The Self-Similar Solutions of a Diffusion Equation

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Abstract: This paper discusses the diffusion equation with a damping term as follows

$$u_t = div(|Du^m|^{p-2}Du^m) - u^{q_1}|Du^m|^{p_1},$$

where $p > 2, m > 1$, and $p > p_1, q_1 + p_1 m > m(p-1) > 1$. By the standard Picard iteration method, a sufficient condition is given to the existence of the singular self-similar solutions. Moreover, the paper gives a classification for these singular self-similar solutions.

Key–Words: Diffusion equation, Damping term, Picard iteration method, Self-similar solution, Singular solution

1 Introduction

Consider the following diffusion equation

$$u_t = div(|Du^m|^{p-2}Du^m) - u^{q_1}|Du^m|^{p_1} \qquad (1)$$

where $-u^{q_1}|Du^m|^{p_1}$ is the damping term. When $q_1 = 0$, $m = 1, p = 2$, (1) is the well-known Hamilton-Jacobi [1] equation. When $p_1 = 0$, (1) is the well-known evolutionary p-Laplacian equation with the absorption term. When $p = 2$, $p_1 = 0$, (1) is the well-known porous media equation with the absorption term. These equations come from many fields such as physics, fluid mechanics et al.

For example, in the study of water infiltration through porous media, Darcy's linear relation

$$V = -K(\theta)\nabla\phi$$

satisfactorily describes flow conditions provided the velocities are small. Here, V represents the seepage velocity of water, θ is the volumetric moisture content, $K(\theta)$ is the hydraulic conductivity and ϕ is the total potential, which can be expressed as the sum of a hydrostatic potential $\psi(\theta)$ and a gravitational potential z

$$\phi = \psi(\theta) + z. \qquad (2)$$

However, (2) fails to describe the flow for large velocities. To get a more accurate description of the flow in this case, several nonlinear versions of (2) have been proposed. One of these versions is

$$V^\alpha = -K(\theta)\nabla\phi \qquad (3)$$

where α is a positive constant, cf. [2-4] and the references therein. If it is assumed that infiltration takes place in a horizontal column of the medium, by the continuity equation

$$\frac{\partial\theta}{\partial t} + \frac{\partial V}{\partial x} = 0,$$

(2) and (3) give

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}(D(\theta)^p|\theta_x|^{p-1}\theta_x)$$

with $\frac{1}{p} = \alpha$ and $D(\theta) = K(\theta)\psi'(\theta)$. Choosing $D(\theta) = D_0\theta^{m-1}$ (cf. [5-6]), one obtains (1) without the damping term $-u^{q_1}|Du^m|^{p_1}$, u being the volumetric moisture content.

Another example where equation (1) appears is the one-dimensional turbulent flow of gas in a porous medium (cf. [7]), where u stands for the density, and the pressure is proportional to u^{m-1}; see also [8]. Typical values of p are 1 for laminar (non-turbulent) flow and $\frac{1}{2}$ for completely turbulent flow.

Due to its degeneracy or singularity, we are only able to get the existence of the weak solution of (1) in general, there are many references, for examples [9,10] and the reference therein, to probe the existence or the uniqueness of the weak solutions for the Cauchy problem or the initial boundary value problem of (1). At the same time, if $p = 2$, $p_1 = 0$, there are also a lot of papers to study the existence of the self-similar solutions of (1) such as [11-20]. Several years ago, [21] had studied the existence of the self-similar singular solutions of the following quasilinear parabolic equations with the nonlinear gradient term

$$u_t = \triangle u - u^{q_1}|Du|^{p_1}, \tag{4}$$

and

$$u_t = \triangle u^m - u^{q_1}|Du|^{p_1}. \tag{5}$$

Clearly, equation (4) is the same as equation (1) with $m = 1, p = 2$, and equation (5) is just slightly different from equation (1) with $p = 2$ in the format. From these known results, if one wants to get the existence of the self-similar solutions of the equations, which are with the format as (1), one needs to use Picard's iteration method and some fixed point theory. Generally speaking, the essential difficulties come from the following two aspects. The first difficulty comes from that how to describes the relationship between the exponents m, p of the function u and the exponents q_1, p_1 of the gradient term. The second difficulty comes from that how the nonlinear damping term $u^{q_1}|Du^m|^{p_1}$ affects the existence of the self-similar solution. However, one should construct some special functions or some special inequalities to get the self-similar solutions of (1) according to the specific exponents m, p, p_1, q_1. This is the main reason of that the authors of [21] deal with (4) and (5) into two cases respectively. To the best knowledge of the author, all the methods of the above references are not able to deal with the case of (1) with $m = 1, p \neq 2$, an essential difficulty comes that the difference between $\int |z'(r)|^{p-1}dr$ and $\int z'(r)dr = z + c$. In other words, all above references only need to deal with the simple integral $\int z'(r)dr = z + c$, while, if $p \neq 2$, we should face the more difficult integral $\int |z'(r)|^{p-1}dr$ in many estimates we needed.

In the paper, we always assume that $p > 2, m > 1$, and

$$p > p_1, q_1 + p_1 m > m(p-1) > 1. \tag{6}$$

Let us introduce some definitions and some known results.

Definition 1 *The self-similar solution means that the solution $u(x,t)$ of (1) has the form as*

$$u(x,t) = t^{-\alpha} f(|x|t^{-\beta}). \tag{7}$$

By a directly calculation, we have

$$\alpha = \frac{p - p_1}{p(q_1 + (p_1 - p + 1)m) - (1 + m - mp)(p - p_1)},$$

$$\beta = \frac{q_1 + (p_1 - p + 1)m}{p(q_1 + (p_1 - p + 1)m) - (1 + m - mp)(p - p_1)}.$$

The condition (6) makes sure of that $\alpha > 0, \beta > 0$.

