Optimization of conical shells of piece wise constant thickness

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Abstract: - Conical shells with piece wise constant thickness subjected to the distributed transverse pressure and loaded by a rigid central boss are studied. In the paper the both, elastic and inelastic shells are considered. In the case of inelastic shells it is assumed that the material obeys the Hill's plasticity condition and associated flow rule. The optimization problem is posed in a general form involving as particular cases several different problems. Resorting to the variational methods necessary optimality conditions are derived. The problems regarding to the maximization of the plastic limit load and to the minimum weight design are studied in a greater detail.

Key-Words: - thin walled shell, optimal design, yield condition, associated flow law, elasticity

1 Introduction

Thin walled conical shells are widely used in the machinery. Therefore, it is important to study the behaviour of optimized shells. Problems of optimization of thin walled shells are investigated by several authors. The reviews of earlier works can be find in [1, 9-11, 22]. Optimal designs of conical shells made of inelastic materials are established by Lellep and Puman [12 - 16] whereas spherical caps made of von Mises material are investigated by Lellep and Tungel [16, 18]. Axisymmetric plates are studied by Lellep and Polikarpus [17].

It is well known that the especially high ratio of the strength-to-weight ratio is achieved by the use of composite materials which exhibit the orthotropic and anisotropic behaviour. Exploiting the upper bound theorem of limit analysis the anisotropic behaviour of structures was studied by Capsoni, Corradi, Vena [2]; Corradi, Luzzi, Vena [4]; Pan, Seshadri [16].

In the present paper conical shells with piece wise constant thickness are considered. An optimal design method is developed for conical shells made of elastic or inelastic materials.

2 Problem formulation

Let us study the behaviour of an axisymmetric conical shell (Fig. 1) subjected to an axisymmetric loading. We confine our attention to sandwich shells with the total thickness H and the thickness of rims h. Let the inner radius of the mid surface be a and the outer radius R, respectively. Different combinations of end conditions will be investigated in this paper.

First of all, the case when the outer edge is clamped or simply supported and the inner edge is free is studied in a greater detail. In this case the shell is loaded by a uniformly distributed transverse pressure of intensity P. In the alternative case the shell is loaded by a central rigid boss.

It is assumed that the face sheet thickness is piece wise constant, e.g. $h = h_j$ for $r \in (a_j, a_{j+1})$ where $j = 0,\dots,n$. It is reasonable to denote $a_0 = a$, $a_{n+1} = R$. The parameters a_j and $h_j \left(j = 1,\dots,n\right)$ will be treated as preliminarily unknown design parameters.

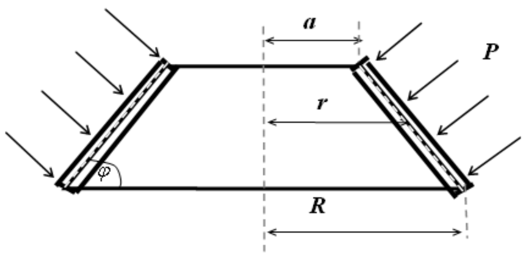

Fig 1. Shell geometry

The aim of the paper is to determine the design parameters so that the given cost function attains its minimum value. In the particular case of minimization of the weight of the shell the cost function can be presented as

$$V = \frac{\pi}{\cos\varphi} \sum_{j=0}^{n} h_j \left(a_{j+1}^2 - a_j^2 \right) \tag{1}$$

where φ stands for the angle of inclination of the middle surface. However, we shall consider herein a more general case of the optimization problem which involves a series of particular problems which can be solved from a unique point of view.

The cost function to be minimized will be presented as

$$J = G(a_1, \ldots, a_n, h_0, \ldots, h_n, W(r_*), U(r_*), M_1(r_*), N_1(r_*)) + \int_a^R F(W, U, N_1, M_1)\, dr \tag{2}$$

were F and G are given differentiable functions whereas W and U stand for displacements in the two orthogonal directions. Here and henceforth M_1, M_2 stand for bending moments and N_1, N_2 for membrane forces in the radial and tangential direction, respectively. The quantity r_* is assumed to be a given value of the current radius $r \in [a, R]$.

It is assumed that the optimal solution satisfies the isoperimetric constraint

$$\int_a^R T(W, U, N_1, M_1)\, dr = T_0 \tag{3}$$

Here the function T is assumed to be a given continuous and differentiable function whereas T_0 stands for a given constant.

The meaning of the cost function (2) and the constraint (3) can be explained by following examples.

If, for instance, $G = W(r_*), F = 0,$ then the optimization problem consists in the minimization of the radial deflection at $r = r_*$. If, however, $F = 0$ and $G = V,$ then one has the minimum weight problem for a stepped conical shell.

At the outer edge of the shell following boundary conditions must be satisfied

$$W(R) = 0, M_1(R) = 0, N_1(R) = 0 \tag{4}$$

When minimizing the cost function (2) one has to take into account additional constraints (3) as well as the governing equations which consist of the equilibrium equations and of constitutive relations. The constitutive equations can be presented via strain components. In the case of a conical element the strain components have the form [5, 7]

$$\begin{aligned} \varepsilon_1 &= \frac{dU}{dr} \cos\varphi, \\ \varepsilon_2 &= \frac{1}{r}\left(U \cos\varphi + W \sin\varphi \right), \\ \kappa_1 &= -\frac{d^2 W}{dr^2} \cos^2\varphi, \\ \kappa_2 &= -\frac{1}{r}\frac{dW}{dr} \cos^2\varphi, \end{aligned} \tag{5}$$

where $\varepsilon_1, \varepsilon_2$ stand for linear extension ratios and κ_1, κ_2 are curvatures of the middle surface of the shell.

