geq 0}$, i.e. $\omega(U) = \lim\limits_{t\to\infty} t^{-1} \log \|U(t)\|$.

Let $(U(t))_{t\geq 0}$ be a strongly continuous semigroup with generator T on a Banach space X. The essential spectral of T is defined by

$$\sigma_{\text{ess}}(T) = \big\{\lambda \in \sigma(T); \lambda \text{ is not an isolate eigenvalue of finite algebraic multiplicity}\big\}. \tag{11}$$

For a bounded linear operator B, the essential spectral radius of B is defined by

$$r_{\text{ess}}(B) = \sup\{|\lambda|; \lambda \in \sigma_{\text{ess}}(B)\}.$$

There is connection between the essential spectral radius and the measure of non-compactness of B defined by

$$\|B\|_m = \inf_{C\in C(X)} ||T - C||$$

where $C(X)$ is the subspace of $\mathcal{B}(X)$ consisted of all compact linear operators. Clearly, $\|\cdot\|_m$ is a semi-norm on $\mathcal{B}(X)$ with property

$$\|B\|_m = 0 \text{ if and only if } B \text{ is compact}$$

and $r_{\text{ess}}(B) = \lim\limits_{n\to\infty} \|B^n\|_m^{\frac{1}{n}}$. Then

$$\begin{aligned}\omega_{\text{ess}}(U)) &= \lim_{t\to\infty} t^{-1} \log \|U(t)\|_m \\ &= \inf\{\lambda \in \mathbb{R}; \exists M, \|U(t)\|_m \leq Me^{\lambda t}\}\end{aligned} \tag{12}$$

exists and $r_{\text{ess}}(U) = e^{t\omega_{\text{ess}}(U)}$ $(t \geq 0)$. The number $\omega_{\text{ess}}(U) \in [-\infty, \omega(U)]$ is called the essential growth type of $(U(t))_{t\geq 0}$.

For $K \in \mathcal{B}(X)$, $T + K$ generates a C_0-semigroup $(V(t))_{t\geq 0}$ given by (2). If $R_k(t)$ is compact, then the operators $U(t)$ and $V(t)$ have the same essential growth type. Therefore there are only isolated points in the spectrum of the perturbed semigroup $(V(t))_{t\geq 0}$ outside the circle $|\mu| = e^{t\omega_{\text{ess}}(U)}$, all these points being eigenvalues with finite algebraic multiplicity. Therefore, for any $v > \omega_{\text{ess}}(U)$, $\sigma(T + K) \cup \{\lambda \in \mathbb{C}; \mathrm{Re}\lambda \geq v\}$ consists of finitely many isolated eigenvalues $\{\lambda_1, \cdots, \lambda_q\}$. Let

$$\beta_1 = \sup\{\mathrm{Re}\lambda; \lambda \in \sigma(T + K)\}$$

and

$$\beta_2 = \min\{\mathrm{Re}\lambda_j, 1 \leq j \leq q\}.$$

The solution of the problem (1) satisfies

$$\lim_{t\to\infty} e^{-\beta t}\Big\|\varphi(t) - \sum_{j=1}^{q} e^{\lambda_j t} e^{D_j t} P_j \varphi_0\Big\| = 0, \quad \beta_2 < \beta < \beta_1,$$

where P_j and D_j denote the Riesz spectral projection and the nilpotent operator associated with λ_j, $j = 1, 2, \cdots q$, respectively.

Consider the linear space $\widetilde{X} := \ell^\infty(X)$ of all bounded sequences in X, endowed with the norm $\|(x_n)_{n\in\mathbb{N}}\| = \sup_{n\in\mathbb{N}} \|x_n\|$. Clearly, $\widetilde{X}$ also is a Banach space. We extend the C_0-semigroup $(U(t))_{t\geq 0}$ on X to $\widetilde{X}$ defined by for each $t \geq 0$,

$$\widetilde{U}(t)\widetilde{x} := (U(t)x_n)_{n\in\mathbb{N}} \text{ for } \widetilde{x} = (x_n)_{n\in\mathbb{N}} \quad (13)$$

and obtain a new semigroup $(\widetilde{U}(t))_{t\geq 0}$. Note that the semigroup $(\widetilde{U}(t))_{t\geq 0}$ is strongly continuous only if and only if $(U(t))_{t\geq 0}$ is uniformly continuous, hence T is bounded.

Let $\widetilde{X}_T$ be the subspace of $\ell^\infty(X)$ defined by

$$\widetilde{X}_T := \{\widetilde{x} \in \ell^\infty(X); \lim_{t\to 0} \|\widetilde{U}(t)\widetilde{x} - \widetilde{x}\| = 0\}$$

which is the subspace conserving strong continuity of $(\widetilde{U}(t))_{t\geq 0}$.