The equation (1) can be transformed to the following ordinary differential equation

$$(|(f^m)'|^{p-2}(f^m)')' + \frac{n-1}{r}|(f^m)'|^{p-2}(f^m)'$$
$$+\beta r f' + \alpha f - f^{q_1}|(f^m)'|^{p_1} = 0, \tag{8}$$

with variable $r = |x|t^{-\beta}$, and we introduce the initial condition of (8) as

$$f(0) = a > 0, f'(0) = 0. \tag{9}$$

Definition 2 *The singular solution $u(x,t)$ of (1) means that $u(x,t)$ is continuous in $R^N \times (0, +\infty) \setminus (0,0)$, $u(x,t) \geq 0$, which is not identical to zero and satisfies that*

$$\lim_{t\to 0} \sup_{|x|>\varepsilon} u(x,t) = 0, \forall \varepsilon > 0. \tag{10}$$

If moreover,

$$\lim_{t\to 0} \int_{|x|\leq \varepsilon} u(x,t) = \infty, \forall \varepsilon > 0, \tag{11}$$

then, $u(x,t)$ is called a strong singular solution of (1).

By the definition of the self-similar solution, (10) is equivalent to that

$$\lim_{t\to 0} r^{\frac{\alpha}{\beta}} f(r) = 0, \tag{12}$$

and the condition (11) is equivalent to that

$$\lim_{t\to 0} r^{n\beta-\alpha} \int_{r\leq \varepsilon t^{-\beta}} f(r)dr = 0. \forall \varepsilon > 0. \tag{13}$$

If $N\beta < \alpha$, and the solution f of equation (8) satisfies (12), then $f \in L^1(0, \infty; r^{N-1}dr)$, and so f satisfies (13). This fact means that the function $u(x,t)$ defined as (7) satisfies (10) and (11), i.e. $u(x,t)$ is a strong singular solution of (1).

For many special cases of the equation (1), an important way to show the large time behavior of the global solutions as $t \to \infty$, is to compare them with their singular self-similar solutions, one refers to the references [22-26]. However, Papers [22-26] are base on the assumption the uniqueness of the strong singular solutions to the corresponding equations. But, in general, since equation (1) contains the damping term $-u^{mq_1}|\nabla u^{mp_1}|$, the uniqueness of the solutions generally is not true, one can refer to [31]-[35]. Thus how to compared to its singular self-similar solution, base the large time behavior of the global solution of (1), remains open for a long time.

However, recently, in [36], the author had based the large time behavior of the global weak solution of

(1) in another way. In details, we had introduced the following definition of the global weak solution of (1) with initial value

$$u(x,0) = u_0(x), x \in R^N. \quad (14)$$

Definition 3 *A nonnegative function $u(x,t)$ is called a weak solution of (1)-(14), if u satisfies*

(i)

$$u \in C(0,T; L^1(R^N)) \cap L^\infty(R^N \times (\tau,T)),$$

$$u^m \in L^p_{\text{loc}}(0,T; W^{1,p}(R^N)), \quad (15)$$

$$u_t \in L^1(R^N \times (\tau,T)),\ \forall \tau > 0; \quad (16)$$

(ii)

$$\int_S [u(x,t)\varphi_t(x,t) - \mid Du^m \mid^{p-2} Du^m \cdot D\varphi$$

$$-u^{q_1} \mid Du^m \mid^{p_1} \varphi] dxdt = 0,\ \ \forall \varphi \in C_0^1(S) \quad (17)$$

where $S = R^N \times (0,\infty)$.

(iii)

$$\lim_{t\to 0} \int_{R^N} \mid u(x,t) - u_0(x) \mid dx = 0. \quad (18)$$

The method of [36] bases on comparing the weak solution of (1)-(14) to the Barenblatt-type solution of (1) without the damping term $-u^{mq_1}|\nabla u^{mp_1}|$. It is not difficult to verify that

$$E_c = t^{\frac{-1}{\mu}} \{[b - \frac{m(p-1)-1}{mp} (N\mu)^{\frac{-1}{(p-1)}}$$

$$\times (\mid x \mid t^{\frac{-l}{N\mu}})^{\frac{p}{p-1}}]_+\}^{\frac{p-1}{m(p-1)-1}}$$

is the Barenblatt-type solution of the Cauchy problem

$$u_t = \text{div}(\mid Du^m \mid^{p-2} Du^m),\ (x,t) \in S, \quad (19)$$

$$u(x,0) = c\delta(x),\ x \in R^N, \quad (20)$$

where

$$\mu = m(p-1) - 1 + \frac{p}{N}),$$

$$c = \int_{R^N} u_0(x)dx - \int_0^\infty \int_{R^N} u^{q_1} \mid Du^m \mid^{p_1} dxdt$$

and b is a constant such that

$$b = \int_{R^N} E_c(x,t)dx,$$

and δ denotes the usual Dirac Delta function. In [37], we have got the following theorems.

Theorem 4 *Let $m(p-1) > 1$. If E_c is a unique solution of (19)-(20), then the solution u of (1) with (14) satisfies*

$$t^{\frac{l}{\mu}} \mid u(x,t) - E_c(x,t) \mid \to 0, \text{as } t \to \infty, \quad (21)$$

uniformly on the sets $\{x \in R^N :\mid x \mid < at^{\frac{-l}{\mu N}}, a > 0\}$.

Theorem 5 *Suppose $m(p-1) > 1, q_1 + mp_1 > m(p-1) - 1$ and*

$$\mid x \mid^\alpha u_0(x) \le B,\ \lim_{|x|\to\infty} \mid x \mid^\alpha u_0(x) = C,$$

where α, B and C are constants with $\alpha \in (0, \frac{p-p_1}{q_1+mp_1})$. If the solution $u(x,t)$ of (1) and (14) satisfies

$$\mid Du^m \mid \ge 1, (x,t) \in S, \quad (22)$$

then

$$t^{\frac{1}{q_1-1}} u(x,t) \to C^*, \text{as } t \to \infty, \quad (23)$$

uniformly on the sets

$$\{x \in R^N : \mid x \mid \le at^{\frac{1}{\beta}}, a > 0\},$$

where

$$C^* = (\frac{1}{q_1 - 1})^{\frac{1}{q_1-1}}$$

and

$$\beta = \frac{p(q_1 + mp_1 - 1) - p_1(m(p-1)-1)}{q_1 + mp_1 - m(p-1)}.$$

Theorem 6 *Suppose $1 < m(p-1) < q_1 + mp_1 < m(p-1) + \frac{p}{N}$ and $\alpha > \frac{p-p_1}{q_1+mp_1-m(p-1)}$,*