In what follows we will treat the shells made of different materials including elastic and inelastic materials. It is well known that in the case of an elastic material the Hooke's law holds good. The latter can be presented as (see Hodge [5], Ventsel and Krauthammer [17])

$$\begin{aligned} N_1 &= D_{oj}\left(\varepsilon_1 + \nu\varepsilon_2 \right), \\ N_2 &= D_{oj}\left(\varepsilon_2 + \nu\varepsilon_1 \right), \\ M_1 &= D_j\left(\kappa_1 + \nu\kappa_2 \right), \\ M_2 &= D_j\left(\kappa_2 + \nu\kappa_1 \right) \end{aligned} \tag{6}$$

for $r \in (a_j, a_{j+1}),\ j = 0, \ldots, n,$ where ν stands for the Poisson's modulus and

$$D_j = \frac{EH^2 h_j}{2(1-\nu^2)}, \quad D_{0j} = \frac{2Eh_j}{1-\nu^2}.$$

Here E denotes the Young modulus.

Conical shells loaded beyond the elastic limit are investigated, as well. In this case it is assumed that the shell is fully plastic and the material obeys the Hill's yield criterion which can be presented as [2 - 4]

$$\begin{aligned} &\frac{1}{M_{0j}^2}\left(M_1^2 - \alpha M_1 M_2 + \beta M_2^2 \right) + \frac{1}{N_{0j}^2} \cdot \\ &\cdot \left(N_1^2 - \alpha N_1 N_2 + \beta N_2^2 \right) - Y_1^2 \le 0 \end{aligned} \tag{7}$$

for the section $r \in (a_j, a_{j+1})$. In (7) the quantities α, β and Y_1 are certain material parameters whereas N_{0j} and M_{0j} stand for the yield force and the yield moment for $r \in (a_j, a_{j+1})$. Evidently, in the case of a sandwhich shell

$$M_{0j} = \sigma_0 H h_j, \quad N_{0j} = 2\sigma_0 h_j$$

where σ_0 is the yield stress of the material. It is known from the theory of plasticity that in a plastic region the associated flow law holds good. According to the associated gradientality law and the Hill's yield criterion (7) one has

$$\varepsilon_1 = \frac{\lambda_j}{N_{0j}^2}\left(2N_1 - \alpha N_2\right),$$
$$\varepsilon_2 = \frac{\lambda_j}{N_{0j}^2}\left(2\beta N_2 - \alpha N_1\right),$$
$$\kappa_1 = \frac{\lambda_j}{M_{0j}^2}\left(2M_1 - \alpha M_2\right), \tag{8}$$
$$\kappa_2 = \frac{\lambda_j}{M_{0j}^2}\left(2\beta M_2 - \alpha M_1\right),$$

for $r \in \left(a_j, a_{j+1}\right)$, $j = 0,\ldots,n$. Here λ_j stands for a non-negative scalar multiplier. Combining the obtained equations with (4) results in the system of equations

$$\frac{dW}{dr} = -\frac{\lambda_j r}{\cos^2\varphi\, M_{0j}^2}\left(2\beta M_2 - M_1\right),$$
$$\frac{d^2W}{dr^2} = -\frac{\lambda_j}{\cos^2\varphi\, M_{0j}^2}\left(2M_1 - \alpha M_2\right), \tag{9}$$
$$\frac{dU}{dr} = \frac{\lambda_j}{\cos\varphi\, N_{0j}^2}\left(2N_1 - \alpha N_2\right),$$

and

$$\frac{1}{r}\left(U\cos\varphi + W\sin\varphi\right) = \frac{\lambda_j}{N_{0j}^2}\left(2\beta N_2 - \alpha N_1\right) \tag{10}$$

The equilibrium of a shell element furnishes equations (see Hodge [5])

$$N_1' = \frac{1}{r}\left(N_2 - N_1\right)$$
$$M_1' = -\frac{1}{r}\left(M_1 - M_2\right) - N_1\frac{\sin\varphi}{\cos^2\varphi} - \frac{P\left(r^2 - a^2\right)}{2\cos^2\varphi\cdot r} \tag{11}$$

where prims denote the differentiation with respect to r.

In the case of an elastic material the governing equations can be expressed as differential equations depending on displacements U and W. It follows from (5), (6) that

$$N_1 = D_{oj}\left(U'\cos\varphi + \frac{\nu}{r}\left(U\cos\varphi + W\sin\varphi\right)\right)$$
$$N_2 = D_{oj}\left(\frac{1}{r}\left(U\cos\varphi + W\sin\varphi\right) + \nu U'\cos\varphi\right)$$
$$M_1 = -D_j\left(W''\cos^2\varphi + \frac{\nu}{r}W'\cos^2\varphi\right) \tag{12}$$
$$M_2 = -D_j\left(\frac{W'}{r}\cos^2\varphi + \nu W''\cos^2\varphi\right)$$

It is easy to show that the equations (11), (12) can be presented as a set of differential equations

$$\frac{dW}{dr} = Z,$$
$$\frac{dZ}{dr} = -\frac{M_1}{D_j\cos^2\varphi} - \frac{\nu}{r}Z,$$
$$\frac{dU}{dr} = \frac{N_1}{D_{0j}\cos\varphi} - \frac{\nu}{r}\left(U + W\tan\varphi\right), \tag{13}$$
$$\frac{dN_1}{dr} = \frac{1}{r}\left(N_2 - N_1\right),$$
$$\frac{dM_1}{dr} = \frac{1}{r}\left(M_2 - M_1\right) - N_1\frac{\sin\varphi}{\cos^2\varphi} - \frac{P\left(r^2 - a^2\right)}{2r\cos^2\varphi},$$

and a system of algebraic equations

$$\left(N_2 - \nu N_1\right)\frac{1}{D_{0j}} - \frac{\left(1 - \nu^2\right)}{r}\left(W\sin\varphi + U\cos\varphi\right) = 0,$$
$$\left(M_2 - \nu M_1\right)\frac{1}{D_j} + \frac{\left(1 - \nu^2\right)}{r}Z\cos^2\varphi = 0. \tag{14}$$

The last system of equations holds good for each $r \in \left(a_j, a_{j+1}\right)$ for $j = 0,\ldots,n$ in the case of shells made of an elastic material.