It is not difficult to prove that the subspace $\widetilde{X}_T$ is closed and $(\widetilde{U}(t))_{t\geq 0}$-invariant and therefore allows the following quotient construction.

Definition 1 *On the quotient space $\widehat{X} := \widetilde{X}/\widetilde{X}_T$ we define the semigroup $(\widehat{U}(t))_{t\geq 0}$ by*

$$\widehat{U}(t)\widehat{x} := U(t)x_n + \widetilde{X}_T \text{ for } \widehat{x} := (x_n)_{n\in\mathbb{N}} + \widetilde{X}_T \in \widehat{X}. \quad (14)$$

This is a semigroup of bounded linear operators on $\widehat{X}$. The critical spectrum of $(U(t))_{t\geq 0}$ is then defined as

$$\sigma_{\text{crit}}(U(t)) = \sigma(\widehat{U}(t)) \quad (15)$$

while its critical spectral radius and critical growth type are defined as

$$r_{\text{crit}}(U(t)) = r(\widehat{U}(t)),\ \omega_{\text{crit}}(U) = \omega(\widehat{U}), \quad (16)$$

respectively.

The mapping $0 \leq t \mapsto \widehat{U}(t)\widehat{x}$ is continuous if and only if $\widehat{x} = 0$. Moreover, the following theorem holds.

Theorem 2 [23] *For a strongly continuous semigroup $(U(t))_{t\geq 0}$ with generator T, the following statements hold.*

(1) $\sigma_{\text{crit}}(U(t)) \subset \sigma_{\text{ess}}(U(t)) \subset \sigma(U(t))$.
(2) $r_{\text{crit}}(U(t)) = e^{t\omega_{\text{crit}}(U)}$.
(3) $\sigma(U(t)) \setminus \{0\} = e^{t\sigma(T)} \cup \sigma_{\text{crit}}(U(t)) \setminus \{0\}$.
(4) $\omega(U) = \max\{s(T), \omega_{\text{crit}}(U)\}$.

In general, the spectra between a C_0-semigroup $(U(t))_{t\geq 0}$ and its generator T have the following relationship $\sigma(U(t)) \setminus \{0\} \supset e^{t\sigma(T)}$, we can write

$$\sigma(U(t)) \setminus \{0\} = e^{t\sigma(T)} \cup \sigma_?(U(t)) \setminus \{0\} \quad (17)$$

for some complex number set $\sigma_?(U(t))$. Since the equation $\sigma(U(t)) \setminus \{0\} = e^{t\sigma(T)}$ always is true for the point spectrum, we may takes $\sigma_?(U(t))$ as the essential spectrum $\sigma_{\text{ess}}(U(t))$. However the essential spectrum is not related to the semigroup structure, and even for bounded T it is unnecessarily big in order to yield the above identity (17). Therefore we proposed critical spectrum $\sigma_{\text{crit}}(U(t))$ which yields in an optimal way a spectral mapping theorem, see Theorem 2 (3). In addition, it is known [24] that the spectrum determined growth condition $\omega(U) = s(T)$ is an important criterion for exponential stability of the C_0-semigroup $(U(t))_{t\geq 0}$. A sufficient condition of the spectrum determined growth assumption can be obtained from Theorem 2, that is, if $\omega_{\text{crit}}(U(t)) < \omega(U)$, then $\omega(U) = s(T)$.

Remark 3 *Recall that the approximate spectrum $\sigma_{\text{ap}}(T)$ of a closed densely linear operator T in a Banach space X is defined as*

$$\sigma_{\text{ap}}(T) = \big\{\lambda \in \mathbb{C}; \exists (x_n)_n \subset D(T), \|x_n\| = 1, \|Tx_n - \lambda x_n\| \to 0 \text{ as } n \to \infty\big\}.$$

Let $\{\lambda_n\}_n \subset \sigma_{\text{ap}}(T)$ be a sequence satisfying $\lim_{n\to\infty} |\text{Im}\lambda_n| = \infty$ and $\lim_{n\to\infty} e^{t\lambda_n} = \mu$. Then $\mu \in \sigma_{\text{crit}}(U(t))$.

The critical spectrum plays a crucial role in spectral mapping theorem and has nice perturbation properties.

Theorem 4 [1] *Let $(U(t))_{t\geq 0}$ be a C_0-semigroup with generator T and $(V(t))_{t\geq 0}$ be the C_0-semigroup with generator $T + K$. If the mapping $0 \leq t \mapsto R_1(t)$ is norm continuous for $t \geq 0$, then one has*

$$\omega(V) = \max\{s(T + K), \omega_{\text{crit}}(U(t))\}$$

and

$$\sigma(V(t))\backslash\{0\} = e^{t\sigma(T+K)} \cup \sigma_{\text{crit}}(U(t))\backslash\{0\}.$$

The compactness and norm continuity of the remainder term (3) are linked in [25].

