Some New Delay Integral Inequalities On Time Scales Arising In The Theory Of Dynamic Equations

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Abstract: In this paper, some new types of delay integral inequalities on time scales are established, which can be used as a handy tool in the investigation of making estimates for bounds of solutions of delay dynamic equations on time scales. Our results generalize the main results in [15, 16, 17], and some of the results in [18, 19].

Key–Words: Delay integral inequality; Time scale; Integral equation; Differential equation; Dynamic equation; Bounded

1 Introduction

The development of the theory of time scales was initiated by Hilger [1]. A time scale is an arbitrary nonempty closed subset of the real numbers. Many integral inequalities on time scales have been established since then, for example [2-12], which have been designed in order to unify continuous and discrete analysis. But to our knowledge, delay integral inequalities on time scales have been scarcely payed attention to in the literature so far.

Our aim in this paper is to establish some new delay integral inequalities on time scales, and present some applications for them.

For two given sets G, H, we denote the set of maps from G to H by (G, H), while denote the definition domain and the image of a function f by $Dom(f)$ and $Im(f)$ respectively.

In the rest of the paper, R denotes the set of real numbers and $R_+ = [0, \infty)$. $\mathbb{Z}$ denotes the set of integers. $\mathbb{T}$ denotes an arbitrary time scale and $\mathbb{T}_0 = [t_0, \infty) \bigcap \mathbb{T}$, where $t_0 \in \mathbb{T}$. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum, otherwise it is $\mathbb{T}$ without the left-scattered maximum. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma(t) \in (\mathbb{T}, \mathbb{T})$ and $\rho(t) \in (\mathbb{T}, \mathbb{T})$ such that $\sigma(t) = inf\{s \in \mathbb{T}, s > t\}$, $\rho(t) = sup\{s \in \mathbb{T}, s < t\}$.

Definition 1 *The grainless $\mu \in (\mathbb{T}, \mathbb{R}_+)$ is defined by $\mu(t) = \sigma(t) - t$.*

Remark 2 *Obviously, $\mu(t) = 0$ if $\mathbb{T} = \mathbb{R}$ while $\mu(t) = 1$ if $\mathbb{T} = \mathbb{Z}$.*

Definition 3 *A point $t \in \mathbb{T}$ with $t > inf\mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$.*

Definition 4 *The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum, otherwise it is $\mathbb{T}$ without the left-scattered maximum.*

Definition 5 *A function $f \in (\mathbb{T}, \mathbb{R})$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1+\mu(t)f(t) \neq 0$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f | f \in \mathfrak{R},\ 1 + \mu(t)f(t) > 0,\ \forall t \in \mathbb{T}\}$.*

Definition 6 *For some $t \in \mathbb{T}^\kappa$, and a function $f \in (\mathbb{T}, \mathbb{R})$, the delta derivative of f is denoted by $f^\Delta(t)$, and satisfies*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for $\forall\varepsilon > 0$*, where* $s \in \mathfrak{U}$*, and* $\mathfrak{U}$ *is a neighborhood of* t*. The function* f *is called delta differential on* $\mathbb{T}^\kappa$*.*

Remark 7 *If* $\mathbb{T} = \mathbb{R}$*, then* $f^\Delta(t)$ *becomes the usual derivative* $f'(t)$*, while* $f^\Delta(t) = f(t+1) - f(t)$ *if* $\mathbb{T} = \mathbb{Z}$*, which represents the forward difference.*

Definition 8 *If* $F^\Delta(t) = f(t),\ t \in \mathbb{T}^\kappa$*, then* F *is called an antiderivative of* f*, and the Cauchy integral of* f *is defined by*

$$\int_a^b f(t)\Delta t = F(b) - F(a),$$

where $a,\ b \in \mathbb{T}^\kappa$*.*

The following two theorems include some important properties for *delta derivative* and *Cauchy integral* on time scales.

Theorem 9 *[11, Theorem 1.1] If* $f,\ g \in (\mathbb{T}, R)$*, and* $t \in \mathbb{T}^\kappa$*, then*

$$(i)\ f^\Delta(t) = \begin{cases} \frac{f(\sigma(t)) - f(t)}{\mu(t)} & if \quad \mu(t) \neq 0, \\ \lim\limits_{s \to t} \frac{f(t) - f(s)}{t - s} & if \quad \mu(t) = 0. \end{cases}$$

(ii) *If* $f,\ g$ *are delta differential at* t*, then* fg *is also delta differential at* t*, and*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t).$$

Theorem 10 *[11, Theorem 1.2] If* $a,\ b,\ c \in \mathbb{T},\ \alpha \in \mathbb{R}$*, and* $f,\ g \in C_{rd}$*, then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$,

(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$,

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,

(v) $\int_a^a f(t)\Delta t = 0$,

(vi) *if* $f(t) \geq 0$ *for all* $a \leq t \leq b$*, then* $\int_a^b f(t)\Delta t \geq 0$.

Remark 11 *If* $b = \infty$*, then all conclusions of Theorem 10 still hold.*

Definition 12 *The cylinder transformation* ξ_h *is defined by*

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h}, & if\ h \neq 0\ (for\ z \neq -\frac{1}{h}), \\ z, & if\ h = 0, \end{cases}$$

where Log is the principal logarithm function.