3 Necessary conditions for optimality

Let us consider first the case of an elastic shell. In this case the governing equations are presented by equations (13), (14).

In order to minimize the cost function (2) constrained by (3), (4) among the trajectories of the system (13), (14) let us introduce an augmented functional

$$
\begin{aligned}
J_* = G + \int_a^R (F + \psi_0 T)dr + \sum_{j=0}^{n} \int_{a_j}^{a_{j+1}} \Bigg\{ \psi_1 \left(\frac{dW}{dr} - Z \right) + \\
+ \psi_2 \left(\frac{dZ}{dr} + \frac{M_1}{D_j cos^2 \varphi} + \frac{\nu}{r} Z \right) + \\
+ \psi_3 \left(\frac{dU}{dr} - \frac{N_1}{D_{0j} cos \varphi} + \frac{\nu}{r}(U + W\, tan \varphi) \right) + \\
+ \psi_5 \left(\frac{dM_1}{dr} - \frac{1}{r}(M_2 - M_1) + N_1 \frac{sin \varphi}{cos^2 \varphi} + \frac{P(r^2 - a^2)}{2r\, cos^2 \varphi} \right) + \\
+ \mu_{1j} \left(\frac{1}{D_{0j}}(N_2 - \nu N_1) - \frac{(1-\nu^2)}{r}(W\, sin \varphi + U\, cos \varphi) \right) + \\
+ \mu_{2j} \left(\frac{1}{D_j}(M_2 - \nu M_1) + \frac{(1-\nu^2)}{r} Z\, cos^2 \varphi \right) \Bigg\} dr
\end{aligned}
\tag{15}
$$

In (15) $\psi_1, \ldots, \psi_5$ stand for conjugate (adjoint) variables whereas μ_{1j}, μ_{2j} for Lagrangean multipliers.

It should be mentioned that the multiplier ψ_0 is an uknown constant.

Calculating the total variation of the functional (15) and taking into account (2), (3) one has

$$
\begin{aligned}
\Delta J_* = \Delta G + \sum_{j=0}^{n} \int_{a_j}^{a_{j+1}} \Bigg\{ \frac{\partial F_*}{\partial U} \delta U + \frac{\partial F_*}{\partial W} \delta W + \frac{\partial F_*}{\partial N_1} \delta N_1 + \\
+ \frac{\partial F_*}{\partial M_1} \delta M_1 + \psi_1 \left(\delta \frac{dW}{dr} - \delta Z \right) + \psi_2 \Bigg(\delta \frac{dZ}{dr} + \frac{\delta M_1}{D_j cos^2 \varphi} - \\
- \frac{M_1 \delta D_j}{D_j^2 cos^2 \varphi} + \frac{\nu}{r} \delta Z \Bigg) + \psi_3 \Bigg(\delta \frac{dU}{dr} - \frac{\delta N_1}{D_{0j} cos \varphi} + \\
+ \frac{N_1 \delta D_{0j}}{D_{0j}^2 cos \varphi} + \frac{\nu}{r} \delta U + \frac{\nu}{r} tan \varphi \delta W \Bigg) + \\
+ \psi_4 \left(\delta \frac{dN_1}{dr} - \frac{1}{r} \delta N_2 + \frac{1}{r} \delta N_1 \right) + \\
+ \psi_5 \left(\delta \frac{dM_1}{dr} - \frac{1}{r} \delta M_2 + \frac{1}{r} \delta M_1 + \delta N_1 \frac{sin \varphi}{cos^2 \varphi} \right) + \\
+ \mu_{1j} \Bigg\{ \frac{1}{D_{0j}} (\delta N_2 - \nu \delta N_1) - \frac{\delta D_{0j}}{D_{0j}^2} (N_2 - \nu N_1) - \\
- \frac{(1-\nu^2)}{r} (\delta W\, sin \varphi + \delta U\, cos \varphi) \Bigg\} + \\
+ \mu_{2j} \Bigg\{ \frac{1}{D_j} (\delta M_2 - \nu \delta M_1) - \frac{\delta D_j}{D_j^2} (M_2 - \nu \delta M_1) + \\
+ \frac{(1-\nu^2)}{r} cos^2 \varphi \delta Z \Bigg\} \Bigg\} dr
\end{aligned}
\tag{16}
$$

where

$$
F_* = F + \psi_0 T \tag{17}
$$

and

$$
\begin{aligned}
\Delta G = \sum_{i=1}^{n} \frac{\partial G}{\partial a_i} \Delta a_i + \sum_{j=0}^{n} \frac{\partial G}{\partial h_j} \Delta h_j + \frac{\partial G}{\partial W(r_*)} \Delta W(r_*) + \\
+ \frac{\partial G}{\partial U(r_*)} \Delta U(r_*) + \frac{\partial G}{\partial M_1(r_*)} \Delta M_1(r_*) + \\
+ \frac{\partial G}{\partial N_1(r_*)} \Delta N_1(r_*).
\end{aligned}
\tag{18}
$$

In (16), (18) $\delta U, \delta W, \delta Z, \delta M_1, \delta N_1$ stand for variations of state variables for $r \in [a, R]$ whereas $\Delta W(r_*), \Delta U(r_*), \Delta M_1(r_*), \Delta N_1(r_*)$ are total variations of corresponding variables at $r = r_*$. The symbols $\Delta a_i, \Delta h_j$ denote arbitrary increments of a_i and h_j, respectively.

