On the global stability of the nonlinear difference equation $x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-m} + \alpha_3 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-m} + \beta_3 x_{n-k}}$

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Abstract: The main objective of this paper is to study the boundedness character, the periodicity character, the convergence and the global stability of positive solutions of the difference equation

$$x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-m} + \alpha_3 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-m} + \beta_3 x_{n-k}}, \quad n = 0, 1, 2, \cdots$$

where the coefficients $\alpha_i, \beta_i \in (0, \infty)$ for $i = 0, 1, 2, 3,$ and l, m, k are positive integers. The initial conditions x_{-k},...,x_{-m},...,x_{-l}, ...,x_{-1}, x_0 are arbitrary positive real numbers such that $l < m < k$. Some numerical experiments are presented.

Key–Words: Difference equations, boundedness, period two solutions, convergence, global stability.

1 Introduction

Our goal in this paper is to investigate the boundedness character, the periodicity character, the convergence and the global stability of positive solutions of the difference equation

$$x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-m} + \alpha_3 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-m} + \beta_3 x_{n-k}}, \quad n = 0, 1, 2, ... \quad (1)$$

where the coefficients $\alpha_i, \beta_i \in (0, \infty)$ for $i = 0, 1, 2, 3,$ and l, m, k are positive integers. The initial conditions x_{-k},...,x_{-m},...,x_{-l}, ...,x_{-1}, x_0 are arbitrary positive real numbers such that $l < m < k$. We consider numerical examples which represent different types of solutions to Eq.(1). The case when any of α_i, β_i for $i = 0, 1, 2, 3$ allowed to be zero gives different special cases of Eq.(1) which are studied by many authors, (see for example [1– 16]). For the related work see [17–46]. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore, the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations. Note that Eq.(1) can be considered as a generalization of that obtained in [9,35,45].

Definition 1 *A difference equation of order $(k + 1)$ is of the form*

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-l}, ..., x_{n-m}, ..., x_{n-k}), \quad n = 0, 1, 2, ... \quad (2)$$

with $l < m < k$ where F is a continuous function which maps some set J^{k+1} into J and J is a set of real numbers. An equilibrium point $\widetilde{x}$ of this equation is a point that satisfies the condition $\widetilde{x} = F(\widetilde{x}, \widetilde{x}, ..., \widetilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \widetilde{x}$ for all $n \geq -k$ is a solution of that equation.

Definition 2 *Let $\widetilde{x} \in (0, \infty)$ be an equilibrium point of the difference equation (2). Then we have the following:*

(i) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if x_{-k}, ..., x_{-1}, $x_0 \in (0, \infty)$ with $|x_{-k} - \widetilde{x}| + ... + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \delta$, then $|x_n - \widetilde{x}| < \varepsilon$ for all $n \geq -k$.

(ii) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if x_{-k}, ..., x_{-1}, $x_0 \in (0, \infty)$ with $|x_{-k} - \widetilde{x}| + ... + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \gamma$, then

$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iii) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called a global attractor if for every x_{-k}, ..., x_{-1}, $x_0 \in (0, \infty)$ we have

$$\lim_{n \to \infty} x_n = \widetilde{x}.$$

(iv) An equilibrium point $\widetilde{x}$ of the equation (2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called unstable if it is not locally stable.

Definition 3 *We say that a sequence $\{x_n\}_{n=-k}^{\infty}$ is bounded and persisting if there exist positive constants m and M such that*

$$m \leq x_n \leq M, \quad n \geq -k.$$

Definition 4 *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period r if $x_{n+r} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period r if r is the smallest positive integer having this property.*

2 Local stability of the equilibrium point

In this section we study the local stability character of the solutions of Eq.(1). Assume that $\widetilde{a} = \sum_{i=0}^{3} \alpha_i$ and $\widetilde{b} = \sum_{i=0}^{3} \beta_i$. Then the positive equilibrium point $\widetilde{x}$ of Eq.(1) is given by

$$\widetilde{x} = \widetilde{a}/\widetilde{b}.$$

Let $F : (0, +\infty)^4 \longrightarrow (0, +\infty)$ be a continuous function defined by

$$F(u_0, u_1, u_2, u_3) = \frac{\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3}{\beta_0 u_0 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3}. \tag{3}$$

Then the linearized equation associated with Eq.(1) about the positive equilibrium point $\widetilde{x}$ takes the form

$$y_{n+1} + a_3 y_n + a_2 y_{n-l} + a_1 y_{n-m} + a_0 y_{n-k} = 0, \tag{4}$$

where

$$\begin{cases} \frac{(\alpha_0\beta_1 - \alpha_1\beta_0) + (\alpha_0\beta_2 - \alpha_2\beta_0) + (\alpha_0\beta_3 - \alpha_3\beta_0)}{\widetilde{a}\,\widetilde{b}} = -\,a_3, \\ \frac{(\alpha_1\beta_0 - \alpha_0\beta_1) + (\alpha_1\beta_2 - \alpha_2\beta_1) + (\alpha_1\beta_3 - \alpha_3\beta_1)}{\widetilde{a}\,\widetilde{b}} = -\,a_2, \\ \frac{(\alpha_2\beta_0 - \alpha_0\beta_2) + (\alpha_2\beta_1 - \alpha_1\beta_2) + (\alpha_2\beta_3 - \alpha_3\beta_2)}{\widetilde{a}\,\widetilde{b}} = -\,a_1, \\ \frac{(\alpha_3\beta_0 - \alpha_0\beta_3) + (\alpha_3\beta_1 - \alpha_1\beta_3) + (\alpha_3\beta_2 - \alpha_2\beta_3)}{\widetilde{a}\,\widetilde{b}} = -\,a_0. \end{cases} \tag{5}$$

The characteristic equation of the linearized equation (4) is

$$\lambda^{n+1} + a_3\lambda^n + a_2\lambda^{n-l} + a_1\lambda^{n-m} + a_0\lambda^{n-k} = 0. \tag{6}$$

Theorem 1 *([16,20] The linearized stability theorem) Suppose F is a continuously differentiable function defined on an open neighborhood of the equilibrium point $\widetilde{x}$. Then the following statements are true.*
(i) If all roots of the characteristic equation (6) of the linearized equation (4) have absolute value less than one, then the equilibrium point $\widetilde{x}$ is locally asymptotically stable.
(ii) If at least one root of Eq.(6) has absolute value greater than one, then the equilibrium point $\widetilde{x}$ is unstable.

