On using the He's polynomials for solving the nonlinear coupled evolution equations in mathematical physics

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Abstract: In this article, we apply the modified variational iteration method for solving the (1+1)- dimensional Ramani equations and the (1+1)-dimensional Joulent Moidek (JM) equations together with the initial conditions. The proposed method is modified the variational iteration method by the introducing He's polynomials in the correction functional. The analytical results are calculated in terms of convergent series with easily computated components.

Key–Words: Variational iteration method, Homotopy perturbation methods, Coupled nonlinear evaluation equations, Exact solutions

1 Introduction

The nonlinear coupled evolution equations have many wide array of applications of many fields, which described the motion of the isolated waves, localized in a small part of space, in many fields such as physics, mechanics, biology, hydrodynamics, plasma physics, etc.. To further explain some physical phenomena, searching for exact solutions of nonlinear partial differential equations is very important. Up to now, many researches in mathematical physics have paid attention to these topics, and a lot of powerful methods have been presented such as the modified extended tanh-function method [7,10,12,38], generalized F-expansion method [34], Adomian decomposition method [1-3,39], homotopy analysis method [5,8,35], Jacobi elliptic function method [36], the tanh-hyperbolic function method [25-26], the extended F-expansion method [23]. He [15-22] developed the variational iteration method and homotopy perturbation method for solving linear and nonlinear initial and boundary value problems. It is worth mentioning that the origin of the variational iteration method can be traced by Inokuti [24], but the real potential of this method was explored by He [15]. Moreover, He realized the physical significance of the variational iteration method, its compatibility with the physical problems and applied this promising technique to a wide class of linear and nonlinear, ordinary, partial, deterministic or stochastic differential equation. The homotopy perturbation method [9,13,17,19-22] was also developed by He by merging two techniques, the standard homotopy and the perturbation. The homotopy perturbation method was formulated by taking full advantage of the standard homotopy and perturbation methods. The variational iteration and homotopy perturbation methods have been applied to a wide class of functional equations. In these methods the solution is given in an infinite series usually converging to an accurate solution. In a later work Ghorbani et.al. [14] splited the nonlinear term into a series of polynomials calling them as the He's polynomials. Most recently, Noor and Mohyud- Din used this concept for solving nonlinear boundary value problems (see Ref.[29-31]) and anther authors [4]. The basic motivation of this paper is an extension of the modified variational iteration method which is formulated by the coupled of variational iteration method and He's polynomials for solving the (1+1)-dimensional Ramani equations and the (1+1)-dimensional Jaulent-Miodek (JM) equations. The MVIM provides the solution in a rapid convergent series which may lead the solution to a closed form. In this method, the correct functional is developed [15,31-33] and the Lagrange multipliers are calculated optimally via variational theory. The use of Lagrange multipliers reduce the successive application of the integral operator and the cumbersome of huge computational work while still maintaining a very high level of accuracy. Finally, He's polynomials are introduced in the correction functional and the comparison of like powers of p gives solutions of various orders. In this paper, we use the modified variational iteration method (MVIM) to solve the (1+1)-dimensional Ramani equations [37]

$$u_{6x} + 15u_{xx}u_{3x} + 15u_xu_{4x} + 45u_x^2u_{xx} - 5(u_{3xt} + 3u_{xx}u_t + 3u_xu_{xt}) - 5u_{tt} + 18v_x = 0,$$
$$v_t - v_{3x} - 3v_xu_x - 3vu_{xx} = 0, \quad (1)$$

and the (1+1)-dimensional Jaulent-Miodek (JM) equations[11]

$$u_t + u_{xxx} + \frac{3}{2}vv_{xxx} + \frac{9}{2}v_xv_{xx} - 6uu_x - 6uvv_x - \frac{3}{2}u_xv^2 = 0,$$
$$v_t + v_{xxx} - 6u_xv - 6uv_x - \frac{15}{2}v_xv^2 = 0. \quad (2)$$

2 Variational iteration method

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g. \quad (3)$$

where L is a linear operator, N is a nonlinear operator and g is the forcing term. According to variational iteration method [15,31-33], we can construct a correct functional as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)[Lu_n(x,\tau) + N\widetilde{u}_n(x,\tau) - g]d\tau, (n \geq 0), \quad (4)$$

where λ is a Lagrange multiplier [15], which can be identified optimally via variational iteration method. The subscript n denotes the nth approximation, $\widetilde{u}$ is considered as restricted variation. i.e. $\delta\widetilde{u} = 0$. Eq.(4) is called as a correct functional. The solution of the linear problem can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principals of variational iteration method and its applicability for various kinds of differential equations are given in [31-33]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation $u_{n+1}, n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0. Consequently, the solution is given by