Integrating the terms of type $\psi \delta \frac{dy}{dr}$ by parts and making use of the necessary optimality condition $\Delta J_* = 0$ leads to the adjoint system

$$
\begin{aligned}
\frac{d\psi_1}{dr} &= \frac{\partial F_*}{\partial W} + \psi_3 \frac{\nu}{r} tan \varphi - \frac{\mu_{1j}}{r}(1-\nu^2) sin \varphi, \\
\frac{d\psi_2}{dr} &= \frac{\partial F_*}{\partial U} + \frac{\psi_3}{r}\frac{\nu}{r} - \frac{\mu_{1j}}{r}(1-\nu^2) cos \varphi, \\
\frac{d\psi_3}{dr} &= -\psi_1 + \psi_2 \frac{\nu}{r} + \frac{\mu_{2j}}{r}(1-\nu^2) cos^2 \varphi, \\
\frac{d\psi_4}{dr} &= \frac{\partial F_*}{\partial N_1} - \frac{\psi_3}{D_{0j} cos \varphi} + \frac{\psi_4}{r} - \frac{\mu_{1j}}{D_{0j}} \nu, \\
\frac{d\psi_5}{dr} &= \frac{\partial F_*}{\partial M_1} + \frac{\psi_2}{D_j cos^2 \varphi} + \frac{\psi_5}{r} - \frac{\mu_{2j}}{D_j} \nu.
\end{aligned}
\tag{19}
$$

It is easy to calculate

$$
\begin{aligned}
\Delta D_j &= \frac{EH^2}{2(1-\nu^2)} \Delta h_j, \\
\Delta D_{0j} &= \frac{2E}{1-\nu^2} \Delta h_j,
\end{aligned}
\tag{20}
$$

for $j = 0, \ldots, n$.

Substituting (20) in (16) and taking (20) into account one can obtain the following equations for determination of quantities h_j

$$\frac{\partial G}{\partial h_j} - \frac{EH^2}{2(1-\nu^2)} \int_{a_j}^{a_{j+1}} \left(\frac{\psi_2 M_1}{D_j^2 \cos^2\varphi} + \frac{\mu_{2j}}{D_j^2}(M_2 - \nu M_1) \right) dr + \\ + \frac{2E}{1-\nu^2} \int_{a_j}^{a_{j+1}} \left(\frac{\psi_3 N_1}{D_{0j}^2 \cos\varphi} + \frac{\mu_{1j}}{D_{0j}^2}(N_2 - \nu N_1) \right) dr = 0 \tag{21}$$

for $j = 0,\ldots,n.$

Since the variations $\delta M_2, \delta N_2$ are independent in (16) one has

$$-\frac{\psi_4}{r} + \frac{\mu_{1j}}{D_{0j}} = 0, \\ -\frac{\psi_5}{r} + \frac{\mu_{2j}}{D_j} = 0. \tag{22}$$

The arbitrariness of total variations of state variables at $r = r_*$ results in the following continuity requirements

$$[\psi_1(r_*)] = \frac{\partial G}{\partial W(r_*)}, \\ [\psi_2(r_*)] = 0, \\ [\psi_3(r_*)] = \frac{\partial G}{\partial U(r_*)}, \\ [\psi_4(r_*)] = \frac{\partial G}{\partial N_1(r_*)}, \\ [\psi_5(r_*)] = \frac{\partial G}{\partial M_1(r_*)}. \tag{23}$$

However, at $r = a_j \quad (j = 1,\ldots,n)$

$$[\psi_i(a_j)] = 0 \tag{24}$$

provided $r_* \neq a_j$. In (24) $i = 1,\ldots,5;$ $j = 1,\ldots,n.$ The square brackets in (23), (24) denote finite jumps of corresponding variables, e.g.

$$\psi(a) = \psi(a+0) - \psi(a-0). \tag{25}$$

Finally, the arbitrariness of increments Δa_j leads to equations

$$[L(a_j)] + \frac{\partial G}{\partial a_j} = 0 \tag{26}$$

for $j = 1,\ldots,n.$ In (26) the function L is the extended Lagrangian function defined as

$$L = \psi_1 Z + \psi_2 \left(-\frac{M_1}{D_j \cos^2\varphi} - \frac{\nu}{r} Z \right) + \\ + \psi_3 \left(\frac{N_1}{D_{0j} \cos\varphi} - \frac{\nu}{r}(U + W \tan\varphi) \right) + \frac{\psi_4}{r}(N_2 - N_1) + \\ + \psi_5 \left(\frac{1}{r}(M_2 - M_1) - N_1 \frac{\sin\varphi}{\cos^2\varphi} - \frac{P(r^2 - a^2)}{2r\cos^2\varphi} \right) + \\ + \mu_{1j} \left\{ \frac{1}{D_j}(N_2 - \nu N_1) - \frac{1}{r}(1-\nu^2)(W \sin\varphi + U \cos\varphi) \right\} + \\ + \mu_{2j} \left\{ \frac{1}{D_j}(M_2 - \nu M_1) + \frac{1}{r}(1-\nu^2) Z \cos^2\varphi \right\}. \tag{27}$$

It is reasonable to rewrite the system (9) as [8, 9]

$$\frac{dW}{dr} = Z, \\ \frac{dZ}{dr} = \frac{Z(2M_1 - \alpha M_2)}{r(2\beta M_2 - \alpha M_1)}, \\ \frac{dU}{dr} = -\frac{M_{oj}^2 \cdot Z(2N_1 - \alpha N_2)}{N_{oj}^2 \cdot r(2\beta M_2 - \alpha M_1)}, \tag{28}$$

where Z is an auxiliary variable and

$$\lambda_j = -\frac{\cos^2\varphi \cdot M_{0j}^2 \cdot Z}{r(2\beta M_2 - \alpha M_1)} \tag{29}$$