Definition 13 *For* $p \in \mathfrak{R}$*, the exponential function is defined by*

$$e_p(t, s) = exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$$

for $s, t \in \mathbb{T}$*.*

Definition 14 *If* $\sup\limits_{t \in \mathbb{T}} t = \infty$*,* $p \in \mathfrak{R}$*, we define*

$$e_p(\infty, s) = exp\left(\int_s^\infty \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$$

for $t \in \mathbb{T}$*.*

Remark 15 *If* $\mathbb{T} = \mathbb{R}$*, then for*

$$\begin{cases} e_p(t, s) = exp(\int_s^t p(\tau)d\tau), & for \quad s,\ t \in \mathbb{R}, \\ e_p(\infty, s) = exp(\int_s^\infty p(\tau)d\tau), & for \quad s \in \mathbb{R}. \end{cases}$$

If $\mathbb{T} = \mathbb{Z}$*, then*

$$\begin{cases} e_p(t, s) = \prod\limits_{\tau=s}^{t-1} [1 + p(\tau)], & for\ s,\ t \in \mathbb{Z} \ \ and\ s < t, \\ e_p(\infty, s) = \prod\limits_{\tau=s}^{\infty} [1 + p(\tau)], & for \quad s \in \mathbb{Z}. \end{cases}$$

The following two theorems include some known properties on the *exponentialfunction.*

Theorem 16 *[12, Theorem 5.2] If* $p \in \mathfrak{R}$*, then the following conclusions hold*

(i) $e_p(t, t) \equiv 1$*, and* $e_0(t, s) \equiv 1$*,*

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$*,*

(iii) *If* $p \in \mathfrak{R}^+$*, then* $e_p(t, s) > 0$ *for* $\forall s,\ t \in \mathbb{T}$*,*

(iv) *If* $p \in \mathfrak{R}^+$*, then* $\ominus p \in \mathfrak{R}^+$*,*

(v) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$*,*

where $\ominus p = -\frac{p}{1+\mu p}$*.*

Remark 17 *If* $s = \infty$*, then Theorem 16(iii)(v) still hold.*

Theorem 18 *[12, Theorem 5.1] If* $p \in \mathfrak{R}$*, and fix* $t_0 \in \mathbb{T}$*, then the exponentialfunction* $e_p(t, t_0)$ *is the unique solution of the following initial value problem*

$$\begin{cases} y^\Delta(t) = p(t)y(t), \\ y(t_0) = 1. \end{cases}$$

For more details about the calculus of time scales, we advise to refer to [13].

2 Main Results

The following Lemma is useful for proving our results.

Lemma 19 *([13], Gronwall's inequality) Suppose $u,\ a,\ b \in C_{rd},\ m \in \mathfrak{R}_+,\ m \geq 0$. Then*

$$u(t) \leq a(t) + b(t)\int_{t_0}^{t} m(s)u(s)\Delta s,\ t \in \mathbb{T}_0$$

implies

$$u(t) \leq a(t)+b(t)\int_{t_0}^{t} e_{\widetilde{m}}(t,\sigma(s))a(s)m(s)\Delta s,\ t \in \mathbb{T}_0,$$

where $\widetilde{m}(t) = m(t)b(t)$, and $e_{\widetilde{m}}(t,t_0)$ is the unique solution of the following equation

$$y^{\Delta}(t) = \widetilde{m}(t)y(t),\ y(t_0) = 1.$$

Lemma 20 *Under the conditions of Lemma 19, furthermore, if $b(t) \equiv 1$, and $a(t)$ is nondecreasing on $\mathbb{T}_0$, then*

$$u(t) \leq a(t)e_m(t,t_0),\ t \in \mathbb{T}_0.$$

Proof: Since $b(t) \equiv 1$, and $a(t)$ is nondecreasing on $\mathbb{T}_0$, then $\widetilde{m} = m$, and

$$u(t) \leq a(t) + \int_{t_0}^{t} e_m(t,\sigma(s))a(s)m(s)\Delta s$$

$$\leq a(t)[1 + \int_{t_0}^{t} e_m(t,\sigma(s))m(s)\Delta s].$$

From [13, Theorem 2.39 and 2.36 (i)], we have

$$\int_{t_0}^{t} e_m(t,\sigma(s))m(s)\Delta s$$

$$= e_m(t,t_0) - e_m(t,t) = e_m(t,t_0) - 1,$$

Combining the above information we can obtain the desired inequality. □

Lemma 21 *[14] Assume that $a \geq 0, p \geq q \geq 0$, and $p \neq 0$, then for any $K > 0$,*

$$a^{\frac{q}{p}} \leq \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}}.$$

Firstly we will study the following Gronwall-Bellman type delay integral inequality of the following form

$$u^p(t) \leq a(t) + b(t)\int_{t_0}^{t}[m(s) + f(s)u^p(\tau_1(s))$$

$$+g(s)u^q(\tau_2(s)) + \int_{t_0}^{s} h(\xi)u^r(\tau_3(\xi))\Delta\xi]\Delta s, \quad (1)$$

Theorem 22 *Suppose $u,\ f,\ g,\ h,\ m,\ a,\ b \in C_{rd}(\mathbb{T}_0, R_+)$ and $a,\ b$ are nondecreasing, $p,\ q,\ r$ are constants, and $p \geq q \geq 0,\ p \geq r \geq 0,\ p \neq 0,\ \tau_i \in (\mathbb{T}_0, \mathbb{T})$ with $\tau_i(t) \leq t,\ i = 1,2,3,$ and $-\infty < \alpha = inf\{min\{\tau_i(t),\ i = 1,2,3\}, t \in \mathbb{T}_0\} \leq t_0$. $\eta \in C_{rd}([\alpha,t_0]\bigcap\mathbb{T}, R_+)$. If for $t \in \mathbb{T}_0$, $u(t)$ satisfies (1) with the initial condition*

$$\begin{cases} u(t) = \eta(t), t \in [\alpha,t_0]\bigcap\mathbb{T}, \\ \eta(\tau_i(t)) \leq a^{\frac{1}{p}}(t),\ \forall t \in \mathbb{T}_0,\ \tau_i(t) \leq t_0,\ i = 1,2,3, \end{cases} \quad (2)$$

then

$$u(t) \leq \{[a(t)+b(t)H_2(t)e_{H_3}(t,t_0)]H_1(t)\}^{\frac{1}{p}},\quad t \in \mathbb{T}_0, \quad (3)$$