Theorem 5 [25] *Let $v > \omega(U)$ and $k \in \mathbb{N}$. Then the following statements are equivalent.*

(1) *$R_k(t)$ is compact for all $t \geq 0$.*
(2) *$0 \leq t \mapsto R_k(t)$ is norm continuous and*

$$R(v + i\gamma, T)(KR(v + i\gamma, T))^k$$

is compact for all $\gamma \in \mathbb{R}$.

Naturally, a question is: what conditions imply the compactness and norm continuity of the remainder term (3)? The following Theorem gives an answer to this question.

Theorem 6 [4] *Let X be a Hilbert space. Assume there exist $v > \omega(U)$ and $m \in \mathbb{N}$ such that*

$$\|R(v+i\gamma,T)^i KR(v+i\gamma,T)^{m-i}\| \to 0, \quad |\gamma| \to \infty \ \ (i = 0,1,\cdots,m).$$

Then $0 \le t \mapsto R_1(T)$ is norm continuous and consequently $\sigma_{\text{crit}}(V(t)) = \sigma_{\text{crit}}(U(t))$ for all $t \ge 0$.

Theorem 7 [4] *Assume that T is dissipative and that for some $v > 0$,*

$$\|K^* R(v+i\gamma,T)K\| \to 0$$

and

$$\|KR(v+i\gamma,T)K^*\| \to 0$$

as $|\gamma| \to \infty$, where K^* denotes the adjoint operator of K. Then $0 \le t \mapsto R_1(t)$ is norm continuous.

Above theorems give the conditions for the norm continuous of $R_1(t)$. The following Theorem gives a condition of compactness about $R_1(t)$.

Theorem 8 [4] *Let X be a Hilbert space. Assume there exist $v > \omega(U)$ and $m \in \mathbb{N}$ such that*

$$\|R(v+i\gamma,T)^i KR(v+i\gamma,T)^{m-i}\| \to 0, \quad |\gamma| \to \infty \ \ (i = 0,1,\cdots,m).$$

and $R(v+i\gamma,T)KR(v+i\gamma,T)$ is compact for all $\gamma \in \mathbb{R}$. Then $R_1(t)$ is compact and consequently $\sigma_{\text{ess}}(V(t)) = \sigma_{\text{ess}}(U(t))$ for all $t \ge 0$.

The compactness result in Banach space has been obtained in [6].

Theorem 9 [6] *Assume that there exist $m \in \mathbb{N}$ and $v > \omega(U)$ satisfying*

(1) *$R(v+i\gamma,T)(KR(v+i\gamma,T))^m$ is compact for all $\gamma \in \mathbb{R}$;*

(2) *$|\gamma|\|R(v+i\gamma,T)(KR(v+i\gamma,T))^m\| \to 0$ as $\gamma \to \infty$.*

Then $R_{2m+1}(t)$ is compact on X for each $t > 0$ and therefore $V(t)$ and $U(t)$ have the same essential growth type.

We observe that the resolvent norm decay implies the norm continuity in the Hilbert space (see [26]), but it is not true in Banach spaces (see, [27])

3 Applications to Transport Models

In this section we discuss the stability of the essential or the critical spectrum of the streaming semigroup when its generator is perturbed by a collision operator.

In transport theory, a collision operator is in general an integral operator of the form

$$\varphi \in X_p \mapsto K\varphi = \int_V k(x,v,v')\varphi(x,v')\mathrm{d}\mu(v') \in X_p$$

where $X_p = L^p(\Omega \times V)$. Thus, naturally regard K as an operator valued mapping

$$x \in \Omega \mapsto K(x) \in \mathcal{B}(L^p(V)),$$

where

$$K(x) : \varphi \in L^p(V) \mapsto \int_V k(x,v,v')\varphi(v')\mathrm{d}\mu(v') \in L^p(V).$$

Assume that K is strongly measurable, i.e. for every $\psi \in L^p(V)$,

$$x \in \Omega \mapsto K(x)\psi \in L^p(V)$$

is measurable, and $x \in \Omega \mapsto \|K(x)\|_{\mathcal{B}(L^p(V))}$ is essentially bounded on Ω. Now the collision operator K can be redefined as

$$K : \varphi \in X_p \mapsto K(x)\varphi(x),$$

where make the identification

$$X_p = L^p(\Omega \times V) = L^p(\Omega; L^p(V)).$$

It follows easily that $K \in \mathcal{B}(X_p)$ and

$$\|K\|_{\mathcal{B}(X_p)} = \operatorname*{ess\,sup}_{x\in\Omega} \|K(x)\|_{\mathcal{B}(L^p(V))}.$$

In what follows, we need the concept of regular operator.