Theorem 2 *([16]). Assume that $p_i \in R$, $i = 1, 2, ...k$. Then*

$$\sum_{i=1}^{k} |p_i| < 1 \tag{7}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, 2, ... \tag{8}$$

Theorem 3 *Assume that*

$$\begin{aligned} &|(\alpha_0\beta_1 - \alpha_1\beta_0) + (\alpha_0\beta_2 - \alpha_2\beta_0) + (\alpha_0\beta_3 - \alpha_3\beta_0)| \\ &+ |(\alpha_1\beta_0 - \alpha_0\beta_1) + (\alpha_1\beta_2 - \alpha_2\beta_1) + (\alpha_1\beta_3 - \alpha_3\beta_1)| \\ &+ |(\alpha_2\beta_0 - \alpha_0\beta_2) + (\alpha_2\beta_1 - \alpha_1\beta_2) + (\alpha_2\beta_3 - \alpha_3\beta_2)| \\ &+ |(\alpha_3\beta_0 - \alpha_0\beta_3) + (\alpha_3\beta_1 - \alpha_1\beta_3) + (\alpha_3\beta_2 - \alpha_2\beta_3)| \\ &< \widetilde{a}\,\widetilde{b}. \end{aligned}$$

Then the positive equilibrium point $\widetilde{x}$ of Eq.(1) is locally asymptotically stable.

Proof. It is obvious from (5) and the assumption of Theorem 3 that

$$\sum_{i=0}^{3} |a_i| < 1.$$

It follows by Theorem 2 that Eq.(1) is asymptotically stable.

3 Boundedness of the solutions

In this section we study the boundedness and persisting character of the positive solutions of Eq.(1) .

Theorem 4 *Every solution of Eq.(1) is bounded and persisting.*

Proof. Let

$$\begin{cases} m = \min\{\alpha_i,\ i = 0, ..., 3\}, \\ M = \max\{\alpha_i,\ i = 0, ..., 3\}, \\ l = \min\{\beta_i,\ i = 0, ..., 3\}, \\ L = \max\{\beta_i,\ i = 0, ..., 3\}, \end{cases} \tag{9}$$

we have

$$\begin{aligned} &\frac{m(x_n+x_{n-l}+x_{n-m}+x_{n-k})}{L(x_n+x_{n-l}+x_{n-m}+x_{n-k})} \\ &\leq x_{n+1} \leq \frac{M(x_n+x_{n-l}+x_{n-m}+x_{n-k})}{l(x_n+x_{n-l}+x_{n-m}+x_{n-k})} \end{aligned} \tag{10}$$

$$\Leftrightarrow \frac{m}{L} \leq x_{n+1} \leq \frac{M}{l}$$

which implies that every solution of Eq.(1) is bounded and persisting. Now, the proof is completed.

4 Periodicity of the solutions

In this section we study the periodic character of the positive solutions of Eq.(1).

Theorem 5 *Eq.(1) has no positive solutions of prime period two if one of the following conditions holds:*

(1) The positive integers l, m and k are even.

(2) The positive integers l, m are even and the positive integer k is odd provided $\alpha_0+\alpha_1+\alpha_2 \geq \alpha_3$.

(3) The positive integers l, m are odd and the positive integer k is even provided $\alpha_0+\alpha_3 \geq \alpha_1+\alpha_2$.

(4) The positive integers l, k are even and the positive integer m is odd provided $\alpha_0+\alpha_1+\alpha_3 \geq \alpha_2$.

(5) The positive integers m, k are even and the positive integer l is odd provided $\alpha_0+\alpha_2+\alpha_3 \geq \alpha_1$.

(6) The positive integers m, k are odd and the positive integer l is even provided $\alpha_0+\alpha_1 \geq \alpha_2+\alpha_3$.

(7) The positive integers l, k are even and the positive integer m is odd provided $\alpha_0+\alpha_2 \geq \alpha_1+\alpha_3$.

(8) The positive integers l, m, k are odd, $\alpha_1+\alpha_2+\alpha_3 \geq \alpha_0$ and $\beta_1+\beta_2+\beta_3 > \beta_0$.

Proof. Suppose that there exist positive distinctive solutions of prime period two

$$......., P, Q, P, Q, ...$$

of Eq.(1). Now, we discuss the following cases:

Case 1. l, m and k are even positive integers. In this case $x_n = x_{n-l} = x_{n-m} = x_{n-k}$. Then there exist a positive period two solution $\{x_n\}$ such that

$$\begin{aligned} x_{2q} &= P, \quad q = -1, 0, 1, ... \\ x_{2q+1} &= Q, \quad q = -1, 0, 1, ... \end{aligned}$$

and $P \neq Q$. From Eq.(1) we have

$$P = Q = \widetilde{a}/\widetilde{b}.$$

This is a contradiction. Thus, Eq.(1) has no prime period two solution.

Case 2. l, m are positive even integers and k is a positive odd integer. In this case $x_n = x_{n-l} = x_{n-m}$ and $x_{n+1} = x_{n-k}$. From Eq.(1) we have

$$P = \frac{(\alpha_0+\alpha_1+\alpha_2)\,Q+\alpha_3 P}{(\beta_0+\beta_1+\beta_2)\,Q+\beta_3 P},$$

$$Q = \frac{(\alpha_0+\alpha_1+\alpha_2)\,P+\alpha_3 Q}{(\beta_0+\beta_1+\beta_2)\,P+\beta_3 Q}.$$

Consequently, we obtain

$$(\alpha_0+\alpha_1+\alpha_2)\,Q+\alpha_3 P = (\beta_0+\beta_1+\beta_2)\,PQ+\beta_3 P^2$$

and

$$(\alpha_0+\alpha_1+\alpha_2)\,P+\alpha_3 Q = (\beta_0+\beta_1+\beta_2)\,PQ+\beta_3 Q^2.$$

By subtracting we have

$$P+Q = -\,\frac{[(\alpha_0+\alpha_1+\alpha_2)-\alpha_3]}{\beta_3}.$$

Since $\alpha_0+\alpha_1+\alpha_2 \geq \alpha_3$, we have $P+Q \leq 0$. Thus, we have a contradiction.