where

$$\begin{array}{l} H_1(t) = 1 + b(t)\int_{t_0}^{t} e_{\widetilde{f}}(t,\sigma(s))f(s)\Delta s, \\ \widetilde{f}(t) = f(t)b(t), \end{array} \quad (4)$$

$$H_2(t) = \int_{t_0}^{t}\{m(s)+g(s)[\frac{q}{p}K^{\frac{q-p}{p}}a(s)+\frac{p-q}{p}K^{\frac{q}{p}}]H_1(s)^{\frac{q}{p}}\Delta s$$

$$+\int_{t_0}^{t}\int_{t_0}^{s} h(\xi)[\frac{r}{p}K^{\frac{r-p}{p}}a(\xi)+\frac{p-r}{p}K^{\frac{r}{p}}]H_1(\xi)^{\frac{r}{p}}\Delta\xi\}\Delta s,$$

$$\forall K > 0, \quad (5)$$

$$H_3(t) = g(t)\frac{q}{p}K^{\frac{q-p}{p}}b(t)H_1(t)^{\frac{q}{p}}+\int_{t_0}^{t} h(\xi)\frac{r}{p}K^{\frac{r-p}{p}}b(\xi)$$

$$H_1(\xi)^{\frac{r}{p}}\Delta\xi,\quad \forall K > 0. \quad (6)$$

Proof: Let the right side of (1) be $v(t)$, then

$$u(t) \leq v^{\frac{1}{p}}(t),\quad t \in \mathbb{T}_0. \quad (7)$$

For $t \in \mathbb{T}_0$, if $\tau_i(t) \geq t_0$, then $\tau_i(t) \in \mathbb{T}_0$, so from (7), considering $\tau_i(t) \leq t$, we have

$$u(\tau_i(t)) \leq v^{\frac{1}{p}}(\tau_i(t)) \leq v^{\frac{1}{p}}(t). \quad (8)$$

If $\tau_i(t) \leq t_0$, from (2) we have

$$u(\tau_i(t))) = \eta(\tau_i(t)) \leq a^{\frac{1}{p}}(t) \leq v^{\frac{1}{p}}(t). \quad (9)$$

So from (8) and (9) we always have

$$u(\tau_i(t)) \leq v^{\frac{1}{p}}(t),\ i = 1,2 \quad for\ \ \forall t \in \mathbb{T}_0. \quad (10)$$

Furthermore

$$v(t) \leq a(t) + b(t)\int_{t_0}^{t}[m(s) + f(s)v(s) + g(s)v^{\frac{q}{p}}(s)$$

$$+\int_{t_0}^{s} h(\xi)v^{\frac{r}{p}}(\xi)\Delta\xi]\Delta s. \tag{11}$$

Let

$$c(t) = a(t) + b(t)\int_{t_0}^{t}[m(s) + g(s)v^{\frac{q}{p}}(s)$$

$$+\int_{t_0}^{s} h(\xi)v^{\frac{r}{p}}(\xi)\Delta\xi]\Delta s, \tag{12}$$

then we have

$$v(t) \le c(t) + b(t)\int_{t_0}^{t} f(s)v(s)\Delta s, \quad t \in \mathbb{T}_0. \tag{13}$$

From Lemma 19, and considering $c(t)$ is nondecreasing on $\mathbb{T}_0$, we can obtain

$$v(t) \le c(t) + b(t)\int_{t_0}^{t} e_{\widetilde{f}}(t,\sigma(s))c(s)f(s)\Delta s$$

$$\le c(t)[1 + b(t)\int_{t_0}^{t} e_{\widetilde{f}}(t,\sigma(s))f(s)\Delta s] = c(t)H_1(t),$$

$$t \in \mathbb{T}_0, \tag{14}$$

where $H_1(t),\ \widetilde{f}(t)$ are defined in (4).

Let

$$y(t) = \int_{t_0}^{t}[m(s)+g(s)v^{\frac{q}{p}}(s)+\int_{t_0}^{s} h(\xi)u^{\frac{r}{p}}(\xi)\Delta\xi]\Delta s, \tag{15}$$

then

$$c(t) = a(t) + b(t)y(t), \tag{16}$$

From Lemma 21, for $\forall K > 0$ we have

$$\begin{cases} (a(t) + b(t)y(t))^{\frac{q}{p}} \\ \le \frac{q}{p}K^{\frac{q-p}{p}}(a(t) + b(t)y(t)) + \frac{p-q}{p}K^{\frac{q}{p}}, \\ (a(t) + b(t)y(t))^{\frac{r}{p}} \\ \le \frac{r}{p}K^{\frac{r-p}{p}}(a(t) + b(t)y(t)) + \frac{p-r}{p}K^{\frac{r}{p}}. \end{cases} \tag{17}$$

Combining (14), (15) (16), (17) we have

$$y(t) \le \int_{t_0}^{t}[m(s) + g(s)(c(s)H_1(s))^{\frac{q}{p}}$$

$$+\int_{t_0}^{s} h(\xi)(c(\xi)H_1(\xi))^{\frac{r}{p}}\Delta\xi]\Delta s$$

$$\le \int_{t_0}^{t}\{m(s) + g(s)[(a(s) + b(s)y(s))H_1(s)]^{\frac{q}{p}}$$

$$+\int_{t_0}^{s} h(\xi)[(a(\xi) + b(\xi)y(\xi))H_1(\xi)]^{\frac{r}{p}}\Delta\xi\}\Delta s$$

$$\le \int_{t_0}^{t}\{m(s) + g(s)[\frac{q}{p}K^{\frac{q-p}{p}}(a(s) + b(s)y(s))$$

$$+\frac{p-q}{p}K^{\frac{q}{p}}]H_1(s)^{\frac{q}{p}}\}\Delta s$$

$$+\int_{t_0}^{t}\int_{t_0}^{s} h(\xi)[\frac{r}{p}K^{\frac{r-p}{p}}(a(\xi) + b(\xi)y(\xi))$$