Definition 10 *Let $1 \le p < \infty$. A collision operator K is said to be regular if the following conditions are satisfied*

(1) *$\{K(x); x \in \Omega\}$ is a set of collectively compact operators on $L^p(V)$, i.e.*

$$\{K(x)\psi; x \in \Omega, \|\psi\|_{L^p(V)} \le 1\}$$

is relatively compact in $L^p(V)$;

(2) *For every $\psi' \in L^q(V)$, $\{K'(x)\psi'; x \in \Omega\}$ is relatively compact in $L^q(V)$.*

where q denotes the conjugate number of p defined by $1/p + 1/q = 1$, and $K'(x)$ denotes the dual operator of $K(x)$.

The following Lemma gives the description for the regular collision operators.

Lemma 11 *The class of regular collision operators is the closure in the operator norm of the class of collision operators with separable kernels*

$$k(x, v, v') = \sum_{i \in I} \alpha_i(x) f_i(v) g_i(v'), \qquad (18)$$

with $f_i \in L^p(V)$, $g_i \in L^q(V)$ $(1/p + 1/q = 1)$ *and* $\alpha_i \in L^\infty(\Omega)$ (I *is a finite set*).

Remark 12 *Since* $1 < p < \infty$, *we note that the set* $C_c(V)$ *of continuous functions with compact support in* V *is dense in* $L^q(V)$ *as well as in* $L^p(V)(1/p + 1/q = 1)$. *Consequently, we may assume in the above definition that* $f_i(\cdot)$ *and* $g_i(\cdot)$ *are continuous functions with compact supports in* V.

Firstly, we recall some applications of the results in section 2 in transport equations with no-reentry boundary conditions (4)-(5).

Lemma 13 [5] *Let* $1 \le p < \infty$. *For bounded* Ω, *the essential spectrum of* $(U(t))_{t \ge 0}$ *is given by*

$$\sigma_{\text{ess}}(U(t)) = \{\mu \in \mathbb{C}; |\mu| \le e^{-\lambda^* t}\} \qquad (19)$$

where

$$\lambda^* = \lim_{t \to \infty} \inf_{\{(x,v); t < t_-(x,v)\}} \frac{1}{t} \int_0^t \sigma(x - sv, v) \mathrm{d}s, \qquad (20)$$

where $t_-(x, v)$ *is defined by* (7).

Lemma 14 [8, 28] *Let* $1 \le p < \infty$, *and* Ω *be a convex domain in* $\mathbb{R}^N$.

(1) *If* $\Omega \subsetneqq \mathbb{R}^N$ *and if the hyperplanes have zero* μ*-measure (i.e. for each* $e \in S^{N-1}$, $d\mu\{v \in \mathbb{R}^N; v \cdot e = 0\} = 0$, *where* S^{N-1} *denotes the unit sphere of* $\mathbb{R}^N$*), then*

$$\sigma_{\text{crit}}(U(t)) = \sigma(U(t)) = \{\mu \in \mathbb{C}; |\mu| \le e^{-\lambda^* t}\} \qquad (21)$$

where λ^* *is given by* (20).

(2) *If* $\Omega = \mathbb{R}^N$ *and the collision frequency* σ *is space homogeneous* (*i.e.* $\sigma(x, v) = \sigma(v)$), *then* $\sigma_{\text{crit}}(U(t))$ *consists of a set of disjoint slabs*

$$\sigma_{\text{crit}}(U(t)) = \sigma(U(t)) = \cup_{i \in I} \Lambda_i \qquad (22)$$

of the form $\Lambda_i = \{\lambda; a_i \le \text{Re}\lambda \le b_i\}$ $(a_i \le b_i)$, *where*

$$\inf_{i \in I} a_i = -\theta^{**} =: -\text{ess sup}\, \sigma(v),$$

$$\sup_{i \in I} b_i = -\theta^* =: -\text{ess inf}\, \sigma(v).$$

Theorem 15 [5] *Let* $1 < p < \infty$ *and let* Ω *be of finite Lebesgue measure. Assume that the affine hyperplanes have zero* μ*-measure and that the collision operator is regular. Then* $R_1(t)$ *is compact on* X_p *and consequently* $\sigma_{\text{ess}}(V(t)) = \sigma_{\text{ess}}(U(t))$ *for all* $t \ge 0$.