$$\mid x \mid^\alpha u_0(x) \le B, \int_{R^N} u_0(x)dx > 0.$$

Assume that (1) has a unique very singular solution $U(x,t)$. Then the solution of (1) with (14) satisfies

$$t^{\frac{1}{q_1-1}} \mid u(x,t) - U(x,t) \mid \to 0, \text{as } t \to \infty, \quad (24)$$

uniformly on the sets

$$\{x \in R^N : \mid x \mid \le at^{\frac{1}{\beta}}\}.$$

In this paper we will discuss the existence of the singular self-similar solutions of (1) and give a complete classification for them.

At the end of this introduction section, we would like to point out that, when $m(p-1) < 1$, there is few reference on the self-similar solution in this case, though for some special q_1, p_1, the well posedness of the weak solution had been studied in [30]. By the way, there are many references devoted to the existence of the self-similar solutions of the general parabolic equation, for examples, see [27-29].

2 The self-similar solutions and the classification

In this section, according to the relationship between the solutions of (8) and the initial value $f(0)$, we will give a classification of the solutions of (8). Base on this classification, we can get the existence of the singular self-similar solution of (1) and the corresponding classification.

Let $z = f^m$. Then (8)-(9) are changed to

$$(|z'|^{p-2}z')' + \frac{n-1}{r}|z'|^{p-2}z' + \beta r(z^{\frac{1}{m}})' +\alpha z^{\frac{1}{m}} - z^{\frac{q_1}{m}}|z'|^{p_1} = 0, \quad (25)$$

$$z(0) = b = a^m > 0, z'(0) = 0. \quad (26)$$

If we transform (25)-(26) into the equivalent integral equation, and use the classical Picard's iteration, for any given $b > 0$, we know that there is an unique solution $f(r) = f(r, b)$ to (25)-(26). Assuming that $(0, R(b))$ is the largest interval such that $z > 0$, then

$$z'(r) < 0, r \in (0, R(b)).$$

Moreover, we have

(i) $R(a) = \infty$, or

(ii) $R(b) < \infty, f(R(a)) = 0$.

At the same time, (12) is transformed to

$$\lim_{r\to\infty} r^{\frac{\alpha}{\beta}} z^{\frac{1}{m}}(r) = 0. \quad (27)$$

Let $v = |z'|^{p-2}z' = -|z'|^{p-1}$. Then (25) can be rewritten as

$$v' = -\frac{n-1}{r}v + \frac{\beta}{m}rz^{\frac{1}{m}-1}|v|^{\frac{1}{p-1}-\alpha z^{\frac{1}{m}}} + z^{\frac{q_1}{m}}|v|^{\frac{p_1}{p-1}}. \quad (28)$$

For any $\lambda, \eta > 0$, we set

$$S_{\lambda,\eta} = \{(z, v) \mid 0 < z \le \eta, -\lambda z^\theta < -|v|^{\frac{1}{p-1}} < 0\},$$

$$S_\lambda = \{(z, v) \mid 0 < z, -\lambda z^\theta < -|v|^{\frac{1}{p-1}} < 0\},$$

where the constant θ satisfies

$$\frac{1}{m} < \theta < \frac{m+1}{2m}. \quad (29)$$

If we choose the constants

$$r_{\lambda,\eta} = \frac{m\alpha}{\beta\lambda}\eta^{1-\theta} + \frac{m\theta\lambda}{\beta}\eta^{\theta-\frac{1}{m}},$$

then, by (28) and using (29), when $r > r_{\lambda,\eta}$,

$$\frac{v'}{(z^\theta)'} < \frac{-\beta r}{m\theta}z^{\frac{1}{m}-\theta} + \frac{\alpha z^{\frac{1}{m}+1-\theta}}{\theta} - |v|^{\frac{1}{p-1}}$$

$$< [-\frac{\alpha}{\theta\lambda}\eta^{1-\theta} - \lambda\eta^{\theta-\frac{1}{m}}]z^{\frac{1}{m}-\theta} + \frac{\alpha}{\theta}z^{\frac{1}{m}-2\theta+1} < -\lambda. \quad (30)$$

Thus, we have the following lemma.

Lemma 7 *For any $\lambda, \eta > 0$, there is a constant $r_{\lambda,\eta}$ such that, when $r > r_{\lambda,\eta}$, $S_{\lambda,\eta}$ is a positive fixed set, i.e. if $(z(r_{\lambda,\eta}), v(r_{\lambda,\eta})) \in S_{\lambda,\eta}$, then, when $r > r_{\lambda,\eta}$, the orbit of equation (28) $(z(r), v(r))$ is in $S_{\lambda,\eta}$.*

Supposing that $(0, R(a))$ is the largest interval of the existence of the positive solution for (25), we can prove that if $R(a) = \infty$, then the orbit of equation (28) will enter in S_1 at last. Otherwise, by Lemma 7, there is a large enough constant r_0, such that

$$-z^\theta(r) \ge v(r), \forall r \ge r_0,$$

and so, when $r \to \infty$,

$$\frac{z^{1-\theta}}{1-\theta} \le \frac{z^{1-\theta}(r_0)}{1-\theta} - (r - r_0) \to -\infty,$$

This is a contradiction.

Now, we define the following three sets:

$$A = \{a > 0 \mid R(a) < \infty, z'(R(a)) < 0\},$$

$$B = \left\{ a > 0 \mid \begin{array}{l} \text{the orbit } (z, v) \text{ of equation (28)} \\ \text{will enter in} S_1 \text{from the point} \\ (a, 0) \end{array} \right\}$$

and

$$C = \{a > 0 \mid R(a) < \infty, z'(R(a)) = 0\}.$$

Since for any $a \in B$, when $r < R(a)$ but close to $R(a)$, the corresponding orbit of (28) satisfies that

$$z' + z^\theta > 0,$$

which implies that $R(a) = \infty$. It means that the sets A, B and C are not intersection each other, and moreover, $A \cup B \cup C = (0, \infty)$.