Substituting (13) in (10) yields the equation

$$U \cos\varphi + W \sin\varphi + \\ + \frac{M_{0j}^2 \cos^2\varphi}{N_{0j}^2} \cdot \frac{(2\beta N_2 - \alpha N_1) Z}{(2\beta M_2 - \alpha M_1)} = 0 \tag{30}$$

In order to minimize the cost function (2) under contraints (11) - (14) let us introduce the extended (augmented) functional [1, 8 – 12]

$$J_* = G + \int_a^R F_* dr + \sum_{j=0}^{n} \int_{a_i}^{a_{i+1}} \left\{ \psi_1 \left(\frac{dW}{dr} - Z \right) + \right. \\ + \psi_2 \left(\frac{dZ}{dr} - \frac{Z(2M_1 - \alpha M_2)}{r(2\beta M_2 - \alpha M_1)} \right) + \\ + \psi_3 \left(\frac{dU}{dr} + \frac{M_{0j}^2 (2N_1 - \alpha N_2) Z}{N_{0j}^2 r(2\beta M_2 - \alpha M_1)} \right) + \\ + \psi_4 \left(\frac{d}{dr} N_1 - \frac{N_2 - N_1}{r} \right) + \tag{31}$$

$$+\psi_5\left(\frac{d}{dr}M_1-\frac{M_2-M_1}{r}-\right.$$
$$\left.-\frac{1}{r\cos^2\varphi}\left(rN_1\sin\varphi-\frac{P}{2\pi}\right)\right)+\nu_j\left(U\cos\varphi+\right.$$
$$\left.+W\sin\varphi+\frac{M_{0j}^2\cos^2\varphi}{N_{0j}^2}\frac{Z(2\beta N_2-\alpha N_1)}{(2\beta M_2-\alpha M_1)}\right)+$$
$$\varphi_j\left\{\frac{1}{M_{0j}^2}\left(M_1^2-\alpha M_1M_2+\beta M_2^2\right)+\right.$$
$$\left.\left.+\frac{1}{N_{0j}^2}\left(N_1^2-\alpha N_1N_2+\beta N_2^2\right)-Y_1^2\right\}\right\}dr$$

Here $\psi_1,\ldots,\psi_5$ stand for conjugate (adjoint) variables whereas $\nu_j,\lambda_j,\ (j=0,\ldots,n)$ and ψ_{0i}, $(i=1,\ldots,m)$ are Lagrangean multipliers. It is well known that $\psi_{0i}=const$ (see Bryson [2], Hull [2]) as multipliers corresponding to isoperimetric constraints must be constant.

Calculating the total variation of (31) and applying the optimality condition $\Delta J_*=0$ leads to a set of differential and algebraic equations with integral terms. First of all, one obtains the system of adjoint equations

$$\begin{aligned}
\frac{d\psi_1}{dr}&=\frac{\partial F}{\partial W}+\nu_j\sin\varphi,\\
\frac{d\psi_2}{dr}&=-\frac{\psi_2}{r}\frac{2M_1-\alpha M_2}{2\beta M_2-\alpha M_1}+\frac{\psi_3}{r}\frac{M_{0j}^2}{N_{0j}^2}\frac{2N_1-\alpha N_2}{2\beta M_2-\alpha M_1}+\\
&\quad+\frac{\nu_jM_{0j}^2}{N_{0j}^2}\cos^2\varphi\frac{2\beta N_2-\alpha N_1}{2\beta M_2-\alpha M_1},\\
\frac{d\psi_3}{dr}&=\frac{\partial F}{\partial U}+\nu_j\cos\varphi, \qquad\qquad (32)\\
\frac{d\psi_4}{dr}&=\frac{\partial F}{\partial N_1}+\frac{\psi_3}{r}\frac{M_{0j}^2}{N_{0j}^2}\frac{2Z}{2\beta M_2-\alpha M_1}+\frac{\psi_4}{r}-\frac{\psi_5\sin\varphi}{\cos^2\varphi}+\\
&\quad+\frac{\nu_jM_{0j}^2}{N_{0j}^2}\frac{\cos^2\varphi(-\alpha Z)}{2\beta M_2-\alpha M_1}+\frac{\varphi_j}{N_{0j}^2}(2N_1-\alpha N_2),\\
\frac{d\psi_5}{dr}&=\frac{\partial F}{\partial M_1}-\frac{\psi_2}{r}\frac{Z(4\beta-\alpha^2)M_2}{(2\beta M_2-\alpha M_1)^2}+\\
&\quad+\frac{\psi_3}{r}\frac{M_{0j}^2}{N_{0j}^2}\frac{Z\alpha(2N_1-\alpha N_2)}{(2\beta M_2-\alpha M_1)^2}+\frac{\psi_5}{r}+\\
&\quad+\frac{\nu_jM_{0j}^2}{N_{0j}^2}\frac{\cos^2\varphi(2\beta N_2-\alpha N_1)Z\alpha}{(2\beta M_2-\alpha M_1)^2}+\\
&\quad+\frac{\varphi_j}{M_{0j}^2}(2M_1-\alpha M_2).
\end{aligned}$$

Note that the system (32) holds good for $r\in(a_j,a_{j+1})$, where $j=0,\ldots,n$.