$$u = \lim_{n\to\infty} u_n. \quad (5)$$

3 Homotopy perturbation method

The homotopy perturbation method is considered as spacial case of homotopy analysis method. To illustrate the homotopy perturbation method [9,13,27], we consider a general equation of the type,

$$L(u) = 0, \quad (6)$$

where L is any integral or differential operator. We define a convex homotopy $H(u,p)$ by

$$H(u,p) = (1-p)F(u) + pL(u), \quad (7)$$

where $F(u)$ is a functional operator with known solution v_0, which can be obtained easily. It is clear that, for

$$H(u,p) = 0, \quad (8)$$

we have

$$H(u,0) = F(u), \qquad H(u,1) = L(u). \quad (9)$$

This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(f,1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u) = 0$ is continuously deforms the original problem $L(u) = 0$. The embedding parameter $p \in [0,1]$ can be considered as an expanding parameter [9,13,27]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter to obtain

$$u = \sum_{i=0}^{\infty} p^iu_i = u_0 + pu_1 + p^2u_2 + p^3u_3 + ... \quad (10)$$

If $p \to 1$, then (10) corresponds to (7) and becomes the approximate solution of the form,

$$f = \lim_{p\to1} u = \sum_{i=0}^{\infty} u_i. \quad (11)$$

It is well known that the series (10) is convergent for most of the cases and also the rate of convergence is depending on $L(u)$; (see[19-22]). we assume that (10) has a unique solution. The comparisons of like powers of p give solutions of various orders.

4 Modified variational iteration method (MVIM) with He's polynomials

The modified variational iteration method is obtained by the elegant coupling of the correction functional formula (4) with the He's polynomial [16-22]. According to [16-22] He has been considered the solution u of the homptopy equation as a series of p which obtained in Eq. (10), and the method considered the nonlinear term $N(u)$as

$$N(u)=\sum_{i=0}^{\infty}p^iH_i=H_0+pH_1+p^2H_2+..., \quad (12)$$

where $H_n's$ are the so-called He's polynomials [16-22], which can be calculated by using the formula

$$H_n(u_0,u_1,...,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\{N(\sum_{i=0}^{n}p^iu_i)\}_{p=0}, n=0,1,2,.. \quad (13)$$

The modified variational iteration method is obtained by the elegant coupling of correction functional (4) of VIM with He's polynomial [16-22] and is given by

$$\sum_{n=0}^{\infty}p^nu_n=u_0+p\int_0^t\lambda(\tau)[\sum_{n=0}^{\infty}p^nL(u_n)+\sum_{n=0}^{\infty}p^nN(\widetilde{u}_n)]d\tau-\int_0^t\lambda(\tau)gd\tau. \quad (14)$$

Comparisons of like powers of p give solutions of various orders.

5 Applications

The modified variational iteration method is used to solve the (1+1)-dimensional Ramani equations (1), and the (1+1)-dimensional Jaulent-Miodek (JM) equations (2).

5.1 Solving the (1+1)-dimensional Ramani equations using MVIM

In this subsection, we find the solutions $u(x,t)$ and $v(x,t)$ satisfying the coupled nonlinear Ramani equations (1) with the following initial conditions [37]:

$$\begin{aligned}&u(x,0)=a_0+2\alpha\coth(\alpha x),\\&u_t(x,0)+2t\beta\alpha^2\text{csch}^2(\alpha x),\\&v(x,0)=-\frac{4}{9}\beta\alpha^4-\frac{16}{27}\alpha^6+\frac{5}{9}\beta^2\alpha^2-\frac{5}{54}\beta^3+\\&(\frac{20}{9}\beta\alpha^4+\frac{16}{9}\alpha^6-\frac{5}{9}\beta^2\alpha^2)\coth^2(\alpha x). \quad (15)\end{aligned}$$

These initial conditions follow by setting $t=0$ in the following exact solutions of Eqs. (1):

$$\begin{aligned}&u(x,t)=a_0+2\alpha\coth(\alpha\zeta),\\&v(x,t)=-\frac{4}{9}\beta\alpha^4-\frac{16}{27}\alpha^6+\frac{5}{9}\beta^2\alpha^2-\\&\frac{5}{54}\beta^3+(\frac{20}{9}\beta\alpha^4+\frac{16}{9}\alpha^6-\\&\frac{5}{9}\beta^2\alpha^2)\coth^2(\alpha\zeta), \quad (16)\end{aligned}$$

where $\zeta=(x-\beta t)$, a_0, β and α are arbitrary constants. These exact solutions have been derived by Yusufoglu et al. [37] using the tanh method. Let us now solve the initial value problem (1) with the initial conditions (15) using the MVIM. To this end, we convey the basic idea of the modified variational iteration method for the equations (1), let us consider the functional iteration formula