Case 3. l, m are positive odd integers and k is a positive even integer. In this case $x_{n+1} = x_{n-l} = x_{n-m}$ and $x_n = x_{n-k}$. From Eq.(1) we have

$$P = \frac{(\alpha_1+\alpha_2)\,P+(\alpha_0+\alpha_3)\,Q}{(\beta_1+\beta_2)\,P+(\beta_0+\beta_3)\,Q},$$

$$Q = \frac{(\alpha_1+\alpha_2)\,Q+(\alpha_0+\alpha_3)\,P}{(\beta_1+\beta_2)\,Q+(\beta_0+\beta_3)\,P}.$$

Consequently, we obtain

$$(\alpha_1+\alpha_2)\,P+(\alpha_0+\alpha_3)\,Q = (\beta_1+\beta_2)\,P^2+(\beta_0+\beta_3)\,PQ$$

and

$$(\alpha_1+\alpha_2)\,Q+(\alpha_0+\alpha_3)\,P = (\beta_1+\beta_2)\,Q^2+(\beta_0+\beta_3)\,PQ.$$

By subtracting we have

$$P+Q = -\,\frac{[(\alpha_0+\alpha_3)-(\alpha_1+\alpha_2)]}{\beta_1+\beta_2}.$$

Since $(\alpha_0+\alpha_3) \geq (\alpha_1+\alpha_2)$, we have $P+Q \leq 0$. Thus, we have a contradiction.

Case 4. l, k are positive even integers and m is a positive odd integer. In this case $x_n = x_{n-l} = x_{n-k}$ and $x_{n+1} = x_{n-m}$. From Eq.(1) we have

$$P = \frac{(\alpha_0+\alpha_1+\alpha_3)\,Q+\alpha_2 P}{(\beta_0+\beta_1+\beta_3)\,Q+\beta_2 P},$$

$$Q = \frac{(\alpha_0+\alpha_1+\alpha_3)\,P+\alpha_2 Q}{(\beta_0+\beta_1+\beta_3)\,P+\beta_2 Q}.$$

Consequently, we obtain

$$(\alpha_0+\alpha_1+\alpha_3)\,Q+\alpha_2 P=(\beta_0+\beta_1+\beta_3)\,PQ+\beta_2 P^2$$

and

$$(\alpha_0+\alpha_1+\alpha_3)\,P+\alpha_2 Q=(\beta_0+\beta_1+\beta_3)\,PQ+\beta_2 Q^2.$$

By subtracting, we have

$$P+Q=-\frac{[\alpha_0+\alpha_1+\alpha_3-\alpha_2]}{\beta_2}.$$

Since $(\alpha_0+\alpha_1+\alpha_3)\geq\alpha_2$, we have $P+Q\leq 0$. Thus, we have a contradiction.

Case 5. m,k are positive even integers and l is a positive odd integer. In this case $x_n=x_{n-m}=x_{n-k}$ and $x_{n+1}=x_{n-l}$. From Eq.(1) we have

$$P=\frac{(\alpha_0+\alpha_2+\alpha_3)\,Q+\alpha_1 P}{(\beta_0+\beta_2+\beta_3)\,Q+\beta_1 P},$$

$$\frac{(\alpha_0+\alpha_2+\alpha_3)\,P+\alpha_1 Q}{(\beta_0+\beta_2+\beta_3)\,P+\beta_1 Q}.$$

Consequently, we obtain

$$(\alpha_0+\alpha_2+\alpha_3)\,Q+\alpha_1 P=(\beta_0+\beta_2+\beta_3)\,PQ+\beta_1 P^2$$

and

$$(\alpha_0+\alpha_2+\alpha_3)\,P+\alpha_1 Q=(\beta_0+\beta_2+\beta_3)\,PQ+\beta_1 Q^2.$$

By subtracting we have

$$P+Q=-\frac{[\alpha_0+\alpha_2+\alpha_3-\alpha_1]}{\beta_1}.$$

Since $(\alpha_0+\alpha_2+\alpha_3)\geq\alpha_1$, we have $P+Q\leq 0$. Thus, we have a contradiction.

Case 6. m,k are positive odd integers and l is a positive even integer. In this case $x_{n+1}=x_{n-m}=x_{n-k}$ and $x_n=x_{n-l}$. From Eq.(1) we have

$$P=\frac{(\alpha_0+\alpha_1)\,Q+(\alpha_2+\alpha_3)\,P}{(\beta_0+\beta_1)\,Q+(\beta_2+\beta_3)\,P},$$

$$Q=\frac{(\alpha_0+\alpha_1)\,P+(\alpha_2+\alpha_3)\,Q}{(\beta_0+\beta_1)\,P+(\beta_2+\beta_3)\,Q}.$$

Consequently, we obtain

$$(\alpha_0+\alpha_1)\,Q+(\alpha_2+\alpha_3)\,P=(\beta_0+\beta_1)\,PQ+(\beta_2+\beta_3)\,P^2$$

and

$$(\alpha_0+\alpha_1)\,P+(\alpha_2+\alpha_3)\,Q=(\beta_0+\beta_1)\,PQ+(\beta_2+\beta_3)\,Q^2.$$

By subtracting we have

$$P+Q=-\frac{[(\alpha_0+\alpha_1)-(\alpha_2+\alpha_3)]}{(\beta_2+\beta_3)}.$$

Since $(\alpha_0+\alpha_1)\geq(\alpha_2+\alpha_3)$, we have $P+Q\leq 0$. Thus, we have a contradiction.

Case 7. l,k are positive odd integers and m is a positive even integer. In this case $x_{n+1}=x_{n-l}=x_{n-k}$ and $x_n=x_{n-m}$. From Eq.(1) we have

$$P=\frac{(\alpha_0+\alpha_2)\,Q+(\alpha_1+\alpha_3)\,P}{(\beta_0+\beta_2)\,Q+(\beta_1+\beta_3)\,P},$$

$$Q=\frac{(\alpha_0+\alpha_2)\,P+(\alpha_1+\alpha_3)\,Q}{(\beta_0+\beta_2)\,P+(\beta_1+\beta_3)\,Q}.$$

Consequently, we obtain

$$(\alpha_0+\alpha_2)\,Q+(\alpha_1+\alpha_3)\,P=(\beta_0+\beta_2)\,PQ+(\beta_1+\beta_3)\,P^2$$

and

$$(\alpha_0+\alpha_2)\,P+(\alpha_1+\alpha_3)\,Q=(\beta_0+\beta_2)\,PQ+(\beta_1+\beta_3)\,Q^2.$$

By subtracting we have

$$P+Q=-\frac{[(\alpha_0+\alpha_2)-(\alpha_1+\alpha_3)]}{(\beta_1+\beta_3)}.$$

Since $(\alpha_0+\alpha_2)\geq(\alpha_1+\alpha_3)$, we have $P+Q\leq 0$. Thus, we have a contradiction.