The arbitrariness of increments $\Delta h_j\ (j=0,\ldots,n)$ in the equation $\Delta J_*=0$ results in

$$\begin{aligned}
&\frac{\partial G}{\partial h_j}+\int_a^R\sum_{i=1}^m\psi_{0i}\frac{\partial T_i}{\partial h_j}dr+\\
&+\int_{a_j}^{a_{j+1}}\left\{-2\varphi_jM_{0j}^{-3}\delta_0H\left(M_1^2-\alpha M_1M_2+\beta M_2^2\right)-\right. \qquad (33)\\
&\left.-\frac{4\delta_0\varphi_j}{N_{0j}^3}\left(N_1^2-\alpha N_1N_2+\beta N_2^2\right)\right\}dr=0
\end{aligned}$$

for $j=0,\ldots,n.$

For determination of controls M_2,N_2 one obtains the equations

$$\begin{aligned}
&\frac{\psi_2Z(\alpha^2-4\beta)M_1}{r(2\beta M_2-\alpha M_1)^2}-\frac{\psi_3}{r}\frac{M_{0j}^2}{N_{0j}^2}\frac{2Z\beta(2N_1-\alpha N_2)}{(2\beta M_2-\alpha M_1)^2}-\\
&\frac{\psi_5}{r}--\frac{2\beta M_{0j}^2}{N_{0j}^2}\frac{\cos^2\varphi(2\beta N_2-\alpha N_1)Z}{(2\beta M_2-\alpha M_1)^2}+\\
&+\frac{\varphi_j}{M_{0j}^2}(2\beta M_2-\alpha M_1)=0, \qquad\qquad (34)\\
&-\frac{\psi_3}{r}\frac{M_{0j}^2}{N_{0j}^2}\frac{Z\alpha}{(2\beta M_2-\alpha M_1)}-\frac{\psi_2}{r}+\\
&+\frac{2\beta M_{0j}^2}{N_{0j}^2}\frac{\cos^2\varphi Z\nu_j}{(2\beta M_2-\alpha M_1)}+\frac{\varphi_j}{N_{0j}^2}(2\beta N_2-\alpha N_1)=0.
\end{aligned}$$

Due to the arbitrariness on increments $\Delta a_j\ (j=0,\ldots,n)$ in the equation $\Delta J_*=0$ one has

$$\frac{\partial G}{\partial a_j}+\sum_{j=0}^{n}\int_{a_j}^{a_{j+1}}\left(\frac{\partial F}{\partial a_j}+\sum_{i=1}^{m}\psi_{0i}\frac{\partial T_i}{\partial a_j}\right)dr - [\psi_1(a_j)Z(a_j)] - -\left[\frac{\psi_2 Z(a_j)(2M_1(a_j)-\alpha M_2(a_j))}{a_j(2\beta M_2(a_j)-\alpha M_1(a_j))}\right] + \left[\frac{M_{0j}^2\psi_3(a_j)Z(a_j)(2N_1(a_j)-\alpha N_2(a_j))}{N_{0j}^2 a_j(2\beta M_2(a_j)-\alpha M_1(a_j))}\right] - \left[\frac{\psi_4(a_j)N_2(a_j)}{a_j}\right]-\left[\frac{\psi_5(a_j)M_2(a_j)}{a_j}\right]=0 \quad (35)$$

for $j = 1,\dots,n$.

4 Numerical results

Numerical results are presented for conical shells of constant thickness and stepped shells with the unique step. Here following notations are used:

$$\gamma_{0,1}=\frac{h_{0,1}}{h_*},\ \alpha_1=\frac{a_1}{R},\ k=\frac{M_*\cos^2\varphi}{RN_*\sin\varphi},$$
$$p=\frac{PR}{N_*\sin\varphi},\quad m_j=\frac{M_j}{M_*},\quad n_j=\frac{N_j}{N_*}.$$

In these formulae h_* stands for the thickness of carrying layers of the reference shell of constant thickness whereas $N_*=2\delta_0 h_*;\ M_*=\delta_0 h_* H$.

The curves depicted in Fig. 2 – Fig. 5 regard to an elastic shell of constant thickness. The shell is clamped at the outer edge and absolutely free at the inner edge. The dimensions of the shell are: $h=0,02;\ R=1;\ \varphi=16°$. The elastic moduli of the material are $\nu=0,3$ and $E=210GPa$. The results are obtained with a finite element tehnique using beam elements.

In Fig. 2 the distributions of transverse deflections are presented in the cases of various values of the inner radius.

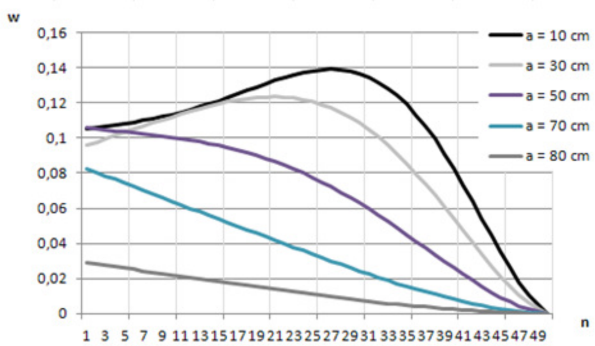

Fig. 2. Transverse deflections

Figures 3, 4, 5 portray the bending moments and the membrane force. It can be seen from Fig. 3 - 5 that the generalized stresses achieve their maximal values at an intermediate point of the interval (a,R), as might be expected. Minimal values of bending moments are achieved at the clamped end at $r=R$.