$$\begin{aligned}u_{n+1}(x,t)&=u_n+\int_0^t\lambda_1(\tau)[-5(u_n)_{\tau\tau}+(\widetilde{u}_n)_{6x}+\\&15(\widetilde{u}_n)_{xx}(\widetilde{u}_n)_{3x}+15(\widetilde{u}_n)_x(\widetilde{u}_n)_{4x}+\\&45(\widetilde{u}_n)_x^2(\widetilde{u}_n)_{xx}-5\{(\widetilde{u}_n)_{3x\tau}+3(\widetilde{u}_n)_{xx}(\widetilde{u}_n)_\tau+\\&3(\widetilde{u}_n)_x(\widetilde{u}_n)_{x\tau}\}+18(\widetilde{v}_n)_x]d\tau,\\v_{n+1}(x,t)&=v_n+\int_0^t\lambda_2(\tau)[(v_n)_\tau-(\widetilde{v}_n)_{3x}-\\&3(\widetilde{v}_n)_x(\widetilde{u}_n)_x-3(\widetilde{v}_n)(\widetilde{u}_n)_{xx}]d\tau. \quad (17)\end{aligned}$$

Making the correct functional stationary, the Lagrange multipliers can be identified as $\lambda_1(\tau)=\frac{t-\tau}{5}$ and $\lambda_2(\tau)=-1$, consequently, we have

$$\begin{aligned}u_{n+1}(x,t)&=u_n+\int_0^t\frac{t-\tau}{5}[-5(u_n)_{\tau\tau}+(u_n)_{6x}+\\&15(u_n)_{xx}(u_n)_{3x}+15(u_n)_x(u_n)_{4x}+\\&45(u_n)_x^2(u_n)_{xx}-5\{(u_n)_{3x\tau}+\\&3(u_n)_{xx}(u_n)_\tau+3(u_n)_x(u_n)_{x\tau}\}+18(v_n)_x]d\tau,\\v_{n+1}(x,t)&=v_n-\int_0^t[(v_n)_\tau-(v_n)_{3x}-\\&3(v_n)_x(u_n)_x-3v_n(u_n)_{xx}]d\tau. \quad (18)\end{aligned}$$

Applying the modified variational iteration method, hence

$$\begin{aligned}\sum_{n=0}^{\infty}p^nu_n&=u_0+p\int_0^t\frac{t-\tau}{5}[-5(\sum_{n=0}^{\infty}p^nu_n)_{\tau\tau}+\\&(\sum_{n=0}^{\infty}p^nu_n)_{6x}+15(\sum_{n=0}^{\infty}p^nu_n)_{xx}(\sum_{n=0}^{\infty}p^nu_n)_{3x}\\&+15(\sum_{n=0}^{\infty}p^nu_n)_x(\sum_{n=0}^{\infty}p^nu_n)_{4x}+\\&45(\sum_{n=0}^{\infty}p^nu_n)_x^2(\sum_{n=0}^{\infty}p^nu_n)_{xx}-\\&5[(\sum_{n=0}^{\infty}p^nu_n)_{(3x)\tau}+\\&3(\sum_{n=0}^{\infty}p^nu_n)_{xx}(\sum_{n=0}^{\infty}p^nu_n)_\tau+\\&3(\sum_{n=0}^{\infty}p^nu_n)_x(\sum_{n=0}^{\infty}p^nu_n)_{x\tau}]+\\&18(\sum_{n=0}^{\infty}p^nv_n)_x]d\tau, \quad (19)\end{aligned}$$

and

$$\sum_{n=0}^{\infty} p^n v_n = v_0 - p\int_0^t [(\sum_{n=0}^{\infty} p^n v_n)_\tau - (\sum_{n=0}^{\infty} p^n v_n)_{3x} - 3(\sum_{n=0}^{\infty} p^n v_n)_x(\sum_{n=0}^{\infty} p^n u_n)_x - 3(\sum_{n=0}^{\infty} p^n v_n)(\sum_{n=0}^{\infty} p^n u_n)_{xx}]d\tau, \quad (20)$$

using Eqs. (19) and (20) to compare the coefficients of like powers of p then we have