Case 8. l,m and k are positive odd integers. In this case $x_{n+1}=x_{n-l}=x_{n-m}=x_{n-k}$. From Eq.(1) we have

$$P=\frac{\alpha_0 Q+(\alpha_1+\alpha_2+\alpha_3)\,P}{\beta_0 Q+(\beta_1+\beta_2+\beta_3)\,P},$$

$$Q=\frac{\alpha_0 P+(\alpha_1+\alpha_2+\alpha_3)\,Q}{\beta_0 P+(\beta_1+\beta_2+\beta_3)\,Q}.$$

Consequently, we obtain

$$\alpha_0 Q+(\alpha_1+\alpha_2+\alpha_3)\,P=\beta_0 PQ+(\beta_1+\beta_2+\beta_3)\,P^2$$

and

$$\alpha_0 P+(\alpha_1+\alpha_2+\alpha_3)\,Q=\beta_0 PQ+(\beta_1+\beta_2+\beta_3)\,Q^2.$$

By subtracting we have

$$P+Q=\frac{[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]}{\beta_1+\beta_2+\beta_3},$$

while, by adding we obtain

$$PQ=-\frac{\alpha_0\,[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]}{(\beta_1+\beta_2+\beta_3)\,[(\beta_1+\beta_2+\beta_3)-\beta_0]}.$$

Since $(\alpha_1+\alpha_2+\alpha_3)>\alpha_0$ and $(\beta_1+\beta_2+\beta_3)>\beta_0$ we have $PQ<0$. Thus, we have a contradiction.

Theorem 6 *If* l,m,k *are odd,* $\alpha_1+\alpha_2+\alpha_3>\alpha_0$ *and* $\beta_1+\beta_2+\beta_3<\beta_0$*, then the necessary and sufficient condition for Eq.(1) to have positive solutions of prime period two is that the inequality*

$$\begin{aligned}&4\alpha_0(\beta_1+\beta_2+\beta_3)\\&<[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0][\beta_0-(\beta_1+\beta_2+\beta_3)]\end{aligned}\tag{11}$$

is valid.

Proof. Suppose that there exist positive distinctive solutions of prime period two

$$......., P, Q, P, Q, ...$$

of Eq.(1). Since l, m, k are odd, we have $x_{n+1} = x_{n-l} = x_{n-m} = x_{n-k}$. From Eq.(1) we have

$$P = \frac{\alpha_0 Q + (\alpha_1 + \alpha_2 + \alpha_3) P}{\beta_0 Q + (\beta_1 + \beta_2 + \beta_3) P},$$

$$Q = \frac{\alpha_0 P + (\alpha_1 + \alpha_2 + \alpha_3) Q}{\beta_0 P + (\beta_1 + \beta_2 + \beta_3) Q}.$$

Consequently, we obtain

$$\alpha_0 Q + (\alpha_1 + \alpha_2 + \alpha_3) P = \beta_0 PQ + (\beta_1 + \beta_2 + \beta_3) P^2$$

and

$$\alpha_0 P + (\alpha_1 + \alpha_2 + \alpha_3) Q = \beta_0 PQ + (\beta_1 + \beta_2 + \beta_3) Q^2.$$

By subtracting we have

$$P + Q = \frac{[(\alpha_1 + \alpha_2 + \alpha_3) - \alpha_0]}{\beta_1 + \beta_2 + \beta_3}, \tag{12}$$

while, by adding we obtain

$$PQ = \frac{\alpha_0 [(\alpha_1 + \alpha_2 + \alpha_3) - \alpha_0]}{(\beta_1 + \beta_2 + \beta_3) [\beta_0 - (\beta_1 + \beta_2 + \beta_3)]} \tag{13}$$

where $(\alpha_1 + \alpha_2 + \alpha_3) > \alpha_0$ and $\beta_0 > (\beta_1 + \beta_2 + \beta_3)$. Assume that P and Q are two positive distinct real roots of the quadratic equation

$$t^2 - (P + Q) t + PQ = 0. \tag{14}$$

Thus, we deduce that

$$\begin{aligned} &\left(\frac{[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]}{\beta_1+\beta_2+\beta_3}\right)^2 > \\ &4\left(\frac{\alpha_0[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]}{(\beta_1+\beta_2+\beta_3)[\beta_0-(\beta_1+\beta_2+\beta_3)]}\right). \end{aligned} \tag{15}$$

From (15), we obtain

$$\begin{aligned} &4\alpha_0 (\beta_1 + \beta_2 + \beta_3) \\ &< [(\alpha_1 + \alpha_2 + \alpha_3) - \alpha_0][\beta_0 - (\beta_1 + \beta_2 + \beta_3)]. \end{aligned}$$

Thus the condition (11) is valid. Conversely, suppose that the condition (11) is valid where $(\alpha_1 + \alpha_2 + \alpha_3) > \alpha_0$ and $\beta_0 > (\beta_1 + \beta_2 + \beta_3)$. Then, we deduce immediately from (11) that the inequality (15) holds. There exist two positive distinctive real numbers P and Q representing two positive roots of Eq.(14) such that

$$P = \frac{[(\alpha_1 + \alpha_2 + \alpha_3) - \alpha_0] + \delta}{2 (\beta_1 + \beta_2 + \beta_3)} \tag{16}$$

and

$$Q = \frac{[(\alpha_1 + \alpha_2 + \alpha_3) - \alpha_0] - \delta}{2 (\beta_1 + \beta_2 + \beta_3)} \tag{17}$$

where

$$\delta^2 = [(\alpha_1 + \alpha_2 + \alpha_3) - \alpha_0]^2 - \frac{4\alpha_0(\beta_1+\beta_2+\beta_3)[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]}{[\beta_0-(\beta_1+\beta_2+\beta_3)]}.$$

Now, we are going to prove that P and Q are positive solutions of prime period two of Eq.(1). To this end, we assume that

$$x_{-k} = Q, ..., x_{-m} = Q, ..., x_{-l} = Q, ..., x_{-1} = Q,$$

and $x_0 = P$. Now, we are going to show that $x_1 = Q$ and $x_2 = P$. From Eq.(1) we deduce that

$$\begin{aligned} x_1 &= \frac{\alpha_0 x_0+\alpha_1 x_{-l}+\alpha_2 x_{-m}+\alpha_3 x_{-k}}{\beta_0 x_0+\beta_1 x_{-l}+\beta_2 x_{-m}+\beta_3 x_{-k}} \\ &= \frac{\alpha_0 P+(\alpha_1+\alpha_2+\alpha_3)Q}{\beta_0 P+(\beta_1+\beta_2+\beta_3)Q}. \end{aligned} \tag{18}$$