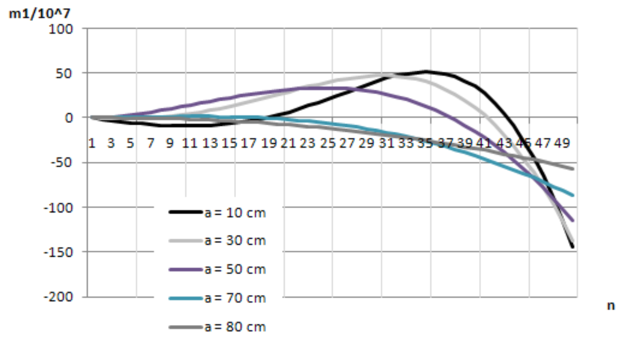

Fig. 3. Radial bending moments

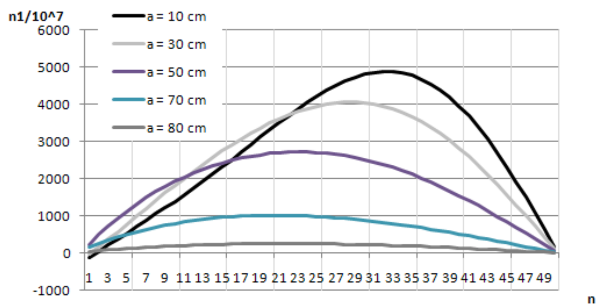

Fig 4. Radial membrane forces

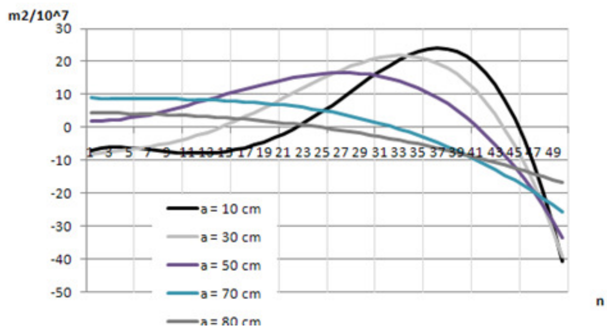

Fig. 5. Circumferential bending moments

In Fig. 6, 7 similar results are depicted for stepped shells with $h_1=0,02$. Solid lines in Fig. 6, 7 correspond to shells of piece wise constant thickness whereas the dashed lines are associated with stepped shells. Here $a = 0{,}1R$ and the step is located at the center of the interval (a, R).

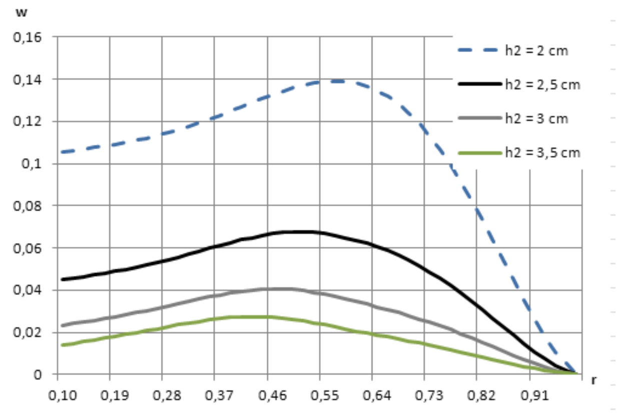

Fig. 6. Transverse deflections for stepped shells

It reveals from Fig. 6 that removing a small amount of the material at the outer region diminishes essentially transverse deflections of the whole shell.

Radial bending moments for stepped shells are depicted in Fig.7 for different values of the thickness in the outer region. Here the step is located at $r = 0{,}5R$. The dashed line in Fig. 7 as well as in Fig. 9-12 corresponds to the shell of constant thickness.

In Fig. 8 the optimal radius of the step is presented versus a/R for different values of the internal thickness $h_0 = \gamma_0 h_*$.

It can be seen from Fig. 8 that when the thickness h_0 tends to h_* then the step location tends to unity.

Fig. 9 portrays the sensitivity of transverse deflections with respect to the location of the step.