$$\begin{aligned}
u_0(x,t) &= u(x,0) + u_t(x,0),\\
u_1(x,t) &= \int_0^t \frac{t-\tau}{5}[(u_0)_{6x} + 15(u_0)_{xx}(u_0)_{3x} + 15(u_0)_x(u_0)_{4x} + 45(u_0)_x^2(u_0)_{xx} - 5\{(u_0)_{(3x)\tau} + 3(u_0)_{xx}(u_0)_\tau + 3(u_0)_x(u_0)_{x\tau}\} + 18(v_0)_x]d\tau,\\
u_2(x,t) &= \int_0^t \frac{t-\tau}{5}[-5(u_1)_{\tau\tau} + (u_1)_{6x} + 15\{(u_0)_{xx}(u_1)_{3x} + (u_1)_{xx}(u_0)_{3x}\} + 15\{(u_0)_x(u_1)_{4x} + (u_1)_x(u_0)_{4x}\} + 45\{(u_0)_x^2(u_1)_{xx} + 2(u_1)_x(u_0)_x(u_0)_{xx}\} - 5\{(u_1)_{(3x)\tau} + 3\{(u_0)_{xx}(u_1)_\tau + (u_1)_{xx}(u_0)_\tau\} + 3\{(u_0)_x(u_1)_{x\tau} + (u_1)_x(u_0)_{x\tau}\}\} + 18(v_1)_x]d\tau, \quad (21)
\end{aligned}$$

$$\begin{aligned}
v_0(x,t) &= v(x,0),\\
v_1(x,t) &= \int_0^t [(v_0)_{3x} + 3(v_0)_x(u_0)_x + 3v_0(u_0)_{xx}]d\tau,\\
v_2(x,t) &= -\int_0^t [(v_1)_\tau - (v_1)_{3x} - 3\{(v_0)_x(u_1)_x + (v_1)_x(u_0)_x\} - 3\{v_0(u_1)_{xx} + v_1(u_0)_{xx}\}]d\tau, \quad (22)
\end{aligned}$$

the other components can be found similarly. After some reduction, we have

$$\begin{aligned}
u_0(x,t) &= a_0 + 2\alpha\coth(\alpha x) + 2t\beta\alpha^2\text{csch}^2(\alpha x),\\
u_1(x,t) &= -2\alpha^3\beta^2t^2\coth(\alpha x)\text{csch}^2(\alpha x)\\
u_2(x,t) &= \frac{2}{3}\alpha^4\beta^3t^3\{2\coth^2(\alpha x)\text{csch}^2(\alpha x) + \text{csch}^4(\alpha x)\}, \quad (23)
\end{aligned}$$

$$\begin{aligned}
v_0(x,t) &= -\frac{4}{9}\beta\alpha^4 - \frac{16}{27}\alpha^6 + \frac{5}{9}\beta^2\alpha^2 - \frac{5}{54}\beta^3 + (\frac{20}{9}\beta\alpha^4 + \frac{16}{9}\alpha^6 - \frac{5}{9}\beta^2\alpha^2)\coth^2(\alpha x),\\
v_1(x,t) &= 2t\alpha\beta(\frac{20}{9}\beta\alpha^4 + \frac{16}{9}\alpha^6 - \frac{5}{9}\beta^2\alpha^2)\coth(\alpha x)\text{csch}^2(\alpha x),\\
v_2(x,t) &= -(\frac{20}{9}\beta\alpha^4 + \frac{16}{9}\alpha^6 - \frac{5}{9}\beta^2\alpha^2)t^2 \times\{2\alpha^2\beta^2\coth^2(\alpha x)\text{csch}^2(\alpha x) + \text{csch}^4(\alpha x)\}. \quad (24)
\end{aligned}$$

Therefore, using the Eq.(10), then approximate solutions of the system of equations of Eqs. (1) take the following forms:

$$u(x,t) = a_0 + 2\alpha\coth(\alpha x) + 2t\beta\alpha^2\text{csch}^2(\alpha x) - 2\alpha^3\beta^2t^2\coth(\alpha x)\text{csch}^2(\alpha x) + \frac{2}{3}\alpha^4\beta^3t^3 \{2\coth^2(\alpha x)\text{csch}^2(\alpha x) + \text{csch}^4(\alpha x)\} + ..., \quad (25)$$

$$v(x,t) = -\frac{4}{9}\beta\alpha^4 - \frac{16}{27}\alpha^6 + \frac{5}{9}\beta^2\alpha^2 - \frac{5}{54}\beta^3 + (\frac{20}{9}\beta\alpha^4 + \frac{16}{9}\alpha^6 - \frac{5}{9}\beta^2\alpha^2) \times\{\coth^2(\alpha x) + 2t\alpha\beta\coth(\alpha x)\text{csch}^2(\alpha x) - 2t^2\alpha^2\beta^2\coth^2(\alpha x)\text{csch}^2(\alpha x) - t^2\text{csch}^4(\alpha x)\} + ... \quad (26)$$

The accuracy of the modified variational iteration method for the Eqs. (1) under conditions (15) is controllable and the absolute errors are very small with the present choice of x, t. These results are listed in Tables 1, 2 and Figures 1-4. It is also clear that when more terms for MVIM are computed, the numerical results are much more closer to the corresponding exact solution.