The main result of the present paper is the following theorem

Theorem 8 *Supposing that $p > 2, m > 1$, and condition (6) is true, then we have*

(I) If $\alpha \le N\beta$, then $R(b) = \infty$, and moreover,

$$\lim_{r\to\infty} \inf r^{\frac{\alpha}{\beta}} f(r; b) > 0.$$

(II) If $\alpha > N\beta$, then there are two open set A, B and a closed set C, such that

$$A \cap B = B \cap C = A \cap C = \emptyset,$$

and

$$A \cup B \cup C = (0, \infty).$$

Moreover, if $0 < b \ll 1$ *and* $(0, b) \subset A$*; if* $b \gg 1$*, and* $(b, \infty) \subset B$*, such that*

(i) if $b \in A$*, then* $R(b) < \infty$*,* $(f^m)'(R(b)) < 0$*;*

(ii) if $b \in B$*, then* $R(b) = \infty$*,*

$$\lim_{r\to\infty} (f(r; b), f'(r, b)) = (0, 0)$$

and there is a constant $k(b) > 0$ *such that*

$$\lim_{r\to\infty} r^{\frac{\alpha}{\beta}} f(r; b) = k(b);$$

(iii) if $b \in C$*, then* $R(b) < \infty$*,*

$$(f^m)'(R(b)) = 0,$$

and

$$\lim_{r\to R(b)} (f^{m-1}(r, b))' = -\frac{m-1}{m}\beta R(b).$$

3 The proof of Theorem

Lemma 9 *The set* A *is a nonempty open set.If* $0 < a \ll 1$*, then* $(0, a) \subset A$*.*

The details of the proof of the lemma can be found in [14] and [22].

Lemma 10 *If* $R(a) = \infty$*, then there is* $k(a) > 0$*, such that* $\lim_{r\to\infty} r^{\frac{\alpha}{\beta}} z^{\frac{1}{m}} = k(a)$*.*

Proof. Since $f'(r) < 0$, $f(r) > 0$, we have

$$\lim_{r\to\infty} f(r) = c \geq 0. \tag{31}$$

Moreover,

$$0 = \lim_{r\to\infty} \frac{f(r)}{r}$$

$$= \lim_{r\to\infty} \frac{\int_0^r f'(r)dr + f(0)}{r} = \lim_{r\to\infty} f'(r), \tag{32}$$

since $z = f^m$,

$$\lim_{r\to\infty} z|z'|^{p-2} = 0. \tag{33}$$

Setting

$$h(r) = \int_{R(a)}^r |z'|^{p-2} z' dr + c = -\int_r^\infty |z'|^{p-2} z' dr + c,$$

$$H(r) = \mu h(r) + r|z'|^{p-2} z',$$

where c and μ are the constants to be chosen later. By the fact that

$$\lim_{r\to\infty} z' = 0,$$

it is easy to know that

$$h'(r) = |z'|^{p-2} z'.$$

We can assert that when $r \gg 1$, the sign of $H(r)$ is fixed. Clearly,

$$H'(r) = (\mu + 1)|z'|^{p-2} z' + r(|z'|^{p-2} z')'$$

$$= (\mu + 1)|z'|^{p-2} z' + r(-\frac{n-1}{r}|z'|^{p-2} z' - \beta r(z^{\frac{1}{m}})'$$

$$-\alpha z^{\frac{1}{m}} + z^{\frac{q_1}{m}} |z'|^{p_1}),$$

$$= (\mu + 2 - n)|z'|^{p-2} z' - \beta r^2 (z^{\frac{1}{m}})'$$

$$-\alpha r z^{\frac{1}{m}} + r z^{\frac{q_1}{m}} |z'|^{p_1}. \tag{34}$$

If there is a $r_0 \gg 1$, $H(r_0) = 0$, then we can choose the constant $c > 0$, such that

$$c > \frac{z^{p-1} m^{p-1}}{\mu(\beta r_0)^{p-1}} + z(r_0).$$

Since

$$\int_r^\infty |z'|^{p-1} dr < \int_r^\infty |z'| dr = z(r),$$

by the above choice of the constant c, we know that

$$\frac{\beta r^2}{m} z^{\frac{1}{m}-1} (\frac{\mu h(r_0)}{r_0})^{\frac{1}{p-1}} - \alpha r_0 z^{\frac{1}{m}} > 0.$$

By (34), we have

$$H'(r_0) > 0. \tag{35}$$

By this fact, it is easy to know that when $r \gg 1$, $H(r)$ is with fixed sign .

If $H(r) < 0$, noticing that

$$\lim_{r\to\infty} \frac{\frac{-r|z'|^{p-1}}{\int_r^\infty |z'|^{p-2} z' dr + c}}{\frac{r|z'|}{z}} = \lim_{r\to\infty} \frac{-z|z'|^{p-2}}{\int_r^\infty |z'|^{p-2} z' dr + c} = 0,$$

then, when $r \gg 1$,

$$\mu < \frac{-r|z'|^{p-1}}{-\int_{R(a)}^r |z'|^{p-2} z' dr + c} < -\frac{rz'}{z},$$

$$-z'(r) > \mu r^{-1} z(r), \lim_{r\to} \frac{-z' r}{z} > \mu. \tag{36}$$

Using equation (25) again,

$$(-|z'|^{p-1})' = -\frac{n-1}{r}|z'|^{p-2} z'$$

$$-\beta r(z^{\frac{1}{m}})' - \alpha z^{\frac{1}{m}} + z^{\frac{q_1}{m}} |z'|^{p_1}$$

$$> -\frac{\beta r}{m} z^{\frac{1}{m}} \frac{z'}{z} - \alpha z^{\frac{1}{m}}$$

$$> (\frac{\beta\mu}{m} - \alpha) z^{\frac{1}{m}} := \delta z^{\frac{1}{m}},$$

where we have chosen the constant μ such that

$$\delta = (\frac{\beta\mu}{m} - \alpha) > 0.$$

By this inequality, it is easy to know that there is a constant c such that

$$-z' \geq c z^{\frac{m+1}{mp}},$$

which implies that there is r_0 such that $z(r_0) = 0$, it contradicts $R(a) = \infty$. So, when $r \gg 1$, $H(r) > 0$. Noticing that

$$\int_r^\infty |z'|^{p-1} dr < z(r),$$

then

$$\frac{|z'|^{p-1}}{z+c} < \frac{|z'|^{p-1}}{\int_{R(a)}^r |z'|^{p-2} z' dr + c} < \frac{\mu}{r}.$$