$$+\frac{p-r}{p}K^{\frac{r}{p}}]H_1(\xi)^{\frac{r}{p}}\Delta\xi\Delta s$$

$$= \int_{t_0}^{t}\{m(s)+g(s)[\frac{q}{p}K^{\frac{q-p}{p}}a(s) + \frac{p-q}{p}K^{\frac{q}{p}}]H_1(s)^{\frac{q}{p}}$$

$$+\int_{t_0}^{s} h(\xi)[\frac{r}{p}K^{\frac{r-p}{p}}a(\xi) + \frac{p-r}{p}K^{\frac{r}{p}}]H_1(\xi)^{\frac{r}{p}}\Delta\xi\}\Delta s$$

$$+\int_{t_0}^{t}[g(s)\frac{q}{p}K^{\frac{q-p}{p}}b(s)H_1(s)^{\frac{q}{p}}y(s)$$

$$+\int_{t_0}^{s} h(\xi)\frac{r}{p}K^{\frac{r-p}{p}}b(\xi)H_1(\xi)^{\frac{r}{p}}y(\xi)\Delta\xi]\Delta s$$

$$\le H_2(t) + \int_{t_0}^{t} H_3(s)y(s)\Delta s, \tag{18}$$

where $H_2(t),\ H_3(t)$ are defined in (5) and (6) respectively.

Considering $H_2(t)$ is nondecreasing on $\mathbb{T}_0$, then according to Lemma 20 we have

$$y(t) \le H_2(t)e_{H_3}(t,t_0), \quad t \in \mathbb{T}_0. \tag{19}$$

Combining (14), (16), (19), we have

$$v(t) \le [a(t) + b(t)H_2(t)e_{H_3}(t,t_0)]H_1(t). \tag{20}$$

From (7), (20) we can obtain the desired inequality (3). □

Remark 23 *If we take $p \ge 1,\ q = 1,\ f(t) = h(t) \equiv 0$, then Theorem 22 reduces to [15, Theorem 1]. and furthermore, if $\mathbb{T} = R$, then Theorem 22 reduces to [16, Theorem 1], which is one case of integral inequality for continuous function.*

Remark 24 *If we take $p \ge 1,\ q = 1,\ h(t) \equiv 0,\ b(t) \equiv 1,\ \tau_1(t) = t$, then according to [13, Theorem 2.39 and 2.36 (i)] we have $H_1(t) = 1 + \int_{t_0}^{t} e_f(t,\sigma(s))f(s)\Delta s = e_f(t,t_0)$, and Theorem 22 coincides with [15, Theorem 4] exactly. Furthermore if $\mathbb{T} = R$, then Theorem 22 reduces to [16, Theorem 2], which is another case of integral inequality for continuous function.*

Remark 25 *In Remark 23 and 24, if we take $\mathbb{T}=\mathbb{Z}$, then Theorem 22 reduces to the theorems established in [17], which are two cases of discrete inequalities.*

Remark 26 *If we take $\mathbb{T}=R$, $b(t)\equiv 1$, $\tau_1(t)=\tau_3(t)=t$, $m(t)\equiv 0$, then Theorem 22 reduces [18, Theorem 2], which is a third case of integral inequality for continuous function. If we take $\mathbb{T}=\mathbb{Z}$, $a(t)\equiv C$, $b(t)\equiv 2$, $h(t)\equiv 0$, $p=2$, $q=1$, then Theorem 22 reduces to [19, Theorem 4(b1)], which is one case of discrete inequality.*

Based on Theorem 22, we will establish two Volterra-Fredholm type delay integral inequality on time scales in the following two theorems.

Theorem 27 *Suppose u, f, g, h, m, p, q, r, τ_i, $i=1,2,3$ are the same as in Theorem 22, $C>0$ is a constant, $T\in\mathbb{T}_0$ is a fixed number. If for $t\in[t_0,T]\bigcap\mathbb{T}$, $u(t)$ satisfies the following inequality*

$$u^p(t)\leq C+\int_{t_0}^{t}[m(s)+f(s)u^p(\tau_1(s))+g(s)u^q(\tau_2(s))+\int_{t_0}^{s}h(\xi)u^r(\tau_3(\xi))\Delta\xi]\Delta s+\int_{t_0}^{T}[m(s)+f(s)u^p(\tau_1(s))+g(s)u^q(\tau_2(s))+\int_{t_0}^{s}h(\xi)u^r(\tau_3(\xi))\Delta\xi]\Delta s, \tag{21}$$

with the initial condition (2), and furthermore $(1+\widetilde{H}_{21}(T)e_{\widetilde{H}_3}(T,t_0))e_f(T,t_0)<2$, then for $t\in[t_0,T]\bigcap\mathbb{T}$,

$$u(t)\leq\{\{[\frac{C+\widetilde{H}_{22}(T)e_{\widetilde{H}_3}(T,t_0)e_f(T,t_0)}{2-(1+\widetilde{H}_{21}(T)e_{\widetilde{H}_3}(T,t_0))e_f(T,t_0)}]\times[1+\widetilde{H}_{21}(t)e_{\widetilde{H}_3}(t,t_0)]+\widetilde{H}_{22}(t)e_{\widetilde{H}_3}(t,t_0)\}e_f(t,t_0)\}^{\frac{1}{p}}, \tag{22}$$

where

$$\widetilde{H}_1(t)=1+\int_{t_0}^{t}e_f(t,\sigma(s))f(s)\Delta s, \tag{23}$$

$$\widetilde{H}_{21}(t)=\int_{t_0}^{t}\{g(s)\frac{q}{p}K^{\frac{q-p}{p}}\widetilde{H}_1(s)^{\frac{q}{p}}+\int_{t_0}^{s}h(\xi)\frac{r}{p}K^{\frac{r-p}{p}}\widetilde{H}_1(\xi)^{\frac{r}{p}}\Delta\xi\}\Delta s,\quad\forall K>0, \tag{24}$$