Table 1. The MVIM results of $u(x,t)$ for the first three approximation in comparison with the exact solution if $a_0 = 1$, $\beta = \alpha = .01$,and $t = 20$ for the solution of the system (1) with the initial conditions(15).

x	u_{Exact}	u_{MVIM}	$\lvert u_{Exact} - u_{MVIM}\rvert$
-50	0.956868	0.956869	1.27251607E-6
-40	0.947597	0.9476	2.48972102E-6
-30	0.931774	0.93178	5.90319012E-6
-20	0.899647	0.899667	1.98912333E-5
-10	0.803242	0.8034	1.58430305E-4
10	1.20473	1.20457	1.58430305E-4
20	1.10233	1.10231	2.00989070E-5
30	1.06909	1.06908	5.94270457E-6
40	1.05288	1.05287	2.50222741E-6
50	1.04343	1.04343	1.27764171E-6

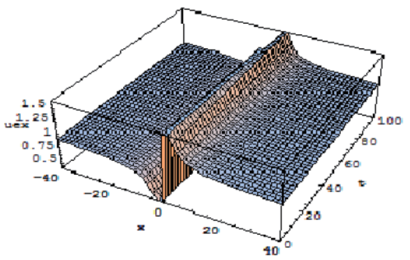

Figure 1: The exact solution of u(x,t) for the equations (1) if $a_0 = 1$, $\beta = \alpha = .01$.

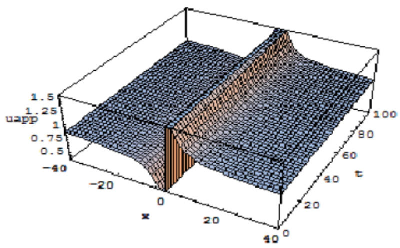

Figure 2: The approximate solution of u(x,t) for the first three approximation of the equations (1) if $a_0 = 1$, $\beta = \alpha = .01$.

Table 2. The MVIM results of $v(x,t)$ for the first three approximation in comparison with the analytical solution if $a_0 = 1$, $\beta = \alpha = .01$,and $t = 20$ for the solution of the system (1) with the initial conditions(15).

x	v_{Exact}	v_{MVIM}	$\lvert v_{Exact} - v_{MVIM}\rvert$
-50	-1.1204856E-7	-1.12232E-7	1.853690E-7
-40	-1.2401512E-7	-1.24375E-7	1.855450E-7
-30	-1.4991042E-7	-1.50764E-7	1.860388E-7
-20	-2.2395349E-7	-2.26833E-7	1.880649E-7
-10	-6.2401173E-7	-6.46982E-7	2.081564E-7
10	-6.2401173E-7	-6.00746E-7	1.619202E-7
20	-2.2395349E-7	-2.21054E-7	1.822860E-7
30	-1.4991042E-7	-1.49052E-7	1.843273E-7
40	-1.2401512E-7	-1.23653E-7	1.848238E-7
50	-1.1204856E-7	-1.11863E-7	1.850006E-7

5.2 Solving the (1+1)-dimensional Jaulent-Miodek (JM) equations using MVIM

In this subsection, we find the solutions $u(x,t)$, $v(x,t)$ satisfying the (1+1)-dimensional Jaulent-Miodek (JM) equations with the following initial conditions [11]:

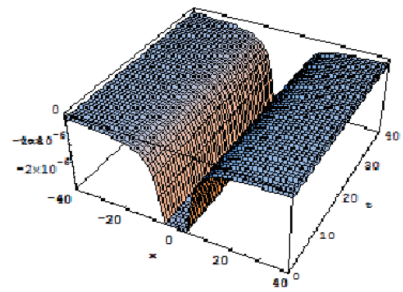

Figure 3: The exact solution of v(x,t) for the equations (1) if $a_0 = 1$, $\beta = \alpha = .01$.

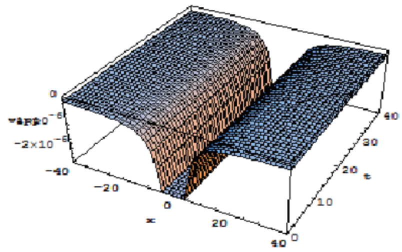

Figure 4: The approximate solution of v(x,t) for the first three approximation for the equations (1) if $a_0 = 1$, $\beta = \alpha = .01$.

$$
\begin{aligned}
u(x,0) &= \frac{1}{4}(c-b^2) - \frac{1}{2}b\sqrt{c}\,\mathrm{sech}(\sqrt{c}\,x) - \\
&\frac{3c}{4}\mathrm{sech}^2(\sqrt{c}\,x), \\
v(x,0) &= b + \sqrt{c}\,\mathrm{sech}(\sqrt{c}\,x). \qquad (27)
\end{aligned}
$$