Since z is bounded, by the above inequality, when $r \gg 1$,

$$|z'|^{p-1} < \frac{\mu(z+c)}{r} < \frac{c_1}{r}. \quad (37)$$

Now, we differentiate the two sides of equation (25), denote $w(r) = v' = (p-1)|z'|^{p-2} z''$, and denote

$$G(r) = -\frac{n-1}{r^2}|z'|^{p-1} + \frac{\beta}{m} z^{\frac{1-m}{m}} |z'|$$

$$+ \frac{\beta(1-m)r}{m^2} z^{\frac{1-2m}{m}} z'$$

$$-\frac{\alpha}{m} z^{\frac{1-m}{m}} z' + \frac{q_1}{m} z^{\frac{q_1-m}{m}} |z'|^{p_1+1},$$

$$F(r) = \frac{n-1}{r} + \frac{\beta r}{m(p-1)} z^{\frac{1}{m}-1} |v|^{\frac{1}{p-1}-1}$$

$$-\frac{p_1}{p-1} z^{\frac{q_1}{m}} |v|^{\frac{p_1}{p-1}-1},$$

then we have

$$w' + F(r) w = G(r). \quad (38)$$

By (37), it is easy to know that, when $r \gg 1$,

$$G(r) > 0, F(r) > 0.$$

Since $z'(r) < 0, \lim_{r\to\infty} z'(r) = 0$, there exists $r_0 \gg 1$ such that $z''(r_0) > 0$. Then $w(r_0) > 0$. By (38), when $r \gg 1$,

$$w(r) > 0, z''(r) > 0. \quad (39)$$

By equation (25),

$$0 < (-|z'|^{p-1})' = -\frac{n-1}{r}|z'|^{p-2} z' - \beta r (z^{\frac{1}{m}})'$$

$$-\alpha z^{\frac{1}{m}} + z^{\frac{q_1}{m}} |z'|^{p_1}, \quad (40)$$

so

$$\lim_{r\to\infty} (-\beta r (z^{\frac{1}{m}})' - \alpha z^{\frac{1}{m}}) \geq 0,$$

and

$$r^{\frac{\alpha}{\beta}} z^{\frac{1}{m}} < c, \lim_{r\to\infty} z = 0. \quad (41)$$

Now, we rewrite (40) as

$$\frac{n-1}{r}|z'|^{p-2} z' + \beta r (z^{\frac{1}{m}})' + \alpha z^{\frac{1}{m}} - z^{\frac{q_1}{m}} |z'|^{p_1} < 0$$

i.e.

$$\beta r (z^{\frac{1}{m}})' + \alpha z^{\frac{1}{m}} - \frac{n-1}{r} m^{p-1} z^{(1-\frac{1}{m})(p-1)} |(z^{|\frac{1}{m}})'|^{p-1}$$

$$-(z^{\frac{1}{m}})^{q_1} m^{p_1} z^{(1-\frac{1}{m})p_1)} |(z^{|\frac{1}{m}})'|^{p_1} < 0. \quad (42)$$

(i) If $p_1 \geq 1$, then

$$-|(z^{|\frac{1}{m}})'|^{p_1} \geq -|(z^{|\frac{1}{m}})'|.$$

For any $\varepsilon > 0, \varepsilon \ll 1$, since $z(r) \to 0, \exists r_\varepsilon \gg 1$, when $r \geq r_\varepsilon$, by (42),

$$\alpha z^{\frac{1}{m}} + \beta(r+\varepsilon)(z^{\frac{1}{m}})' < 0,$$

$$z(r) \leq c(r+\varepsilon)^{-\frac{\alpha m}{\beta}} \leq c_1 r^{-\frac{\alpha m}{\beta}}, \quad (43)$$

$$-z'(r) < \mu r^{-1} z(r) \leq c_2 r^{-1-\frac{\alpha m}{\beta}}. \quad (44)$$

(ii) If $0 < p_1 < 1, q_1 \geq 1$, similar to (i), when $r \geq r_\varepsilon \gg 1$,

$$-(z^{\frac{1}{m}})^{q_1} m^{p_1} z^{(1-\frac{1}{m})p_1} |(z^{|\frac{1}{m}})'|^{p_1} > -\alpha\varepsilon z^{\frac{1}{m}}$$

$$\alpha z^{\frac{1}{m}} + \beta(r+\varepsilon)(z^{\frac{1}{m}})' - \alpha\varepsilon z^{\frac{1}{m}} < 0,$$

$$z(r) \leq c_1 r^{-\frac{(\alpha-\varepsilon)m}{\beta}}, \quad (45)$$

$$-z'(r) << \leq c_2 r^{-1-\frac{(\alpha-\varepsilon)m}{\beta}}. \quad (46)$$

(iii) If $0 < p_1, q_1 < 1$, by $q_1 + p_1 m > m(p-1) > 1$, from (42) and (37), we also know that (45), (46) are true.