Theorem 16 [8] *Let* $1 < p < \infty$. *Assume that the affine hyperplanes have zero* μ*-measure, the collision operator is regular and the collision frequency* σ *is space homogeneous.*

(1) *If* $\Omega = \mathbb{R}^N$ *and if*

$$x \in \Omega \mapsto \langle K(x)\varphi, \psi \rangle \qquad (23)$$

is uniformly continuous with respect to x *for every* $(\varphi, \psi) \in L^p(V) \times L^q(V)$. *Then the mapping* $0 \le t \mapsto R_1(t)$ *is norm continuous.*

(2) *If* $\Omega \subsetneqq \mathbb{R}^N$ *and if* $x \in \Omega \mapsto \langle K(x)\varphi, \psi \rangle$ *is uniformly continuous for every* $(\varphi, \psi) \in L^p(V) \times L^q(V)$ *and its extension* (*by continuous*) *to* $\overline{\Omega}$ *vanishes on* $\partial\Omega$. *Then the mapping* $0 \le t \mapsto R_1(t)$ *is norm continuous.*

Combining Theorem 2, Lemma 14 with Theorem 16 yields the following partial spectral mapping theorem

Theorem 17 [8] *Under the assumptions of Theorem 16 we have:*

(1) *If* $\Omega = \mathbb{R}^N$, *then* $\sigma_{\text{crit}}(V(t)) = \sigma_{\text{crit}}(U(t))$ *and*

$$\sigma(V(t)) \cap \{\mu; |\mu| < e^{-\theta^{**} t} \text{ or } |\mu| > e^{-\theta^* t}\} = e^{t(\sigma(T+K) \cap \{\lambda; \text{Re}\lambda < -\theta^{**} \text{ or } \text{Re}\lambda > -\theta^*\})}.$$

(2) *If* $\Omega \subsetneqq \mathbb{R}^N$, *then* $\sigma_{\text{crit}}(V(t)) = \sigma_{\text{crit}}(U(t))$ *and*

$$\sigma(V(t)) \cap \{\mu; |\mu| > e^{-\lambda^* t}\} = e^{t(\sigma(T+K) \cap \{\lambda; \text{Re}\lambda > -\lambda^*\})}.$$

We introduce a stronger assumption on the measure of velocity, μ, by

$$\int_D e^{iz \cdot v} d\mu(v) \to 0 \text{ as } |z| \to \infty \qquad (24)$$

for all Borel set $D \subset \mathbb{R}^N$ with $\mu(D) < \infty$. This assumption allows us to improve the restrictive condition (23) and to obtain full spectral mapping theorems.

Theorem 18 [8] *Let* $1 < p < \infty$ *and* Ω *be convex. Assume that the collision operator* K *is regular and that* (24) *is satisfied. Then* $0 \le t \mapsto R_2(t)$ *is norm continuous and we have the following assertions*

(1) *If* $\Omega = \mathbb{R}^N$ *and if the essential range of* $\sigma(v)$ *is connected, then*

$$\sigma(V(t)) = e^{t\sigma(T+K)}$$

where the essential range of $\sigma(v)$, $\mathcal{R}_{\rm ess}(\sigma(v))$, *is the set*

$$\left\{u \in \mathbb{C}; \big|\{v \in V; |\sigma(v)-u| < \varepsilon\}\big| \neq 0,\ \forall \varepsilon > 0\right\}$$

where $|A|$ *denotes the Lebesgue measure of the set* A.

(2) *If* $\Omega \subsetneqq \mathbb{R}^N$, *then*

$$\sigma(V(t)) = e^{t\sigma(T+K)} \cup \{0\}.$$

Theorems 15–18 were proved in [5, 8] by the so-called semigroup approach, that is, by analyzing the streaming C_0-semigroups $(U(t))_{t\geq 0}$ to investigate the properties of the remained term (3). The following theorem in [4] was proved by the so-called resolvent approach, that is, by analysis the resolvent of the streaming operator T to investigate the properties of the remained term. It is showed in [4] that The resolvent approach is powerful in transport theory, because it does not need to calculate the expression of $(U(t))_{t\geq 0}$.

Theorem 19 [4] (1) *Let* $p=2$ *and let* Ω *be bounded* (*not necessarily convex*). *Assume that the affine hyperplanes have zero* μ*-measure and that the collision operator is regular. Then* $R_1(t)$ *is compact and consequently* $\sigma_{\rm ess}(V(t)) = \sigma_{\rm ess}(U(t))$ *for all* $t \geq 0$.

(2) *Let* $p=2$ *and let* Ω *be convex* (*not necessarily bounded*). *Assume that the affine hyperplanes have zero* μ*-measure, the collision operator is regular and the collision frequency* σ *is space homogeneous. Then the mapping* $0 \leq t \mapsto R_1(t)$ *is norm continuous and consequently* $\sigma_{\rm crit}(V(t)) = \sigma_{\rm crit}(U(t))$ *for all* $t \geq 0$.

Note that the Theorem 19 is proved in Hilbert space L^2. In fact, it also is valid in L^p-spaces for any $1 < p < \infty$, this was proved in [10] for transport problems.