Substituting (16) and (17) into (18) we deduce that

$$\begin{aligned} x_1 - Q &= \frac{\alpha_0 P+(\alpha_1+\alpha_2+\alpha_3)Q}{\beta_0 P+(\beta_1+\beta_2+\beta_3)Q} \\ &- \frac{\beta_0 PQ+(\beta_1+\beta_2+\beta_3)Q^2}{\beta_0 P+(\beta_1+\beta_2+\beta_3)Q} = \frac{B-C}{D}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} B &= \alpha_0 \left(\frac{[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]+\delta}{2(\beta_1+\beta_2+\beta_3)}\right) \\ &+ (\alpha_1 + \alpha_2 + \alpha_3) \left(\frac{[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]-\delta}{2(\beta_1+\beta_2+\beta_3)}\right) \\ C &= \beta_0 \left(\frac{\alpha_0[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]}{(\beta_1+\beta_2+\beta_3)[\beta_0-(\beta_1+\beta_2+\beta_3)]}\right) \\ &+ (\beta_1 + \beta_2 + \beta_3) \left(\frac{[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]-\delta}{2(\beta_1+\beta_2+\beta_3)}\right)^2, \\ D &= \beta_0 \left(\frac{[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]+\delta}{2(\beta_1+\beta_2+\beta_3)}\right) \\ &+ (\beta_1 + \beta_2 + \beta_3) \left(\frac{[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]-\delta}{2(\beta_1+\beta_2+\beta_3)}\right). \end{aligned}$$

Multiplying the denominator and numerator of (19) by $4 (\beta_1 + \beta_2 + \beta_3)^2$ we get

$$\begin{aligned} x_1 - Q &= \frac{(\beta_1+\beta_2+\beta_3)[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]\left[2\widetilde{a}-G\right]}{E} \\ &- \frac{\frac{4\alpha_0\beta_0(\beta_1+\beta_2+\beta_3)[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]}{[\beta_0-(\beta_1+\beta_2+\beta_3)]}+(\beta_1+\beta_2+\beta_3)\delta^2}{E} \\ &+ \frac{2(\beta_1+\beta_2+\beta_3)[\alpha_0-(\alpha_1+\alpha_2+\alpha_3)+G]\delta}{E} \\ &= \frac{2(\beta_1+\beta_2+\beta_3)[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]\left[\widetilde{a}-G\right]}{E} \\ &- \frac{\frac{4\alpha_0(\beta_1+\beta_2+\beta_3)[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0][\beta_0-(\beta_1+\beta_2+\beta_3)]}{[\beta_0-(\beta_1+\beta_2+\beta_3)]}}{E} \\ &= \frac{4\alpha_0(\beta_1+\beta_2+\beta_3)[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0]}{E} \\ &- \frac{\frac{4\alpha_0(\beta_1+\beta_2+\beta_3)[(\alpha_1+\alpha_2+\alpha_3)-\alpha_0][\beta_0-(\beta_1+\beta_2+\beta_3)]}{[\beta_0-(\beta_1+\beta_2+\beta_3)]}}{E} \\ &= 0, \end{aligned}$$

where

$$\begin{aligned} &E = 2(\beta_1 + \beta_2 + \beta_3) \times \\ &\left[\widetilde{b}[(\alpha_1 + \alpha_2 + \alpha_3) - \alpha_0] + [\beta_0 - (\beta_1 + \beta_2 + \beta_3)]\delta\right], \\ &G = [(\alpha_1 + \alpha_2 + \alpha_3) - \alpha_0] \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} x_2 &= \frac{\alpha_0 x_1 + \alpha_1 x_{-l+1} + \alpha_2 x_{-m+1} + \alpha_3 x_{-k+1}}{\beta_0 x_1 + \beta_1 x_{-l+1} + \beta_2 x_{-m+1} + \beta_3 x_{-k+1}} \\ &= \frac{\alpha_0 Q + (\alpha_1 + \alpha_2 + \alpha_3) P}{\beta_0 Q + (\beta_1 + \beta_2 + \beta_3) P} = P. \end{aligned}$$

By using the mathematical induction, we have

$$x_n = Q \quad and \quad x_{n+1} = P, \quad n \geq -k.$$

Now, the proof is completed.

5 Global stability

In this section we study the global asymptotic stability of the positive solutions of Eq.(1).

Lemma 1 *For any values of the quotient* $\frac{\alpha_i}{\beta_i}$ *for* $i = 0, 1, 2, 3$, *the function* $F(u_0, u_1, u_2, u_3)$ *defined by Eq.(3) is monotonic in its arguments.*

Proof. By differentiating the function $F(u_0, u_1, u_2, u_3)$ given by the formula (3) with respect to u_i $(i = 0, 1, 2, 3)$ we obtain

$$F_{u_0} = \frac{(\alpha_0\beta_1 - \alpha_1\beta_0)u_1 + (\alpha_0\beta_2 - \alpha_2\beta_0)u_2 + L_1u_3}{(\beta_0u_0 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3)^2}, \tag{20}$$

where $L_1 = (\alpha_0\beta_3 - \alpha_3\beta_0)$ and

$$F_{u_1} = \frac{-(\alpha_0\beta_1 - \alpha_1\beta_0)u_0 + (\alpha_1\beta_2 - \alpha_2\beta_1)u_2 + L_2u_3}{(\beta_0u_0 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3)^2}, \tag{21}$$

where $L_2 = (\alpha_1\beta_3 - \alpha_3\beta_1)$

$$F_{u_2} = \frac{-(\alpha_0\beta_2 - \alpha_2\beta_0)u_0 - (\alpha_1\beta_2 - \alpha_2\beta_1)u_1 + L_3u_3}{(\beta_0u_0 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3)^2} \tag{22}$$

where $L_3 = (\alpha_2\beta_3 - \alpha_3\beta_2)$ and

$$F_{u_3} = \frac{-(\alpha_0\beta_3 - \alpha_3\beta_0)u_0 - (\alpha_1\beta_3 - \alpha_3\beta_1)u_1 - L_4u_3}{(\beta_0u_0 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3)^2}. \tag{23}$$

where $L_4 = (\alpha_2\beta_3 - \alpha_3\beta_2)$.