Certainly, by (41), we have

$$0 \leq \lim_{r\to\infty} r^{\frac{\alpha}{\beta}} z^{\frac{1}{m}} = k(a) < \infty. \tag{47}$$

If $k(a) = 0$, multiplying (25) with $r^{\frac{\alpha-\beta}{\beta}}$, and making the integral from r_0 to r

$$-\int_{r_0}^{r} s^{\frac{\alpha-\beta}{\beta}} d|z'|^{p-1} = -|z'|^{p-1} s^{\frac{\alpha-\beta}{\beta}} \mid_{r_0}^{r}$$

$$+\frac{\alpha-\beta}{\beta}\int_{r_0}^{r} s^{\frac{\alpha}{\beta}-2}|z'|^{p-1}ds,$$

$$\int_{r_0}^{r} \frac{n-1}{s}|z'|^{p-2}z' s^{\frac{\alpha}{\beta}-1}ds = \int_{r_0}^{r} \frac{n-1}{s}|z'|^{p-1}s^{\frac{\alpha}{\beta}-1}ds,$$

$$\int_{r_0}^{r} \beta s(z^{\frac{1}{m}})' s^{\frac{\alpha}{\beta}-1}ds = \beta s^{\frac{\alpha}{\beta}} z^{\frac{1}{m}} \mid_{r_0}^{r} -\alpha\int_{r_0}^{r} s^{\frac{\alpha}{\beta}-1} z^{\frac{1}{m}}ds,$$

we obtain

$$-|z'|^{p-1}s^{\frac{\alpha-\beta}{\beta}} \mid_{r_0}^{r} +\beta s^{\frac{\alpha}{\beta}} z^{\frac{1}{m}} \mid_{r_0}^{r}$$

$$= (-\frac{\alpha}{\beta})\int_{r_0}^{r} s^{\frac{\alpha}{\beta}-2}|z'|^{p-1}ds$$

$$+\int_{r_0}^{r} z^{\frac{q_1}{m}}|z'|^{p_1}s^{\frac{\alpha}{\beta}-1}ds. \tag{48}$$

Substituting the above integral as $\int_r^\infty$, we obtain

$$|z'|^{p-1}r^{\frac{\alpha}{\beta}} - \beta r^{\frac{\alpha}{\beta}+1}z^{\frac{1}{m}}$$

$$= (\frac{\alpha}{\beta}-n)r\int_r^\infty |z'|^{p-1}s^{\frac{\alpha}{\beta}-2}ds + r\int_r^\infty z^{\frac{q_1}{m}}|z'|^{p_1}s^{\frac{\alpha}{\beta}-1}ds.$$

From $\lim_{r\to\infty} r^{\frac{\alpha}{\beta}} z^{\frac{1}{m}} = k(a) = 0$, using L'Hospital rule, we have

$$\lim_{r\to\infty} r\int_r^\infty |z'|^{p-1}s^{\frac{\alpha}{\beta}-2}ds = 0,$$

$$\lim_{r\to\infty} r\int_r^\infty z^{\frac{q_1}{m}}|z'|^{p_1}s^{\frac{\alpha}{\beta}-1}ds = \lim_{r\to\infty} z^{\frac{q_1}{m}}|z'|^{p_1}r^{\frac{\alpha}{\beta}+1}.$$

If $p_1 \geq 1$, from (43) and (44) we get

$$\lim_{r\to\infty} z^{\frac{q_1}{m}}|z'|^{p_1}r^{\frac{\alpha}{\beta}+1} = 0.$$

If $p_1 < 1$,

$$\frac{q_1+mp_1-m(p-1)}{p-p_1} < q_1+mp_1-m(p-1)$$

$$< q_1+mp_1-1,$$

$$\frac{\alpha}{\beta}(q_1+mp_1-1) > 1$$

by (45) and (46) we have

$$z^{\frac{q_1}{m}}|z'|^{p_1}r^{\frac{\alpha}{\beta}+1} < r^{-\frac{\alpha-\varepsilon}{\beta}q_1 - p_1(1+\frac{(\alpha-\varepsilon)m}{\beta})+\frac{\alpha}{\beta}+1}$$

$$= r^{\frac{\alpha}{\beta}(q_1+mp_1-1)+\frac{q_1\varepsilon}{\beta}+\frac{p_1 m\varepsilon}{\beta}+1} \to 0,$$

$$\lim_{r\to\infty} r^{\frac{\alpha}{\beta}+1}z^{\frac{1}{m}} = 0.$$

If we repeat this method, for $\forall M > 0$, we get

$$\lim_{r\to\infty} r^M z^{\frac{1}{m}} = 0.$$

It contradicts (41), so $k(a) > 0$.

Lemma 11 *The set B is a nonempty open set. If $a \gg 1$, then $(a,\infty) \subset B$. Moreover, for any $a \in B$*

$$\lim_{r\to\infty} r^{\frac{\alpha}{\beta}} z^{\frac{1}{m}} = k(a) > 0. \tag{49}$$

Proof. We notice that

$$|z'|^{p-2}z'(r) < 0,\ r \in (0, R(a)),$$

and there is r_1 such that

$$(|z'|^{p-2}z')'(r) < 0,\ r \in (0, r_1).$$

If for some $\overline{r} \in (0, R(a))$,

$$(|z'|^{p-2}z')'(\overline{r}) \leq 0,$$

then from (25),

$$z^{\frac{q_1}{m}}|z'|^{p_1} \leq \frac{n-1}{r}v + \beta r(z^{\frac{1}{m}})' + \alpha z^{\frac{1}{m}} < \alpha z^{\frac{1}{m}},$$

$$z^{\frac{q_1-1}{m}}|z'|^{p_1} \leq \alpha. \tag{50}$$

If for some $\overline{r} \in (0, R(a))$,

$$(|z'|^{p-2}z')'(\overline{r}) > 0,$$

then there is b such that

$$(|z'|^{p-2}z')' = 0,\ 0 < b < \overline{r},$$

and in $(b, \overline{r})$,

$$(|z'|^{p-2}z') \geq 0.$$

So, in $(b, \overline{r})$, $|z'|^{p-2}z' = -|z'|^{p-1}$ is monotone increasing, it means that

$$z^{\frac{q_1-1}{m}}|z'|^{p_1}(\overline{r}) \leq z^{\frac{q_1-1}{m}}|z'|^{p_1}(b) \leq \alpha.$$