$$\widetilde{H}_{22}(t)=\int_{t_0}^{t}\{m(s)+g(s)\frac{p-q}{p}K^{\frac{q}{p}}\widetilde{H}_1(s)^{\frac{q}{p}}+\int_{t_0}^{s}h(\xi)\frac{p-r}{p}K^{\frac{r}{p}}\widetilde{H}_1(\xi)^{\frac{r}{p}}\Delta\xi\}\Delta s,\quad\forall K>0, \tag{25}$$

$$\widetilde{H}_3(t)=g(t)\frac{q}{p}K^{\frac{q-p}{p}}\widetilde{H}_1(t)^{\frac{q}{p}}+\int_{t_0}^{t}h(\xi)\frac{r}{p}K^{\frac{r-p}{p}}\widetilde{H}_1(\xi)^{\frac{r}{p}}\Delta\xi,\quad\forall K>0. \tag{26}$$

Proof: Let the right side of (21) be $v(t)$, then

$$u(t)\leq v^{\frac{1}{p}}(t),\quad t\in[t_0,T]\bigcap\mathbb{T}, \tag{27}$$

and similar to the process of (8)-(10) we have

$$u(\tau_i(t))\leq v^{\frac{1}{p}}(t),\ i=1,2\quad t\in[t_0,T]\bigcap\mathbb{T}. \tag{28}$$

Considering

$$v(t_0)=C+\int_{t_0}^{T}[m(s)+f(s)u^p(\tau_1(s))+g(s)u^q(\tau_2(s))+\int_{t_0}^{s}h(\xi)u^r(\tau_3(\xi))\Delta\xi]\Delta s,$$

it follows

$$v(t)=v(t_0)+\int_{t_0}^{t}[m(s)+f(s)u^p(\tau_1(s))+g(s)u^q(\tau_2(s))+\int_{t_0}^{s}h(\xi)u^r(\tau_3(\xi))\Delta\xi]\Delta s$$
$$\leq v(t_0)+\int_{t_0}^{t}[m(s)+f(s)v(s)+g(s)u^{\frac{q}{p}}(s)+\int_{t_0}^{s}h(\xi)u^{\frac{r}{p}}(\xi)\Delta\xi]\Delta s,\quad t\in[t_0,T]\bigcap\mathbb{T}. \tag{29}$$

We notice the structure of (29) is just similar to that of (11), so following in a same manner as the process of (11)-(20) in Theorem 22 (that is, $v(t_0)$ takes the place of $a(t)$ in Theorem 22, and let $b(t)\equiv 1$ in Theorem 22), considering $\widetilde{H}_1(t)=1+\int_{t_0}^{t}e_f(t,\sigma(s))f(s)\Delta s=e_f(t,t_0)$, we can obtain

$$v(t)\leq[v(t_0)(1+\widetilde{H}_{21}(t)e_{\widetilde{H}_3}(t,t_0))+\widetilde{H}_{22}(t)e_{\widetilde{H}_3}(t,t_0)]\times e_f(t,t_0),\quad t\in[t_0,T]\bigcap\mathbb{T}, \tag{30}$$

where $\widetilde{H}_{21}(t)$, $\widetilde{H}_{22}(t)$, $\widetilde{H}_3(t)$ are defined in (24), (25) and (26) respectively.

Setting $t = T$ in (30) we have

$$v(T) \le [v(t_0)(1 + \widetilde{H}_{21}(T)e_{\widetilde{H}_3}(T, t_0)) + \widetilde{H}_{22}(T)e_{\widetilde{H}_3}(T, t_0)]e_f(T, t_0). \quad (31)$$

As $2v(t_0) - C = v(T)$, so it follows

$$2v(t_0) - C = v(T) \le [v(t_0)(1 + \widetilde{H}_{21}(T)e_{\widetilde{H}_3}(T, t_0)) + \widetilde{H}_{22}(T)e_{\widetilde{H}_3}(T, t_0)]e_f(T, t_0),$$

that is,

$$v(t_0) \le \frac{C + \widetilde{H}_{22}(T)e_{\widetilde{H}_3}(T, t_0)e_f(T, t_0)}{2 - (1 + \widetilde{H}_{21}(T)e_{\widetilde{H}_3}(T, t_0))e_f(T, t_0)}. \quad (32)$$

Combining (27), (30), (32) we can obtain the desired inequality (21). □

Theorem 28 *Suppose u, f, h, p, r, τ_i, $i = 1, 2, 3$ are the same as in Theorem 22, $C > 0$ is a constant, $T \in \mathbb{T}_0$ is a fixed number, $L \in C(\mathbb{T}_0 \times R_+, R_+)$, and $0 \le L(s, x) - L(s, y) \le M(s, y)(x - y)$ for $x \ge y \ge 0$, where $M \in C(\mathbb{T}_0 \times R_+, R_+)$. If for $t \in [t_0, T] \bigcap \mathbb{T}$, $u(t)$ satisfies the following inequality*

$$u^p(t) \le C + \int_{t_0}^{t} [f(s)u^p(\tau_1(s)) + L(s, u(\tau_2(s))) + \int_{t_0}^{s} h(\xi)u^r(\tau_3(\xi))\Delta\xi]\Delta s + \int_{t_0}^{T} [f(s)u^p(\tau_1(s)) + L(s, u(\tau_2(s))) + \int_{t_0}^{s} h(\xi)u^r(\tau_3(\xi))\Delta\xi]\Delta s, \quad (33)$$

with the initial condition (2), and furthermore, $e_{\widehat{H}_2}(T, t_0)e_f(T, t_0) < 2$, then

$$u(t) \le \{[\frac{C + \widehat{H}_1(T)e_{\widehat{H}_2}(T, t_0)e_f(T, t_0)}{2 - e_{\widehat{H}_2}(T, t_0)e_f(T, t_0)} + \widehat{H}_1(t)] \times e_{\widehat{H}_2}(t, t_0)e_f(t, t_0)\}^{\frac{1}{p}}, \quad t \in [t_0, T] \bigcap \mathbb{T}. \quad (34)$$