From Eqs.(20)–(23), we see that the function $F(u_0, u_1, u_2, u_3)$ is monotonic in its arguments. Now, the proof is completed.

Theorem 7 *The positive equilibrium point* $\widetilde{x}$ *of Eq.(1) is a global attractor if the conditions*

$$\begin{array}{l} \alpha_0\beta_1 \geq \alpha_1\beta_0,\ \alpha_0\beta_2 \geq \alpha_2\beta_0,\ \alpha_0\beta_3 \geq \alpha_3\beta_0, \\ \alpha_1\beta_2 \geq \alpha_2\beta_1, \\ \alpha_1\beta_3 \geq \alpha_3\beta_1, \\ \alpha_2\beta_3 \geq \alpha_3\beta_2 \ and \ \alpha_3 \geq (\alpha_0 + \alpha_1 + \alpha_2) \end{array} \tag{24}$$

hold.

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of Eq.(1). To prove this theorem, it suffices to prove that $x_n \to \widetilde{x}$, as $n \to \infty$. Let us prove this as follows: Let $F : (0, +\infty)^4 \longrightarrow (0, +\infty)$ be a continuous function defined by the formula (3). With reference to Lemma 7, we notice that if the conditions

$$\alpha_0\beta_1 \geq \alpha_1\beta_0, \alpha_0\beta_2 \geq \alpha_2\beta_0, \alpha_0\beta_3 \geq \alpha_3\beta_0, \alpha_1\beta_2 \geq \alpha_2\beta_1,$$
$$\alpha_1\beta_3 \geq \alpha_3 \ and \ \alpha_2\beta_3 \geq \alpha_3\beta_2$$

hold, then the function $F(u_0, u_1, u_2, u_3)$ is non-decreasing in u_0 and non-increasing in u_3.

Now, from Eq.(1) we have

$$\begin{aligned} x_{n+1} &\leq \frac{\alpha_0x_n + \alpha_1x_{n-l} + \alpha_2x_{n-m} + \alpha_3(0)}{\beta_0x_n + \beta_1x_{n-l} + \beta_2x_{n-m} + \beta_3(0)} \\ &\leq \frac{\alpha_0}{\beta_0} + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}, \quad n \geq 0. \end{aligned}$$

Consequently, we obtain

$$x_n \leq \frac{\alpha_0}{\beta_0} + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} = H, \quad n \geq 1, \tag{25}$$

where H is a positive constant. On the other hand, we deduce from Eq.(1) that

$$\begin{aligned} x_{n+1} &\geq \frac{\alpha_0(0) + \alpha_1(0) + \alpha_2(0) + \alpha_3(H)}{\beta_0(H) + \beta_1(H) + \beta_2(H) + \beta_3(H)} \\ &\geq \frac{\alpha_3}{\beta_0 + \beta_1 + \beta_2 + \beta_3}, \quad n \geq 0. \end{aligned}$$

Consequently, we obtain

$$x_n \geq \frac{\alpha_3}{\beta_0 + \beta_1 + \beta_2 + \beta_3} = h, \quad n \geq 1, \tag{26}$$

where h is a positive constant. From the inequalities (25) and (26), we find that

$$h \leq x_n \leq H, \quad n \geq 1.$$

Thus the sequence $\{x_n\}$ is bounded. It follows by the method of full limiting sequences ([10,16]) that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq.(1) with

$$I = I_0 = \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n = S_0 = S, \tag{27}$$

where

$$I_n, S_n \in [I, S], \quad n = 0, -1, ...$$

On the other hand, it follows from Eq.(1) that

$$\begin{aligned} I &= \frac{\alpha_0I_{-1} + \alpha_1I_{-l-1} + \alpha_2I_{-m-1} + \alpha_3I_{-k-1}}{\beta_0I_{-1} + \beta_1I_{-l-1} + \beta_2I_{-m-1} + \beta_3I_{-k-1}} \\ &\geq \frac{\alpha_0I + \alpha_1I_{-l-1} + \alpha_2I_{-m-1} + \alpha_3S}{\beta_0I + \beta_1I_{-l-1} + \beta_1I_{-m-1} + \beta_2S} \\ &\geq \frac{(\alpha_0 + \alpha_1 + \alpha_2)I + \alpha_3S}{\beta_0I + (\beta_1 + \beta_2 + \beta_3)S}. \end{aligned}$$

Consequently, we have

$$(\alpha_0 + \alpha_1 + \alpha_2)I + \alpha_3S - \beta_0I^2 \leq (\beta_1 + \beta_2 + \beta_3)SI. \tag{28}$$

Similarly, we deduce from Eq.(1) that

$$\begin{aligned} S &= \frac{\alpha_0S_{-1} + \alpha_1S_{-l-1} + \alpha_2S_{-m-1} + \alpha_3S_{-k-1}}{\beta_0S_{-1} + \beta_1S_{-l-1} + \beta_2S_{-m-1} + \beta_3S_{-k-1}} \\ &\leq \frac{\alpha_0S + \alpha_1S_{-l-1} + \alpha_2S_{-m-1} + \alpha_3I}{\beta_0S + \beta_1S_{-l-1} + \beta_2S_{-m-1} + \beta_3I} \\ &\leq \frac{(\alpha_0 + \alpha_1 + \alpha_2)S + \alpha_3I}{\beta_0S + (\beta_1 + \beta_2 + \beta_3)I}. \end{aligned}$$

Consequently, we have

$$(\alpha_0+\alpha_1+\alpha_2)S+\alpha_3 I-\beta_0 S^2 \geq (\beta_1+\beta_2+\beta_3)SI. \quad (29)$$

It follows from the inequalities (28) and (29) that

$$(I-S)\left[\beta_0(I+S)+\alpha_3-(\alpha_0+\alpha_1+\alpha_2)\right] \geq 0. \quad (30)$$

Since $\alpha_3 \geq (\alpha_0+\alpha_1+\alpha_2)$ we deduce from (30) that

$$I \geq S. \quad (31)$$

Consequently, we have $I = S$.

From Theorems 3 and 7, we arrive at the following result:

Theorem 8 *The positive equilibrium point* $\widetilde{x}$ *of Eq.(1) is globally asymptotic stable.*

6 Numerical experiments on the main results

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical experiments in this section. These experiments represent different types of qualitative behavior of solutions to the nonlinear difference equation (1).