If $q_1 \leq 1$, then

$$|z'|^{p_1} \leq \alpha z^{-\frac{q_1-1}{m}} \leq \alpha a^{1-q_1},$$

$$|z'| \le \alpha^{\frac{1}{p_1}} a^{\frac{1}{p_1}} (1-q_1). \tag{51}$$

If $q_1 > 1$, then

$$|z'| z^{\frac{q_1-1}{mp_1}} \le \alpha^{\frac{1}{p_1}}. \tag{52}$$

Denote by $\gamma - 1 = \frac{q_1-1}{mp_1}$, then we have

$$z^{\gamma}(r) \ge a^{m\gamma} - m\gamma\alpha^{\frac{1}{p_1}} r, \forall r \in (0, R(a)). \tag{53}$$

Supposing that r_0 is the first intersection point of the orbit crossing the boundary of S_1, then

$$-|v|^{\frac{1}{p-1}} = -|z'(r_0)| = -z^{\theta}(r_0).$$

Since $q_1 > 1$, we have $\gamma > 1$, and by (52) and (53) we have

$$|z'(r_0)| z^{\gamma-1}(r_0) = z^{\gamma+\theta-1}(r_0) \le m\alpha^{\frac{1}{p_1}}$$

and

$$m\gamma\alpha^{\frac{1}{p_1}} r_0 \ge a^{m\gamma} - (m\alpha^{\frac{1}{p_1}})^{\frac{\gamma}{\gamma+\theta-1}} := \phi_1(a). \tag{54}$$

If $q_1 \le 1$, then we have $\gamma \le 1$, and by (51) we have

$$z(r_0) = |z'(r_0)|^{\frac{1}{\theta}} \le (\alpha^{\frac{1}{p_1}})^{\frac{1}{\theta}},$$

and

$$z(r_0) = z(0) + \int_0^{r_0} z'(s) ds \ge a^m - \alpha^{\frac{1}{p_1}} a^{1-\gamma} r_0,$$

i.e.

$$\alpha^{\frac{1}{p_1}} r_0 \ge a^{1-\gamma}(a^m - z(r_0))$$
$$\ge a^{\gamma}(a^{m-1} - \alpha^{\frac{1}{p_1\theta}} a^{\frac{1-\gamma-\theta}{\theta}}) := \phi_2(a). \tag{55}$$

At the same time, by Lemma 7, we have

$$r_{1,a} = \frac{m\alpha}{\beta} a^{1-\theta} + \frac{m\alpha}{\beta} a^{1-\theta}$$
$$= \frac{ma^{1-\theta}}{\beta}(\alpha + \theta a^{2\theta - \frac{m+1}{m}}).$$

If $q_1 > 1$, by the choice of θ in (29), we have $m\gamma > 1-\theta$. By (54), we have

$$\lim_{a\to\infty} \frac{\phi_1(a)}{r_{1,a}} = \infty.$$

If $q_1 \le 1$, it is clear of that

$$m - 1 > \frac{1-\gamma-\theta}{\theta},$$

$$(m-1)\gamma > 1 - \theta,$$

by (54), we have

$$\lim_{a\to\infty} \frac{\phi_2(a)}{r_{1,a}} = \infty.$$

So, if we choose a large enough, we have $r_0 > r_{1,a}$, it contradicts Lemma 7. The above discussion means that when $a \gg 1$, $(a, \infty) \in B$. Now, by that the solution depends on the initial value continuously, we know that B is an open set.

Lemma 12 *The set C is a nonempty closure set, and for any $a \in C$, the solution $z(r)$ of (25) satisfies that*

$$\lim_{r\to R(a)} \frac{z'(r)}{z^{\frac{1}{m}}(r)} = -\beta R(a). \tag{56}$$

Proof. Multiplying (25) with r^{n-1}, and making the integral from r to $R = R(a)$

$$|z'|^{p-1} r^{n-1} - \beta r^n z^{\frac{1}{m}} = (n\beta + \alpha)\int_r^R z^{\frac{1}{m}} s^{n-1} ds$$
$$+ \int_r^R z^{\frac{q_1}{m}} |z'|^{p_1} s^{n-1} ds. \tag{57}$$

Clearly, when $r \to R$,

$$0 < z^{-\frac{1}{m}}(r) \int_r^R z^{\frac{1}{m}} s^{n-1} ds \le \int_r^R s^{n-1} ds \to 0.$$

Now, we want to prove

$$\lim_{r\to R} \int_r^R z^{\frac{q_1}{m}} |z'|^{p_1} s^{n-1} ds = 0. \tag{58}$$

If $q_1 \ge 1$, then (58) is true clearly. If $0 < q_1 \le 1$, we discuss it with the following two cases.

Case 1. $p_1 \ge 1$.

$$\lim_{r\to R} \frac{\int_r^R z^{\frac{q_1}{m}} |z'|^{p_1} s^{n-1} ds}{z^{\frac{1}{m}}} = \lim_{r\to R} \frac{z^{\frac{q_1}{m}} |z'|^{p_1} s^{n-1}}{\frac{1}{m} z^{\frac{1}{m}-1}(r) z'}$$

$$\lim_{r\to R} z^{\frac{q_1}{m} + n - \frac{1}{m}} |z'|^{p_1 - 1} = 0. \tag{59}$$

Case 2. $0 < p_1 < 1$. Set

$$H(r) = \left(-\int_r^R |z'|^{p-2} z' ds\right)^{\sigma} + |z'|^{p-2} z'.$$

Then

$$H'(r) = \sigma\left(-\int_r^R |z'|^{p-2} z' ds\right)^{\sigma-1} |z'|^{p-2} z'(r)$$
$$+ (|z'|^{p-2} z')'$$

$$= \sigma \left(-\int_r^R |z'|^{p-2} z' ds \right)^{\sigma-1} |z'|^{p-2} z'(r)$$

$$+(-\frac{n-1}{r}|z'|^{p-2}z' - \beta r(z^{\frac{1}{m}})' - \alpha z^{\frac{1}{m}} + z^{\frac{q_1}{m}}|z'|^{p_1}).$$