Now let us consider one-dimensional transport equation

$$\begin{aligned}\frac{\partial\varphi}{\partial t}(x,v,t) = -v\frac{\partial\varphi}{\partial x}(x,v,t) - \sigma(x,v)\varphi(x,v,t)\\ + \int_{-1}^{1} k(x,v,v')\varphi(x,v',t)\mathrm{d}v'\end{aligned} \tag{25}$$

with specular reflection boundary conditions

$$\varphi(\pm a,\mu) = \varphi(\pm a,-\mu) \tag{26}$$

where $(x,v) \in [-a,a]\times[-1,1]$ $(a>0)$. We make the following assumption on the collision frequency $\sigma(x,v)$:

(A1) σ is not bounded and space homogeneous, there exists a closed subset $O \subset (-1,1)$ with zero Lebesgue measure and a constant σ_0 such that $\sigma(\cdot) \in L^\infty_{\rm loc}((-1,1)\setminus O)$, $\sigma(v) > \sigma_0$ a.e. on $(-1,1)$ and $\sigma(v) = \sigma(-v)$, $\forall \mu \in (-1,1)$.

Theorem 20 [11, 12, 13] *Let* (A1) *be satisfied and* K *be a regular operator. Then for all* $t \geq 0$, *the first and second order remainder terms,* $R_1(t)$, $R_2(t)$, *are weakly compact on* $L^1([-a,a]\times[-1,1])$ *and compact on* $L^p([-a,a]\times[-1,1])$ $(1<p<\infty)$.

Remark 21 *If a remainder term* $R_n(t)$ $(n \geq 1)$ *is compact* (*resp. weakly compact*) *for all* $t \geq 0$, *then* $R_{n+1}(t)$ *is compact* (*resp. weakly compact*) *for all* $t \geq 0$. *Similarly, if a mapping* $0 \leq t \mapsto R_n(t)$ $(n \geq 1)$ *is norm continuous, then* $0 \leq t \mapsto R_{n+1}(t)$ *is norm continuous.*

The works in [11, 12, 13] and [17, 18, 19] had shown that Theorem 20 remains true even if consider the transport equation (25) with periodic boundary conditions

$$\varphi(a,\mu) = \varphi(-a,\mu). \tag{27}$$

As we pointed out in the introduction, there were only a few results to be available for higher dimensions transport equation with reentry boundary conditions. Let us consider the transport equation (9) with bounce-back boundary conditions

$$\varphi|_{\Gamma_-}(x,v,t) = \gamma\varphi|_{\Gamma_+}(x,-v,t),\ \ (x,v)\in\Gamma_-. \tag{28}$$

In [22], the spectrum of the streaming operator T and the explicit expression of the streaming semigroup $(U(t))_{t\geq 0}$ have been obtained. Moreover, the compactness of $R_1(t)$ also was proved when Ω is bounded.

For any $(x,v) \in \overline{\Omega}\times\mathbb{R}^N$, define

$$\begin{aligned} t_\pm(x,v) &= \sup\{t>0; x\pm sv \in \Omega, 0<s<t\}\\ &= \inf\{s>0; x\pm sv \notin \Omega\}.\end{aligned}$$

For the sake of convenience, we set

$$\tau(x,v) := t_-(x,v) + t_+(x,v),\ \ (x,v)\in\overline{\Omega}\times\mathbb{R}^N.$$

Hence, for $(x,v) \in \Gamma_\pm$, one has $t_\pm(x,v) = 0$, $t_\mp(x,v) > 0$. In all cases it always holds that $x\mp t_\mp(x,v)v \in \Gamma_\mp, \forall(x,v) \in \overline{\Omega}\times V$. So we can define a function by

$$\kappa(x,v) = \int_{-t_+(x,v)}^{t_-(x,v)} \sigma(x-sv,v)ds,\ \ (x,v)\in\overline{\Omega}\times\mathbb{R}^N.$$

For $p \in [1,\infty)$, we introduce the following Sobolev space

$$W_p = \{\varphi \in X_p; v\cdot\nabla_x\varphi \in X_p\}$$

here we also denote $L^p(\Omega\times\mathbb{R}^N)$ by X_p. A suitable L^p-space for the traces is defined by

$$L^p_\pm := L^p(\Gamma_\pm; \big|\, v\cdot\eta(x)|d\gamma(x)\otimes dv)$$

$d\gamma(\cdot)$ being the Lebesgue measure on $\partial\Omega$.