Experiment 1. Figure 1 shows that Eq.(1) has no prime period two solution if l, m, k are even. Choose $l = 2,\ m = 4, k = 6,\ \ x_{-6} = 1,\ x_{-5} = 2,\ x_{-4} = 3,\ x_{-3} = 4,\ x_{-2} = 5,\ x_{-1} = 6,\ x_0 = 7,\ \alpha_0 = 2,\ \alpha_1 = 10,\ \alpha_2 = 20,\ \alpha_3 = 15,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5.$

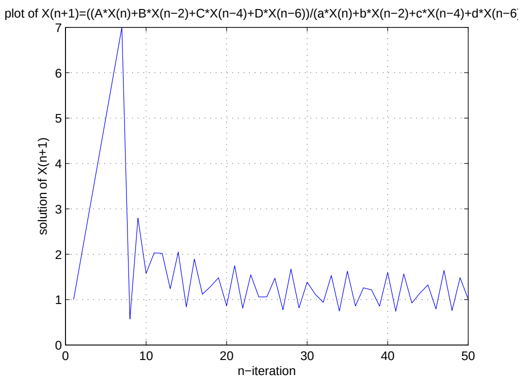

Figure 1

Experiment 2. Figure 2 shows that Eq.(1) has no prime period two solution if l, m are even and k is odd. Choose $l = 2,\ m = 4, k = 1,\ \ x_{-4} = 1,\ x_{-3} = 2,\ x_{-2} = 3,\ x_{-1} = 4,\ x_0 = 5,\ \alpha_0 = 2,\ \alpha_1 = 10,\ \alpha_2 = 20,\ \alpha_3 = 5,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5.$

Experiment 3. Figure 3 shows that Eq.(1) has no prime period two solution if l, m are odd and k is even. Choose $l = 1,\ m = 3, k = 2,\ x_{-3} = 1,\ x_{-2} = 2,\ x_{-1} = 3,\ x_0 =$

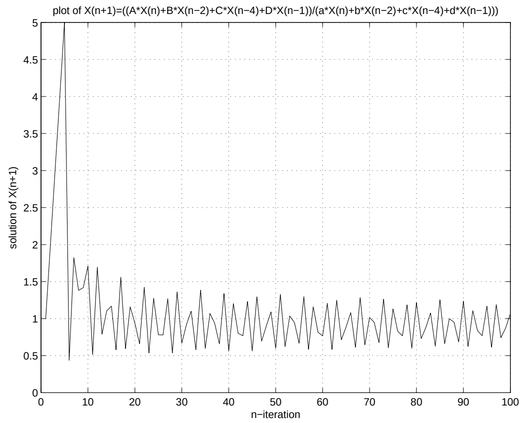

Figure 2

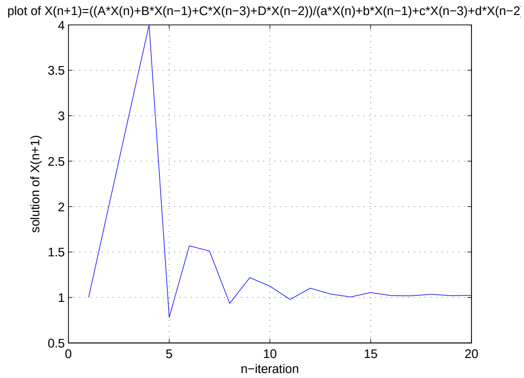

Figure 3

$4,\ \alpha_0 = 15,\ \alpha_1 = 2,\ \alpha_2 = 6,\ \alpha_3 = 20,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5.$

Experiment 4. Figure 4 shows that Eq.(1) has no prime period two solution if l, k are even and m is odd. Choose $l = 2,\ m = 3,\ k = 4,\ \ x_{-4} = 1,\ x_{-3} = 2,\ x_{-2} = 3,\ x_{-1} = 4,\ x_0 = 5,\ \alpha_0 = 15,\ \alpha_1 = 2,\ \alpha_2 = 6,\ \alpha_3 = 20,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5.$

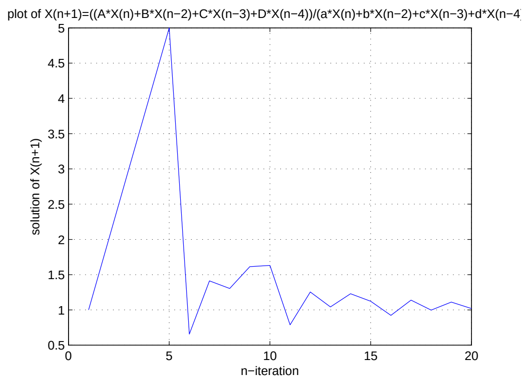

Figure 4

Experiment 5. Figure 5 shows that Eq.(1) has no prime period two solution if m, k are even and l is odd. Choose $l = 1,\ m = 2,\ k = 4,\ \ x_{-4} = 1,\ x_{-3} = 2,\ x_{-2} =$

$3,\ x_{-1} = 4,\ x_0 = 5,\ \alpha_0 = 15,\ \alpha_1 = 2,\ \alpha_2 = 10,\ \alpha_3 = 20,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5.$

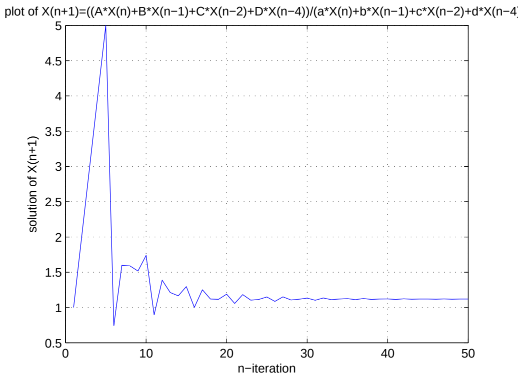

Figure 5

Experiment 6. Figure 6 shows that Eq.(1) has no prime period two solution if m, k are odd and l is even. Choose $l = 2,\ m = 1,\ k = 3,\ x_{-4} = 1,\ x_{-3} = 2,\ x_{-2} = 3,\ x_{-1} = 4,\ x_0 = 5,\ \alpha_0 = 20,\ \alpha_1 = 10,\ \alpha_2 = 15,\ \alpha_3 = 5,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5.$