Assume that there is r_0, when $0 < R - r_0 \ll 1$, $H(r_0) = 0$. For the simplicity, we can denote

$$h(r) = -\int_r^R |z'|^{p-2} z' ds.$$

Then

$$H'(r_0) = -(\sigma h^{\sigma-1} + \frac{n-1}{r_0})h^\sigma + \frac{\beta}{m} r_0 z^{\frac{1}{m}-1} |h|^{\frac{\sigma}{p-1}}$$

$$-\alpha z^{\frac{1}{m}} + z^{\frac{q_1}{m}} |h|^{\frac{\sigma p_1}{p-1}},$$

$$= -\sigma h^{2\sigma-1} - \frac{n-1}{r_0} + \frac{\beta}{m} r_0 z^{\frac{1}{m}-1} |h|^{\frac{\sigma}{p-1}} - \alpha z^{\frac{1}{m}}$$

$$+z^{\frac{q_1}{m}} |h|^{\frac{\sigma p_1}{p-1}}.$$

Since $\lim_{r\to R} z(r) = 0$, if $\sigma > \frac{p-1}{2p-3}$, then

$$\frac{\sigma}{p-1} < 2\sigma - 1, \frac{\sigma}{p-1} < \sigma,$$

and so

$$|h|^{\frac{\sigma}{p-1}} = h^{\frac{\sigma}{p-1}} > h^{2\sigma-1}, h^{\frac{\sigma}{p-1}} > h^\sigma,$$

i.e. we have

$$H'(r_0) > 0.$$

By this fact, it is easy to know that $H(r)$ is with fixed sign in (r_0, R).

If it holds that when $0 < R - r \ll 1$, $H(r) > 0$ i.e. $h^\sigma > |z'|^{p-1}$, according to (59), we can assume that $\lim_{r\to R} |z'| = 0$ without loss of the generality. Otherwise the problem is just the same as the case 1. Thus,

$$h(r) = \int_r^R |z'|^{p-1} ds < -\int_r^R z' ds = z(r), r-R \ll 1,$$

$$|z'| < h^{\frac{\sigma}{p-1}} < z^{\frac{\sigma}{p-1}},$$

$$\frac{|z'|}{z} < z^{\frac{\sigma}{p-1}-1}.$$

$$\lim_{r\to R} \frac{\int_r^R z^{\frac{q_1}{m}} |z'|^{p_1} s^{n-1} ds}{z^{\frac{1}{m}}} \le \lim_{r\to R} \frac{\int_r^R z^{\frac{q_1}{m}} z^{\frac{p_1\sigma}{p-1}} s^{n-1} ds}{z^{\frac{1}{m}}}$$

$$\le \lim_{r\to R} z^{\frac{q_1 p_1 \sigma}{m(p-1)} - \frac{1}{m}} \int_r^R s^{n-1} ds = 0, \quad (60)$$

where we choose $\sigma > \max\{\frac{p-1}{2p-3}, \frac{p-1}{p_1 q_1}\}$, (60) is true.

If it holds that when $0 < R - r \ll 1$, $H(r) < 0$, i.e. $h^\sigma < |z'|^{p-1}$, then

$$h(r) = -\int_r^R |z'|^{p-2} z' ds = -\int_r^R |z'|^{p-2} dz$$

$$= -z|z'|^{p-2} |_r^R - (p-2)\int_r^R z|z'|^{p-3} z'' ds$$

$$= |z'|^{p-2}(r) - (p-2)\int_r^R z|z'|^{p-3} z'' ds < |z'|^{\frac{p-1}{\sigma}}$$

$$|z'|^{p-2-\frac{p-1}{\sigma}}(r) < (p-2)\frac{\int_r^R z|z'|^{p-3} z'' ds}{|z'|^{\frac{p-1}{\sigma}}}. \quad (61)$$

Since

$$\lim_{r\to R} \frac{\int_r^R z|z'|^{p-3} z'' ds}{|z'|^{\frac{p-1}{\sigma}}} = \lim_{r\to R} \frac{-z|z'|^{p-3} z''}{-\frac{p-1}{\sigma}|z'|^{\frac{p-1}{\sigma}-1} z''}$$

$$= \frac{p-1}{\sigma} \lim_{r\to R} z|z'|^{p-2-\frac{p-1}{\sigma}}, \quad (62)$$

we now choose σ close to $\frac{p-1}{2p-3}$, then $\sigma\frac{p-1}{2p-3}$ is very close to $\frac{p-2}{2p-3} < 1$ too, which implies that (61) contradicts (62).

The proof of Theorem 8: (i) If $\alpha \le N\beta$, multiplying the equation (25) with $r^{\frac{\alpha}{\beta}-1}$, we obtain

$$(r^{\frac{\alpha}{\beta}-1}|z'|^{p-2}z' + \beta r^{\frac{\alpha}{\beta}} z^{\frac{1}{m}})' = (n - \frac{\alpha}{\beta}) r^{\frac{\alpha}{\beta}-2} |z'|^{p-1}$$

$$+r^{\frac{\alpha}{\beta}-1} z^{\frac{q_1}{m}} |z'|^{p_1} < 0. \quad (63)$$

Let

$$g(r) = r^{\frac{\alpha}{\beta}-1}|z'|^{p-2}z' + \beta r^{\frac{\alpha}{\beta}} z^{\frac{1}{m}}.$$

It is clear that $\lim_{r\to 0} g(r) = 0$. By (63), for any $r > 0$,

$$g(r) > 0. \quad (64)$$

If $R(a) < \infty$, since $z \equiv 0, z' \equiv 0$ in $[R(a), \infty)$, we have

$$\lim_{r\to R(a)} g(r) = 0.$$

which contradicts (64). So $R(a) = \infty$.

By $g(r) < \beta r^{\frac{\alpha}{\beta}} z^{\frac{1}{m}}$ and the monotonicity of $g(r)$, we have

$$\lim_{r\to\infty} \inf r^{\frac{\alpha}{\beta}} z^{\frac{1}{m}} > 0.$$

At the same time, by Lemma 7, Lemmas 9-12, we can get the conclusions (i), (ii) and (iii) of the theorem.

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