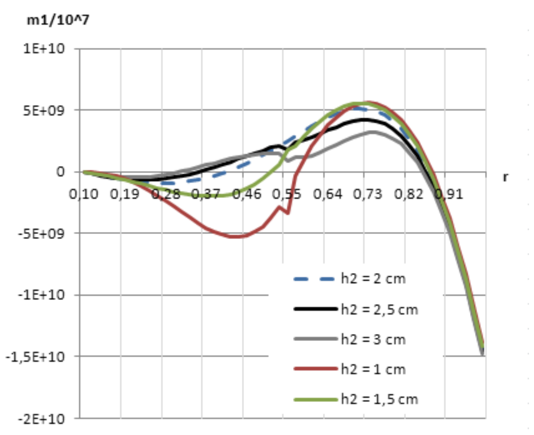

Fig. 7. Radial bending moments for stepped shells

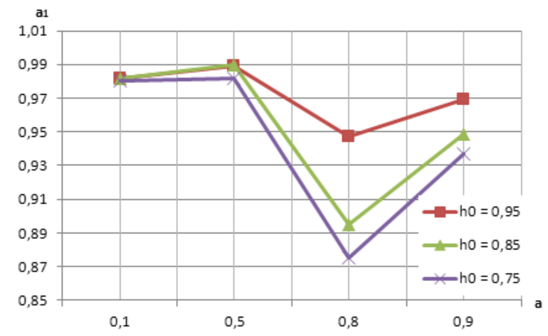

Fig. 8. Optimal location of the step

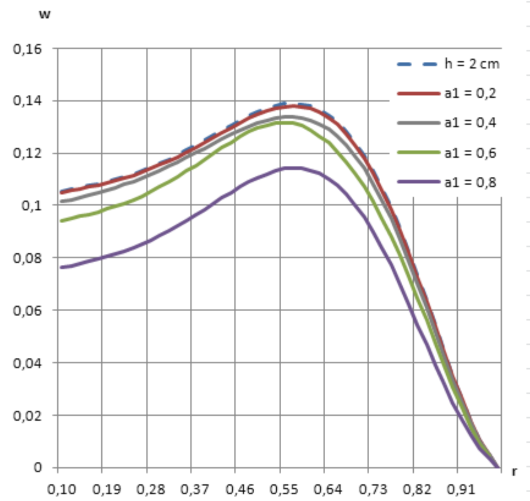

Fig. 9 Transverse deflections for different step locations

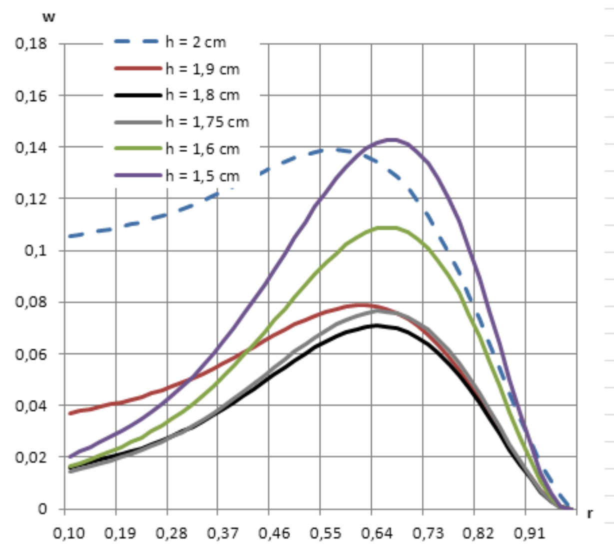

Fig. 10 Optimal transverse deflections for different thicknesses for $a = 0{,}1R$.

In Fig. 10-12 optimal distributions of transverse deflections are presented. Fig. 10, 11 and 12 correspond to shells with the inner radius $a = 0{,}1R$; $a = 0{,}5R$ and $a = 0{,}9R$, respectively. Here the optimal design is the design for which the deflection at the free edge of the shell attains its minimal value.

It can be seen from Fig. 11 that at an inner region of the interval (a, R) the deflection corresponding to an optimized shell can slightly exceed that of the reference shell of constant thickness. However, the deflections of optimized shells at the free edge are less of the deflection of the shell of constant thickness, as might be expected.

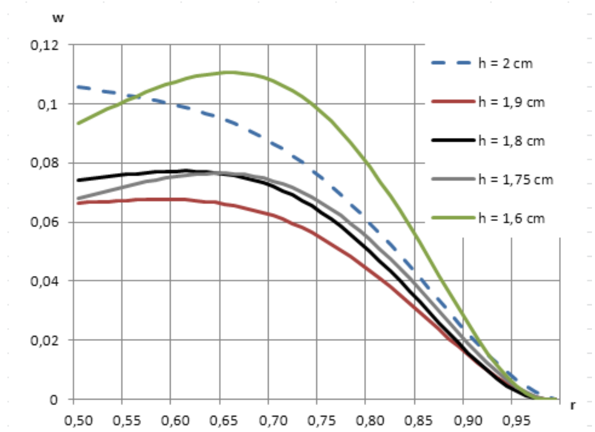

Fig. 11 Optimal transverse deflections for different thicknesses for $a = 0{,}5R$.

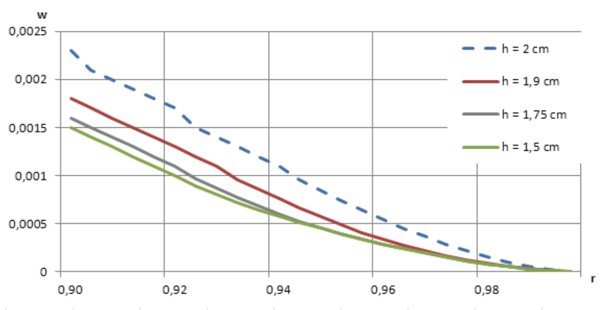

Fig. 12 Optimal transverse deflections for different thicknesses for $a = 0{,}9R$.

In Fig. 13, 14 the optimal parameters of a shell loaded by the rigid central boss are depicted versus the inner radius a.

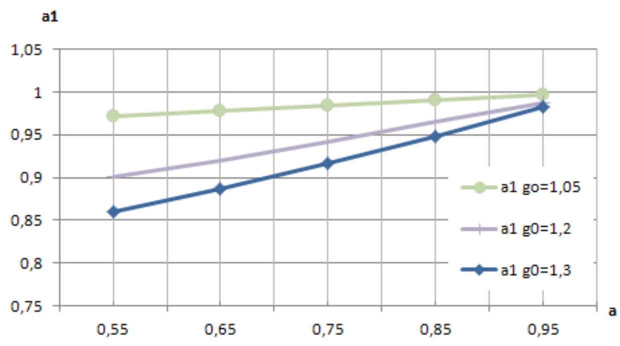

Fig. 13. Optimal location of the step for an inelastic shell

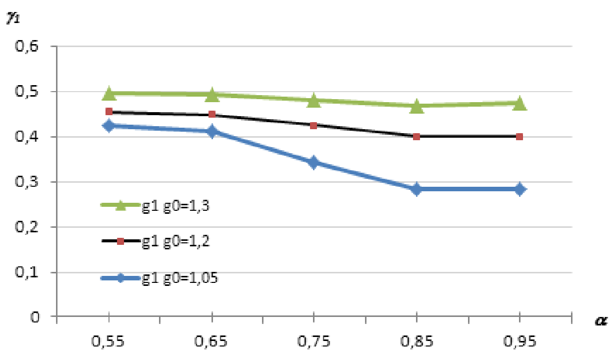

Fig 14. Optimal thickness vrs *a/R* for an inelastic shell

5 Conclusion

Analytical and numerical methods of analysis and optimization of circular conical shells are established. The cases of elastic and inelastic (ideal plastic) materials are investigated. Resorting to the variational methods of the theory of optimal control necessary optimality conditions are obtained.

Calculations carried out showed that in the case of the minimum weight problem a considerable amount of the material can be saved even when using the design of a unique step. Also, in the case of minimization of deflections at the free edge of the shell the design with a single step admits to diminish deflections more than two times. When increasing the number of steps one can achieve the more efficient design of the shell.

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