These initial conditions follow by setting $t = 0$ in the following exact solutions of eqs. (2):

$$
\begin{aligned}
u(x,t) &= \frac{1}{4}(c-b^2) - \frac{1}{2}b\sqrt{c} \\
&\mathrm{sech}(\sqrt{c}\,(x + \frac{(6b^2+c)t}{2})) - \\
&\frac{3c}{4}\mathrm{sech}^2(\sqrt{c}\,(x + \frac{(6b^2+c)t}{2})), \\
v(x,t) &= b + \sqrt{c} \\
&\times \mathrm{sech}(\sqrt{c}\,(x + \frac{(6b^2+c)t}{2})), \qquad (28)
\end{aligned}
$$

where c and b are arbitrary constants. These exact solutions have been derived by Fan [11] using the unified algebraic method. Let us now solve the initial value problem (2) and (27) using the MVIM. To this end, we convey the basic idea of the modified variational iteration method for the equations (2), let us consider the functional iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(\tau)[(u_n)_\tau + (\widetilde{u}_n)_{xxx} + \frac{3}{2}(\widetilde{v}_n)(\widetilde{v}_n)_{xxx} + \frac{9}{2}(\widetilde{v}_n)_x(\widetilde{v}_n)_{xx} - 6\{(\widetilde{u}_n)(\widetilde{u}_n)_x + (\widetilde{u}_n)(\widetilde{v}_n)(\widetilde{v}_n)_x\} - \frac{3}{2}(\widetilde{u}_n)_x(\widetilde{v}_n)^2]d\tau,$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(\tau)[(v_n)_\tau + (\widetilde{v}_n)_{xxx} - 6\{(\widetilde{u}_n)_x(\widetilde{v}_n) + (\widetilde{u}_n)(\widetilde{v}_n)_x\} - \frac{15}{2}(\widetilde{v}_n)_x(\widetilde{v}_n)^2]d\tau. \quad (29)$$

Making the correct functional stationary, the Lagrange multipliers can be identified as $\lambda_1(\tau) = -1 = \lambda_2(\tau) = -1$, consequently

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t [(u_n)_\tau + (u_n)_{xxx} + \frac{3}{2}(v_n)(v_n)_{xxx} + \frac{9}{2}(v_n)_x(v_n)_{xx} - 6\{u_n)(u_n)_x + (u_n)(v_n)(v_n)_x\} - \frac{3}{2}(u_n)_x(v_n)^2]d\tau,$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t [(v_n)_\tau + (v_n)_{xxx} - 6\{(u_n)_x(v_n) + (u_n)(v_n)_x\} - \frac{15}{2}(v_n)_x(v_n)^2]d\tau. \quad (30)$$

Applying the modified variational iteration method, we have

$$\sum_{n=0}^{\infty} p^n u_n = u_0 - p\int_0^t [(\sum_{n=0}^{\infty} p^n u_n)_\tau + (\sum_{n=0}^{\infty} p^n u_n)_{xxx} + \frac{3}{2}(\sum_{n=0}^{\infty} p^n v_n)(\sum_{n=0}^{\infty} p^n v_n)_{xxx} + \frac{9}{2}(\sum_{n=0}^{\infty} p^n v_n)_x(\sum_{n=0}^{\infty} p^n v_n)_{xx} - 6\{\sum_{n=0}^{\infty} p^n u_n)(\sum_{n=0}^{\infty} p^n u_n)_x + (\sum_{n=0}^{\infty} p^n u_n)(\sum_{n=0}^{\infty} p^n v_n)(\sum_{n=0}^{\infty} p^n v_n)_x\} - \frac{3}{2}(\sum_{n=0}^{\infty} p^n u_n)_x(\sum_{n=0}^{\infty} p^n v_n)^2]d\tau,$$

$$\sum_{n=0}^{\infty} p^n v_n = v_0(x,t) - p\int_0^t [(v_n)_\tau + (v_n)_{xxx} - 6\{(u_n)_x(v_n) + (u_n)(v_n)_x\} - \frac{15}{2}(\sum_{n=0}^{\infty} p^n v_n)_x(\sum_{n=0}^{\infty} p^n v_n)^2]d\tau. \quad (31)$$