For any $\varphi \in W_p$, we can define the traces $\varphi|_{\Gamma_\pm}$ on $\Gamma_\pm$, however, the traces do not belong to $L^p_\pm$ but to a certain weighted L^p space. Define

$$\widetilde{W}_p = \{\varphi \in W_p; \varphi|_{\Gamma_-} \in L^p_\pm\}.$$

We assume that the collision frequency $\sigma \in L^\infty(\Omega \times \mathbb{R}^N)$ is a non-negative measurable function on $\Omega\times\mathbb{R}^N$ and an even function in the velocity, i.e. $\sigma(x,v) = \sigma(x,-v)$, for any $(x,v) \in \Omega\times\mathbb{R}^N$. Define the streaming operator with the bounce-back boundary condition by

$$T\varphi(x,v) = -v\cdot\nabla_x\varphi(x,v) - \sigma(x,v)\varphi(x,v)$$

with domain

$$D(T) := \{\varphi \in \widetilde{W}_p; \varphi|_{\Gamma_-}(x,v) = \gamma\varphi|_{\Gamma_+}(x,-v)\}$$

where $0 < \gamma < 1$. It is known from [29] that, T is a generator of a non-negative C_0-semigroup of contractions $(U(t))_{t\geq 0}$ in X_p.

In order to get the resolvent of T, we define the following operators depending on the parameter λ. Let $M_\lambda \in \mathcal{B}(L^p_-, L^p_+)$ be defined by

$$M_\lambda u(x,v) = u(x-\tau(x,v)v, v)\cdot e^{-\int_0^{\tau(x,v)}(\lambda+\sigma(x-sv,v))ds}, \quad (x,v)\in\Gamma_+,$$

and let $B_\lambda \in \mathcal{B}(L^p_-, X_p)$ be defined by

$$B_\lambda u(x,v) = u(x-t_-(x,v)v, v)\cdot e^{-\int_0^{t_-(x,v)}(\lambda+\sigma(x-sv,v))ds}, \ (x,v)\in\Omega\times\mathbb{R}^N.$$

In the same way, let $G_\lambda \in \mathcal{B}(X_p, L^p_+)$ be given as

$$G_\lambda\varphi(x,v) = \int_0^{\tau(x,v)}\varphi(x-sv,v)\cdot e^{-\int_0^s(\lambda+\sigma(x-tv,v))dt}ds, \ (x,v)\in\Gamma_+,$$

and let $C_\lambda \in \mathcal{B}(X_p)$ be given as

$$C_\lambda\varphi(x,v) = \int_0^{t_-(x,v)}\varphi(x-tv,v)\cdot e^{\int_0^t(\lambda+\sigma(x-sv,v))ds}dt, \ (x,v)\in\Omega\times\mathbb{R}^N.$$

Theorem 22 [30] *Let $0 < \gamma < 1$ be fixed and let $H \in \mathcal{B}(L^p_+, L^p_-)$ be defined by*

$$H(\varphi|_{\Gamma_+})(x,v) = \gamma\varphi|_{\Gamma_+}(x,-v), \ (x,v)\in\Gamma_-. \tag{29}$$

If $\lambda\in\mathbb{C}$ is such that $1\in\rho(M_\lambda H)$, then $\lambda\in\rho(T)$ and

$$(\lambda I - T)^{-1} = B_\lambda H(I - M_\lambda H)^{-1}G_\lambda + C_\lambda. \tag{30}$$

In particular, if there is $\lambda_0\in\mathbb{R}$ such that

$$r(M_\lambda H) < 1, \ \forall \mathrm{Re}\lambda > \lambda_0,$$

then $\{\lambda\in\mathbb{C}; \mathrm{Re}\lambda>\lambda_0\}\subset\rho(T)$ and the resolvent of T is given by (30).

Remark 23 *For transport equations with reentry boundary conditions, the resolvent of T has the form* (30)*, in which C_λ is the resolvent of the transport operator with no-reentry boundary conditions.*

Theorem 24 [22] *For any $k\in\mathbb{Z}$, define function*

$$F_k(x,v) = \frac{\log\gamma-\kappa(x,v)}{\tau(x,v)} - i\frac{2k\pi}{\tau(x,v)}, \quad \forall (x,v)\in\Omega\times\mathbb{R}^N. \tag{31}$$

Then,

$$\sigma(T) = \overline{\bigcup_{k\in\mathbb{Z}}\mathcal{R}_{\mathrm{ess}}(F_k)} \tag{32}$$

where $\mathcal{R}_{\mathrm{ess}}(F_k)$ stands for the essential range of F_k, i.e.,

$$\mathcal{R}_{\mathrm{ess}}(F_k) = \Big\{u\in\mathbb{C}; \big|\{(x,v)\in\Gamma_+; |F_k(x,v)-u|<\varepsilon\}\big| \neq 0, \forall\varepsilon>0\Big\}.$$