where

$$\widehat{H}_1(t) = \int_{t_0}^{t} L(s, \frac{p-1}{p} K^{\frac{1}{p}} (e_f(s, t_0))^{\frac{1}{p}})\Delta s + \int_{t_0}^{t}\int_{t_0}^{s} h(\xi)\frac{p-r}{p} K^{\frac{r}{p}} (e_f(\xi, t_0))^{\frac{r}{p}} \Delta\xi\Delta s, \ \forall K > 0, \quad (35)$$

$$\widehat{H}_2(t) = M(t, \frac{p-1}{p} K^{\frac{1}{p}} (e_f(t, t_0))^{\frac{1}{p}}) \frac{1}{p} K^{\frac{1-p}{p}} (e_f(t, t_0))^{\frac{1}{p}} + \int_{t_0}^{t} h(\xi)\frac{r}{p} K^{\frac{r-p}{p}} (e_f(\xi, t_0))^{\frac{r}{p}} \Delta\xi, \ \forall K > 0. \quad (36)$$

Proof: Let the right side of (33) be $v(t)$, then

$$u(t) \le v^{\frac{1}{p}}(t), \quad t \in [t_0, T] \bigcap \mathbb{T}, \quad (37)$$

and similar to the process of (8)-(10) we have

$$u(\tau_i(t)) \le v^{\frac{1}{p}}(t), \ i = 1, 2 \ \ t \in [t_0, T] \bigcap \mathbb{T}. \quad (38)$$

Furthermore, considering

$$v(t_0) = C + \int_{t_0}^{T} [f(s)u^p(\tau_1(s)) + L(s, u(\tau_2(s))) + \int_{t_0}^{s} h(\xi)u^r(\tau_3(\xi))\Delta\xi]\Delta s,$$

we have

$$v(t) = v(t_0) + \int_{t_0}^{t} [f(s)u^p(\tau_1(s)) + L(s, u(\tau_2(s))) + \int_{t_0}^{s} h(\xi)u^r(\tau_3(\xi))\Delta\xi]\Delta s$$
$$\le v(t_0) + \int_{t_0}^{t} [f(s)v(s) + L(s, v^{\frac{1}{p}}(s)) + \int_{t_0}^{s} h(\xi)v^{\frac{r}{p}}(\xi)\Delta\xi]\Delta s. \quad (39)$$

Let

$$c(t) = \int_{t_0}^{t} [L(s, v^{\frac{1}{p}}(s)) + \int_{t_0}^{s} h(\xi)v^{\frac{r}{p}}(\xi)\Delta\xi]\Delta s, \quad (40)$$

then

$$v(t) \le v(t_0) + c(t) + \int_{t_0}^{t} f(s)v(s)\Delta s, \ \ t \in [t_0, T] \bigcap \mathbb{T}. \quad (41)$$

Considering $c(t)$ is nondecreasing on $\mathbb{T}_0$, by Lemma 20 we obtain

$$v(t) \le (v(t_0) + c(t))e_f(t, t_0), \quad t \in [t_0, T] \bigcap \mathbb{T}. \quad (42)$$

Combining (40) and (42) it follows

$$c(t) \le \int_{t_0}^{t} \{L(s, ((v(t_0) + c(s))e_f(s, t_0))^{\frac{1}{p}}) + \int_{t_0}^{s} h(\xi)[(v(t_0) + c(\xi))e_f(\xi, t_0)]^{\frac{r}{p}} \Delta\xi\}\Delta s,$$

$$t \in [t_0, T] \bigcap \mathbb{T}. \quad (43)$$

On the other hand, from Lemma 21 one can see the following inequalities hold for $\forall K > 0$

$$\begin{cases} (v(t_0) + c(t))^{\frac{1}{p}} \le \frac{1}{p} K^{\frac{1-p}{p}} (v(t_0) + c(t)) + \frac{p-1}{p} K^{\frac{1}{p}}, \\ (v(t_0) + c(t))^{\frac{r}{p}} \le \frac{r}{p} K^{\frac{r-p}{p}} (v(t_0) + c(t)) + \frac{p-r}{p} K^{\frac{r}{p}}. \end{cases} \quad (44)$$

So Combining (43) and (44) we have

$$v(t_0)+c(t) \le v(t_0)+\int_{t_0}^{t} L(s, (\frac{1}{p} K^{\frac{1-p}{p}} (v(t_0)+c(s)) + \frac{p-1}{p} K^{\frac{1}{p}})(e_f(s,t_0))^{\frac{1}{p}})\Delta s$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s} h(\xi)[\frac{r}{p} K^{\frac{r-p}{p}} (v(t_0) + c(\xi)) + \frac{p-r}{p} K^{\frac{r}{p}}]\times (e_f(\xi,t_0))^{\frac{r}{p}}\Delta\xi\Delta s$$
$$= v(t_0)+\int_{t_0}^{t}\{L(s, (\frac{1}{p} K^{\frac{1-p}{p}} (v(t_0)+c(s))+\frac{p-1}{p} K^{\frac{1}{p}})\times (e_f(s,t_0))^{\frac{1}{p}}) - L(s, \frac{p-1}{p} K^{\frac{1}{p}} (e_f(s,t_0))^{\frac{1}{p}})$$
$$+L(s, \frac{p-1}{p} K^{\frac{1}{p}} (e_f(s,t_0))^{\frac{1}{p}})\}\Delta s$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s} h(\xi)[\frac{r}{p} K^{\frac{r-p}{p}} (v(t_0) + c(\xi)) + \frac{p-r}{p} K^{\frac{r}{p}}]\times (e_f(\xi,t_0))^{\frac{r}{p}}\Delta\xi\Delta s$$
$$\le v(t_0) + \int_{t_0}^{t}\{M(s, \frac{p-1}{p} K^{\frac{1}{p}}\times (e_f(s,t_0))^{\frac{1}{p}})[\frac{1}{p} K^{\frac{1-p}{p}} (v(t_0) + c(s))(e_f(s,t_0))^{\frac{1}{p}}]$$
$$+L(s, \frac{p-1}{p} K^{\frac{1}{p}} (e_f(s,t_0))^{\frac{1}{p}})\}\Delta s\}\Delta s$$
$$+ \int_{t_0}^{t}[\int_{t_0}^{s} h(\xi)\frac{r}{p} K^{\frac{r-p}{p}} (e_f(\xi,t_0))^{\frac{r}{p}}\Delta\xi](v(t_0)+c(s))\Delta s$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s} h(\xi)\frac{p-r}{p} K^{\frac{r}{p}} (e_f(\xi,t_0))^{\frac{r}{p}}\Delta\xi\Delta s$$
$$= v(t_0)+\widehat{H}_1(t)+\int_{t_0}^{t} \widehat{H}_2(s)(v(t_0)+c(s))\Delta s, \quad (45)$$