Using eqs. (31) to compare the coefficient of like powers of p then we have

$$u_0(x,t) = u(x,0),$$

$$u_1(x,t) = -\int_0^t [(u_0)_{xxx} + \frac{3}{2}(v_0)(v_0)_{xxx} + \frac{9}{2}(v_0)_x(v_0)_{xx} - 6\{u_0)(u_0)_x + (u_0)(v_0)(v_0)_x\} - \frac{3}{2}(u_0)_x(v_0)^2]d\tau,$$

$$u_2(x,t) = -\int_0^t [(u_0)_\tau(u_1)_{xxx} + \frac{3}{2}\{(v_1)(v_0)_{xxx} + (v_0)(v_1)_{xxx}\} + \frac{9}{2}\{(v_0)_x(v_1)_{xx} + (v_1)_x(v_0)_{xx}\} - 6\{\{u_0)(u_0)_x + u_1)(u_1)_x\} + \{(u_1)(v_0)(v_0)_x + (u_0)(v_1)(v_0)_x + (u_0)(v_0)(v_1)_x\}\} - \frac{3}{2}\{(u_1)_x(v_0)^2 + 2(u_0)_x(v_0)(v_1)\}]d\tau, \quad (32)$$

$$v_0(x,t) = v(x,0),$$

$$v_1(x,t) = -\int_0^t [(v_0)_{xxx} - 6\{(u_0)_x(v_0) + (u_0)(v_0)_x\} - \frac{15}{2}(v_0)_x(v_0)^2]d\tau,$$

$$v_2(x,t) = -\int_0^t [(v_1)_\tau - v_1)_{xxx} - 6\{\{(u_0)_x(v_1) + (u_1)_x(v_0)\} + \{(u_0)(v_1)_x + (u_1)(v_0)_x\}\} - \frac{15}{2}\{(v_1)_x(v_0)^2 + 2(v_0)_x v_1 v_0\}]d\tau, \quad (33)$$

The other components can be found similarly. After some reduction, we have

$$u_0(x,t) = \frac{1}{4}(c-b^2) - \frac{1}{2}b\sqrt{c}\,\mathrm{sech}(\sqrt{c}\,x) - \frac{3c}{4}\mathrm{sech}^2(\sqrt{c}\,x),$$

$$u_1(x,t) = -\frac{t}{4}(c+6b^2)\{bc\,\mathrm{sech}(\sqrt{c}\,x)\tanh(\sqrt{c}\,x) + 3c^{3/2}\,\mathrm{sech}^2(\sqrt{c}\,x)\tanh(\sqrt{c}\,x)\},$$

$$u_2(x,t) = -\frac{t^2}{8}(c+6b^2)^2\{bc^{3/2}\,(\mathrm{sech}^3(\sqrt{c}\,x) + \mathrm{sech}(\sqrt{c}\,x)\tanh^2(\sqrt{c}\,x)) + 3c^2(\mathrm{sech}^4(\sqrt{c}\,x) + 2\mathrm{sech}^2(\sqrt{c}\,x)\tanh^2(\sqrt{c}\,x))\}, \quad (34)$$

$$v_0(x,t) = b + \sqrt{c}\,\text{sech}(\sqrt{c}\,x),$$
$$v_1(x,t) = \frac{c\,t}{2}(c+6b^2)\text{sech}(\sqrt{c}\,x)\tanh(\sqrt{c}\,x),$$
$$v_2(x,t) = \frac{c^{3/2}t^2}{4}(c+6b^2)^2\{\text{sech}^3(\sqrt{c}\,x) + \text{sech}(\sqrt{c}\,x)\tanh^2(\sqrt{c}\,x)\}, \quad (35)$$

In this manner the other components can be easily obtained. We construct the solutions $u(x,t)$ and $v(x,t)$ as follows:

$$u(x,t) = \frac{1}{4}(c-b^2) - \frac{1}{2}b\sqrt{c}\,\text{sech}(\sqrt{c}\,x) - \frac{3c}{4}\text{sech}^2(\sqrt{c}\,x) - \frac{t}{4}(c+6b^2)\{bc\,\text{sech}(\sqrt{c}\,x) \times \tanh(\sqrt{c}\,x) + 3c^{3/2}\,\text{sech}^2(\sqrt{c}\,x) \tanh(\sqrt{c}\,x)\} - \frac{c^{3/2}t^2}{8}(c+6b^2)^2\{bc^{3/2} (\,\text{sech}^3(\sqrt{c}\,x) + \,\text{sech}(\sqrt{c}\,x)\tanh^2(\sqrt{c}\,x)) + 3c^2(\text{sech}^4(\sqrt{c}\,x) + 2\text{sech}^2(\sqrt{c}\,x) \times \tanh^2(\sqrt{c}\,x))\} + ..., \quad (36)$$

$$v(x,t) = b + \sqrt{c}\,\text{sech}(\sqrt{c}\,x) + \frac{ct}{2}(c+6b^2) \times \text{sech}(\sqrt{c}\,x)\tanh(\sqrt{c}\,x) + \frac{c^{3/2}\,t^2}{4}(c+6b^2)^2 \times \{\text{sech}^3(\sqrt{c}\,x) + \text{sech}(\sqrt{c}\,x) \times \tanh^2(\sqrt{c}\,x)\} + ... \quad (37)$$

The accuracy of the MVIM for the eqs. (2) under conditions (27) are controllable and the absolute errors are very small with the present choice of x, t. These results are listed in Tables 3, 4 and Figures 5-8. It is also clear that when more terms for homotopy analysis method are computed, the numerical results are much more closer to the corresponding exact solution.