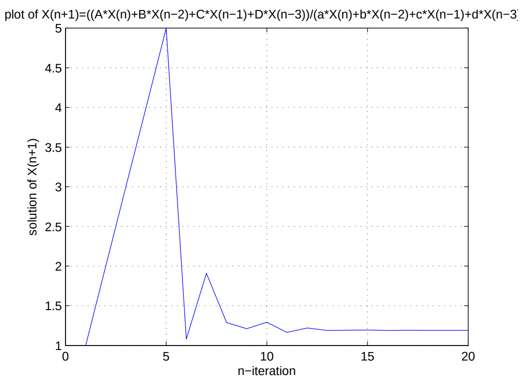

Figure 6

Experiment 7. Figure 7 shows that Eq.(1) has no prime period two solution if l, k are even and m is odd. Choose $l = 2,\ m = 1,\ k = 4,\ x_{-4} = 1,\ x_{-3} = 2,\ x_{-2} = 3,\ x_{-1} = 4,\ x_0 = 5,\ \alpha_0 = 20,\ \alpha_1 = 10,\ \alpha_2 = 15,\ \alpha_3 = 5,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5.$

Experiment 8. Figure 8 shows that Eq.(1) has no prime period two solution if l, m, k are odd. Choose $l = 1,\ m = 3,\ k = 5,\ x_{-5} = 1,\ x_{-4} = 2,\ x_{-3} = 3,\ x_{-2} = 4,\ x_{-1} = 5,\ x_0 = 6,\ \alpha_0 = 2,\ \alpha_1 = 10,\ \alpha_2 = 20,\ \alpha_3 = 5,\ \beta_0 = 3,\ \beta_1 = 30,\ \beta_2 = 4,\ \beta_3 = 5.$

Experiment 9. Figure 9 shows that Eq.(1) has prime period two solution and $l < m < k$. Choose $l = 1,\ m = 3,\ k = 5,\ p = \max\{l, m, k\} = 5,\ x_{-5} = 4.7,\ x_{-4} = 0.09,\ x_{-3} = 4.7,\ x_{-2} = 0.09,\ x_{-1} = 4.7,\ x_0 = 0.09,\ x_1 = 4.7,\ x_2 = 0.09,\ \alpha_0 = 2,\ \alpha_1 = 10,\ \alpha_2 = 20,\ \alpha_3 = 10,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 1.$

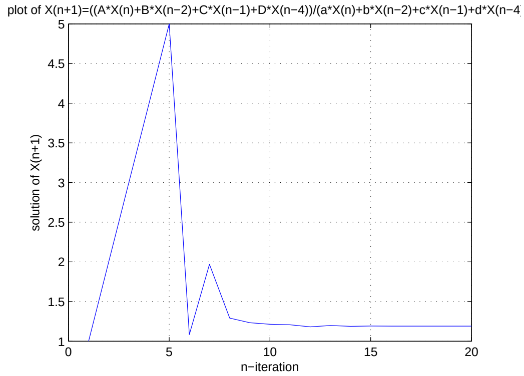

Figure 7

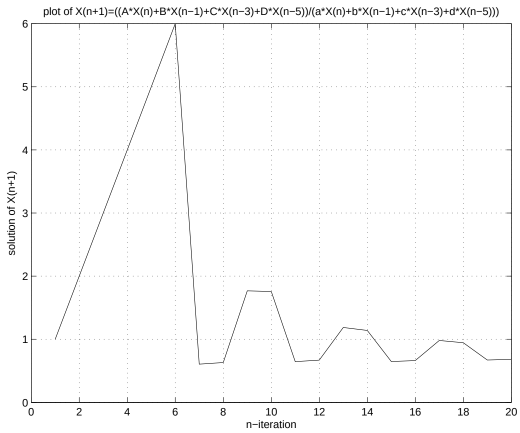

Figure 8

Experiment 10. Figure 10 shows that the solution of Eq.(1) has global stability and $l < m < k$. Choose $l = 2,\ m = 4,\ k = 6,\ x_{-6} = 1,\ x_{-5} = 2,\ x_{-4} = 3,\ x_{-3} = 4,\ x_{-2} = 5,\ x_{-1} = 6,\ x_0 = 7,\ \alpha_0 = 0.5,\ \alpha_1 = 0.25,\ \alpha_2 = 1,\ \alpha_3 = 2,\ \beta_0 = 3,\ \beta_1 = 2,\ \beta_2 = 10,\ \beta_3 = 25.$

Remark Note that experiments 1– 8 verify Theorem 5 which shows that Eq.(1) has no prime period two solution, while experiment 9 verifies Theorem 6 which shows that Eq.(1) has prime period two solution. But experiment 10 verifies Theorems 3,8 which shows that if the conditions $\alpha_0\beta_1 \geq \alpha_1\beta_0,\ \alpha_0\beta_2 \geq \alpha_2\beta_0,\ \alpha_0\beta_3 \geq \alpha_3\beta_0,\ \alpha_1\beta_2 \geq \alpha_2\beta_1,\ \alpha_1\beta_3 \geq \alpha_3\beta_1,\ \alpha_2\beta_3 \geq \alpha_3\beta_2$ *and* $\alpha_3 \geq (\alpha_0+\alpha_1+\alpha_2)$ hold, then the solution of Eq.(1) has globally asymptotic stable.

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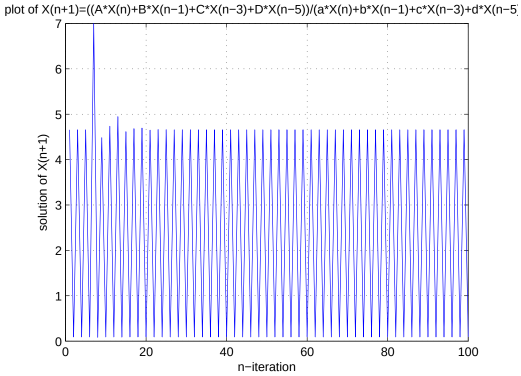

Figure 9

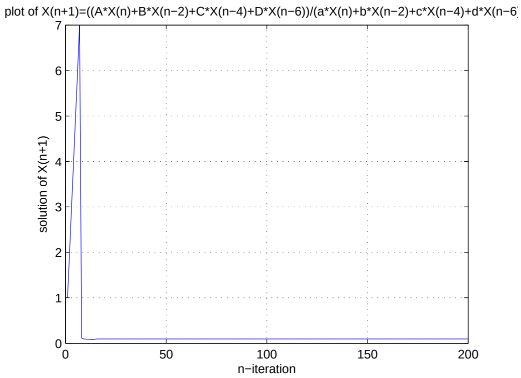

Figure 10

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