where $\widehat{H}_1(t)$, $\widehat{H}_2(t)$ are defined in (35) and (36) respectively.
Considering $\widehat{H}_1(t)$ is nondecreasing on $\mathbb{T}_0$, then by Lemma 20 we have

$$v(t_0) + c(t) \le [v(t_0) + \widehat{H}_1(t)]e_{\widehat{H}_2}(t,t_0), \quad (46)$$

Combining (42) and (46) we obtain

$$v(t) \le [v(t_0) + \widehat{H}_1(t)]e_{\widehat{H}_2}(t,t_0)e_f(t,t_0),$$
$$t \in [t_0, T] \bigcap \mathbb{T}. \quad (47)$$

Take $t = T$ in (47), considering $2v(t_0)-C = v(T)$, then

$$2v(t_0)-C = v(T) \le [v(t_0)+\widehat{H}_1(T)]e_{\widehat{H}_2}(T,t_0)e_f(T,t_0), \quad (48)$$

which is followed by

$$v(t_0) \le \frac{C + \widehat{H}_1(T)e_{\widehat{H}_2}(T,t_0)e_f(T,t_0)}{2 - e_{\widehat{H}_2}(T,t_0)e_f(T,t_0)}. \quad (49)$$

Then combining (37), (47), and (49) we can obtain the desired inequality (34). □

Remark 29 *Compared with the main results in [20, 21], the established results by Theorems 22-28 mainly deal with Gronwall-Bellman type inequalities on time scales including integrals on finite intervals, while in [20, Theorems 21, 22, 24, 26], some Gronwall-Bellman type inequalities on time scales including integrals on infinite intervals are concerned, and in [21, Theorems 2.1-2.3], some Gronwall-Bellman type inequalities on time scales including four iterated integrals are concerned.*

3 Applications

In this section, we will give some applications for the presented results above, and try to give explicit bounds for solutions of certain delay dynamic equations.

Example 1 *Consider the delay dynamic differential equation*

$$(u^p(t))^{\Delta} = F(t, u(\tau_1(t)), u(\tau_2(t)), \int_{t_0}^{t} W(\xi, u(\tau_3(\xi)))\Delta\xi),$$
$$t \in \mathbb{T}_0, \quad (50)$$

with the initial condition

$$\begin{cases} u(t) = \eta(t), t \in [\alpha, t_0] \bigcap \mathbb{T} \\ |\eta(\tau_i(t))| \le |C|^{\frac{1}{p}}, \ \forall t \in \mathbb{T}_0, \ \tau_i(t) \le t_0, \ i = 1, 2, 3, \end{cases} \quad (51)$$

where $u \in C_{rd}(\mathbb{T}_0, R)$, C is a constant with $C = u^p(t_0)$, $p > 0$ is a constant, $\eta \in C_{rd}([\alpha, t_0] \bigcap \mathbb{T}, R)$. α, τ_i, $i = 1, 2, 3$ are the same as in Theorem 22.

Theorem 30 *Suppose $u(t)$ is a solution of (50)-(51), and assume $|F(t,x,y,z)| \le f(t)|x|^p + g(t)|y|^q + |z|$, $|W(t,x)| \le h(t)|x|^r$, where f, g, h, q, r are defined the same as in Theorem 22, then the following inequality holds*

$$u(t) \le \{[|C| + H_2(t)e_{H_3}(t,t_0)]H_1(t)\}^{\frac{1}{p}}, \quad t \in \mathbb{T}_0, \tag{52}$$

where

$$H_1(t) = 1 + \int_{t_0}^{t} e_f(t,\sigma(s))f(s)\Delta s,$$

$$H_2(t) = \int_{t_0}^{t}\{g(s)[\frac{q}{p}K^{\frac{q-p}{p}}|C| + \frac{p-q}{p}K^{\frac{q}{p}}]H_1(s)^{\frac{q}{p}} + \int_{t_0}^{s} h(\xi)[\frac{r}{p}K^{\frac{r-p}{p}}|C| + \frac{p-r}{p}K^{\frac{r}{p}}]H_1(\xi)^{\frac{r}{p}}\Delta\xi\}\Delta s, \quad \forall K > 0$$

$$H_3(t) = g(t)\frac{q}{p}K^{\frac{q-p}{p}}H_1(t)^{\frac{q}{p}} + \int_{t_0}^{t} h(\xi)\frac{r}{p}K^{\frac{r-p}{p}} H_1(\xi)^{\frac{r}{p}}\Delta\xi, \ \forall K > 0.$$

Proof: The equivalent integral equation of (40) can be denoted by

$$u^p(t) = C + \int_{t_0}^{t} F(s, u(\tau_1(s)), u(\tau_2(s)), \int_{t_0}^{s} W(\xi, u(\tau_3(\xi)))\Delta\xi)\Delta s. \tag{53}$$