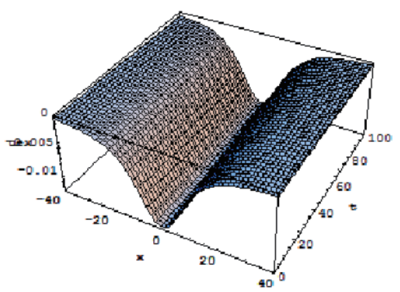

Figure 5: The exact solution of u(x,t) for the equations (2) if $b = .1, c = .01$.

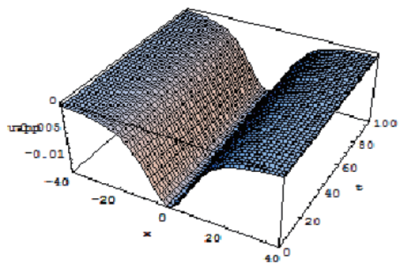

Figure 6: The approximate solution of u(x,t) for the first three approximation for the equations (2) if $b = .1, c = .01$.

Table 3. The MVIM results of $u(x,t)$ for the first three approximation in comparison with the exact solution if $b = .1, c = .01$ and $t = .2$ for the solution of the Eqs. (2) with the initial conditions (27).

x	u_{Exact}	u_{MVIM}	$\lvert u_{Exact} - u_{MVIM}\rvert$
-50	-0.00006878	-0.000068786	1.03236E-9
-40	-0.00019329	-0.000193286	7.98438E-9
-30	-0.00057108	-0.000571017	6.75572E-8
-20	-0.00186051	-0.00185984	6.65449E-7
00	-0.00639517	-0.00638898	6.18910E-6
00	-0.0125	-0.0124967	3.29510E-6
10	-0.00638499	-0.00639128	6.28541E-6
20	-0.00185728	-0.00185806	7.81587E-7
30	-0.000570186	-0.00057026	8.12470E-8
40	-0.00019301	-0.00019302	9.60074E-9
50	-0.000068689	-0.00006869	1.22044E-9

Table 4. The MMVIM results of $v(x,t)$ for the first three approximation in comparison with the exact solution if $b = .1$, $c = .01$ and $t = .2$ for the solution of the eqs. (2) with the initial conditions (27).

x	v_{Exact}	v_{MVIM}	$\lvert v_{Exact} - v_{MVIM}\rvert$
-50	0.101348	0.101347	1.6994901E-6
-40	0.103664	0.10366	4.8481756E-6
-30	0.10994	0.109915	2.5172504E-5
-20	0.126598	0.126426	1.7214243E-4
-10	0.16484	0.163784	1.0560166E-3
00	0.2	0.199999	1.1865000E-6
10	0.164771	0.165828	1.0567760E-3
20	0.126562	0.126735	1.7238910E-4
30	0.109926	0.109951	2.5211635E-5
40	0.103659	0.103664	4.8552982E-6
50	0.101347	0.101348	1.7027722E-6

6 Conclusion

In the present paper, the modified variational iteration method (MVIM) is used to find the solutions of

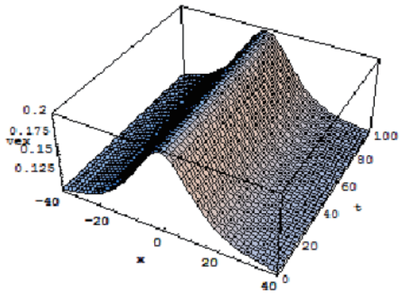

Figure 7: The exact solution of v(x,t) for the equations (2) if $b = .1, c = .01$.

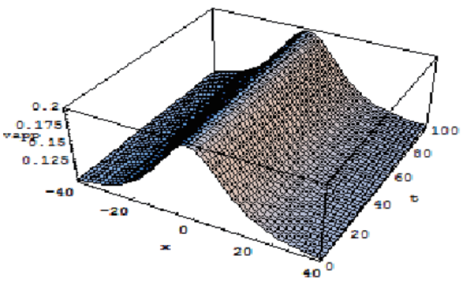

Figure 8: The approximate solution of v(x,t) for the first three approximation for the equations (2) if $b = .1, c = .01$.

the nonlinear coupled equations in the mathematical physics via the (1+1)-dimensional Ramani equations and the (1+1)-dimensional Jaulent-Miodek (JM) equations together with the initial conditions. It can be concluded that the MVIM is very powerful and efficient in finding the exact solutions for wide classes of problems. It is worth pointing out that the MVIM presents rapid convergence solutions.

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