Theorem 25 [22] *Let the collision frequency σ be space homogeneous. Then the C_0-semigroup $(U(t))_{t\geq0}$ generated by T in X_p is given by*

$$U(t) = \sum_{n=0}^{\infty}U_n(t), \ \forall t\geq 0, \tag{33}$$

where, for any fixed $t\geq0$,

$$[U_0(t)\varphi](x,v) = \varphi(x-tv,v)e^{-\sigma(v)t}\chi_{\{t<t_-(x,v)\}},$$

and for any $n\geq0$,

$$[U_{2n+2}(t)\varphi](x,v) = \gamma^{2n+2}e^{-\sigma(v)t}\chi_{I_{2n+1}(x,v)}(t)\cdot \varphi(x-tv+(2n+2)\tau(x,v)v, v)$$

and

$$[U_{2n+1}(t)\varphi](x,v) = \gamma^{2n+1}e^{-\sigma(v)t}\chi_{I_{2n}(x,v)}(t)\cdot \varphi(x+tv-2t_-(x,v)v-2n\tau(x,v)v, -v)$$

for any $\varphi\in X_p$, and any $(x,v)\in\Omega\times\mathbb{R}^N$, with

$$I_k(x,v) = \big[k\tau(x,v)+t_-(x,v); (k+1)\tau(x,v)+t_-(x,v)\big], \ \text{for any } k\in\mathbb{N}.$$

Theorem 26 [22] *Let $1<p<\infty$ and let Ω be a bounded convex set in $\mathbb{R}^N$. If K is a regular operator and the collision frequency σ is space homogeneous, then $R_1(t)$ is compact and consequently $\sigma_{\mathrm{ess}}(V(t)) = \sigma_{\mathrm{ess}}(U(t))$ for all $t\geq0$.*

4 Some open problems

In this section we provide some concerning problems for further discussion.

The researches in L^1 space are less than in L^p $(1 < p < \infty)$ spaces for transport theory. As we know, the most results of the stability of the essential and critical spectra were obtained in Hilbert spaces, so the applications to transport theory also were limited in L^2 space. Thanks to B. Lods and M. Sbihi in [10] we can extend the applications to L^p $(1 < p < \infty)$ spaces by an interpolation argument. Unfortunately, it is invalid in L^1 space. For example, in the L^p $(1 < p < \infty)$ theory, $0 \leq t \mapsto R_1(t)$ is norm continuous [8], but this is never true in L^1, see [9]. So we need to establish more L^1 theory with different technicalities. For instance, in [11, 12, 13], do the weak compactness of the remainder terms $R_1(t)$, $R_2(t)$ in L^1 (Theorem 22) remain true or not for other cases?

From the applications in section 3 we have seen that the space homogeneous of the collision frequency $\sigma(v)$ are required. We wish to extend these results to general collision frequencies $\sigma(x, v)$. Inspired by [4], we assume that the collision frequency can be approximated in L^∞ by degenerate collision frequencies of the form

$$\sum_{i \in I} \sigma_1^i(x)\sigma_2^i(v) \ \ (\ I \text{ is a finite set }).$$

Note that before researching the compactness or norm continuity of the remainder term, one has to prove that the streaming operator T generates a C_0-semigroup $(U(t))_{t\geq 0}$. For no-reentry boundary condition, the expression of $(U(t))_{t\geq 0}$ is given by (6). However, for higher dimensional transport equations with reentry boundary conditions, we only know that the streaming operator can generate a C_0-semigroup. For most situations, the explicit expressions of the streaming semigroups for transport equation with reentry boundary conditions have not been obtained yet. Maybe we can imitate the paper [22] in calculation method to study this problem.

Most of the papers concerning with the reentry boundary conditions mainly investigate the spectrum of the streaming operator T. But the explicit spectrum structure of the streaming semigroup $(U(t))_{t\geq 0}$ have not been derived yet, even for the one-dimensional situation.

It has known from the Theorem 2 that, if $\omega_{\text{crit}}(U(t)) < \omega(U)$, then the spectrum determined growth assumption holds, i.e., $\omega(U) = s(T)$, which is a useful tool to prove the exponential stability of the distributed parameter system, see [24]. However, the condition $\omega_{\text{crit}}(U(t)) < \omega(U)$ is not easy to check in practice. Clearly, further study of the property of the critical spectrum about the C_0-semigroup is necessary from the application point of view. An important thing is that we should obtain an easy test condition.

If Ω is unbounded, due to the losing the compactness, K is not longer to be regular operator [22]. In order to study the spectrum of transport C_0-semigroup $(V(t))_{t\geq 0}$, we should establish a spectral mapping of critical spectrum. So the norm continuity of the mapping $0 \leq t \mapsto R_1(t)$ is necessary. Even in this case, computation of the spectrum and critical spectrum of the streaming semigroup is also a hard work. Therefore, some new tricks need to develop.

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