Then we have

$$\begin{aligned}|u^p(t)| &\le |C| + \int_{t_0}^{t} |F(s, u(\tau_1(s)), u(\tau_2(s)), \int_{t_0}^{s} W(\xi, u(\tau_3(\xi)))\Delta\xi)|\Delta s \\ &\le |C| + \int_{t_0}^{t}[f(s)|u(\tau_1(s))|^p + g(s)|u(\tau_2(s))|^q + |\int_{t_0}^{s} W(\xi, u(\tau_3(\xi)))\Delta\xi|]\Delta s \\ &\le |C| + \int_{t_0}^{t}[f(s)|u(\tau_1(s))|^p + g(s)|u(\tau_2(s))|^q + \int_{t_0}^{s} h(\xi)|u(\tau_3(\xi))|^r\Delta\xi]\Delta s. \end{aligned} \tag{54}$$

A suitable application of Theorem 22 (that is, $|C|$ takes the place of a(t) in Theorem 22, and $b(t) \equiv 1$ in Theorem 22) yields (52). □

Remark 31 *Under the conditions of Theorem 30, Considering $H_1(t) = 1 + \int_{t_0}^{t} e_f(t,\sigma(s))f(s)\Delta s = e_f(t,t_0)$, furthermore we have the following estimate*

$$|u(t)| \le \{[|C| + H_2(t)e_{H_3}(t,t_0)]e_f(t,t_0)\}^{\frac{1}{p}}, \quad t \in \mathbb{T}_0. \tag{55}$$

Example 2 *Consider the delay dynamic integral equation*

$$u^p(t) = C + \int_{t_0}^{t} \widehat{F}(s, u(\tau_1(s)), u(\tau_2(s)), \int_{t_0}^{s} \widehat{W}(\xi, u(\tau_3(\xi)))\Delta\xi)\Delta s + \int_{t_0}^{T} \widehat{F}(s, u(\tau_1(s)), u(\tau_2(s)), \int_{t_0}^{s} \widehat{W}(\xi, u(\tau_3(\xi)))\Delta\xi)\Delta s, \quad t \in \mathbb{T}_0, \tag{56}$$

with the initial condition (51), where $u \in C_{rd}(\mathbb{T}_0, R)$, C is a constant with $C = u^p(t_0)$, $p > 0$ is a constant, $\eta \in C_{rd}([\alpha,t_0]\bigcap\mathbb{T}, R)$. α, τ_i, $i = 1,2,3$ are the same as in Theorem 22.

Theorem 32 *Suppose $u(t)$ is a solution of (56), and assume $|\widehat{F}(t,x,y,z)| \le f(t)|x|^p + L(t,|y|^q) + |z|$, $|\widehat{W}(t,x)| \le h(t)|x|^r$, where f, h, r, L are defined the same as in Theorem 28, then the following inequality holds*

$$u(t) \le [\frac{|C| + \widehat{H}_1(T)e_{\widehat{H}_2}(T,t_0)e_f(T,t_0)}{2 - e_{\widehat{H}_2}(T,t_0)e_f(T,t_0)} + \widehat{H}_1(t)] \times e_{\widehat{H}_2}(t,t_0)e_f(t,t_0), \quad t \in [t_0,T]\bigcap\mathbb{T}, \tag{57}$$

provided that $e_{\widehat{H}_2}(T,t_0)e_f(T,t_0) < 2$, where $\widehat{H}_1(t)$, $\widehat{H}_2(t)$ are the same as in Theorem 28.

Proof: From (56) we have

$$\begin{aligned}|u^p(t)| &\le |C| + \int_{t_0}^{t} |\widehat{F}(s, u(\tau_1(s)), u(\tau_2(s)), \int_{t_0}^{s} \widehat{W}(\xi, u(\tau_3(\xi)))\Delta\xi)|\Delta s \\ &+ \int_{t_0}^{T} |\widehat{F}(s, u(\tau_1(s)), u(\tau_2(s)), \int_{t_0}^{s} \widehat{W}(\xi, u(\tau_3(\xi)))\Delta\xi)|\Delta s \\ &\le |C| + \int_{t_0}^{t}[f(s)|u(\tau_1(s))|^p + L(s, |u(\tau_2(s))|^q)\end{aligned}$$

$$
\begin{aligned}
&\quad +|\int_{t_0}^{s}\widehat{W}(\xi,u(\tau_3(\xi)))\Delta\xi|]\Delta s\\
&+\int_{t_0}^{T}[f(s)|u(\tau_1(s))|^p+L(s,|u(\tau_2(s))|^q)\\
&\quad +|\int_{t_0}^{s}\widehat{W}(\xi,u(\tau_3(\xi)))\Delta\xi|]\Delta s\\
&\le |C|+\int_{t_0}^{t}[f(s)|u(\tau_1(s))|^p+L(s,|u(\tau_2(s))|^q)\\
&\quad +\int_{t_0}^{s}h(\xi)|u(\tau_3(\xi))|^r]\Delta s\\
&+\int_{t_0}^{T}[f(s)|u(\tau_1(s))|^p+L(s,|u(\tau_2(s))|^q)\\
&\quad +\int_{t_0}^{s}h(\xi)|u(\tau_3(\xi))|^r]\Delta s. \qquad (58)
\end{aligned}
$$

Then under the condition $e_{\widehat{H}_2}(T,t_0)e_f(T,t_0)<2$, a suitable application of Theorem 28 yields (57). □

4 Conclusions

We have established several new delay integral inequalities on time scales, which can be used to provide explicit bounds for solutions of certain delay dynamic equations. As one can see from Remark 1-4, Theorem 22 generalize many known results in the literature, and unify some integral inequalities on continuous functions and their corresponding discrete versions to some degree. The process of Theorem 22-28 can be applied to establish delay inequalities with two independent variables on time scales, which are supposed to further research.

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