Some boundary properties of Cauchy type integral in terms of mean oscillation

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Abstract: In this paper radial boundary values of Cauchy type integral are investigated when conditions on integral density are set in terms of mean oscillation of functions. Corresponding statements about boundary values of Poisson integral and conjugate Poisson integral are proved.

Key-Words: Cauchy type integral, Poisson integral, boundary values, singular integrals, mean oscillation

1 Introduction

It is well known that integrals of Cauchy type are closely connected with singular integrals. This connection is expressed by Sokhotskii's formulas [11] (see [1], [2], also). And so, investigation of boundary values of Cauchy type integral requires study of corresponding properties of singular integrals. Structural properties of singular integral operators (multidimensional, in general) in terms of the mean oscillation of functions were investigated by many authors (see, for instance, [4], [7], [9] and papers cited in these works). In one-dimensional case it is a question of Hilbert transform.

In the present work the radial boundary value of the Cauchy type integral over real line $R=(-\infty,\ +\infty)$ is investigated. Conditions on the density of integral are given in terms of the mean oscillation of functions.

The paper is organized as follows. In section 2 we provide necessary preliminaries and notations. In section 3 we obtained upper estimate for the quantity $\left|P_y f(x)-f(I(x,y))\right|$, where $P_y f(x)$ is Poisson integral and $f(I(x,y))$ denoted the average of functions f in the set $I(x,y):=[x-y,\ x+y]$. In section 4 boundary values of conjugate Poisson integral are investigated, while boundary values of Cauchy type integral are studied in section 5. The main results are given in Theorems 3.1, 4.1, 5.1.

2 Preliminary results and definitions

Consider the Cauchy type integral (see [3])

$$\Phi(z)=Kf(z)=\frac{1}{2\pi i}\int_R\left\{\frac{1}{t-z}-\frac{t}{1+t^2}\right\}f(t)dt,$$

where $R=(-\infty,\ +\infty)$ and f is a locally integrable function such that the integral expressing $Kf(z)$ converges. This condition holds, for instance for functions with bounded mean oscillation.

Let's introduce the following designations

$$P_y(x)=\frac{1}{\pi}\frac{y}{x^2+y^2},\quad Q_y(x)=\frac{1}{\pi}\frac{x}{x^2+y^2},$$
$$y>0,\quad x\in R,$$

$$P_y f(x):=\frac{1}{\pi}\int_R\frac{y}{(x-t)^2+y^2}f(t)dt=$$
$$=P_y*f(x),\quad y>0,\quad x\in R,$$

$$Q_y f(x):=\frac{1}{\pi}\int_R\left\{\frac{x-t}{(x-t)^2+y^2}+\frac{t}{1+t^2}\right\}f(t)dt=$$
$$=\int_R\{Q_y(x-t)-Q_1(-t)\}f(t)dt,\quad y>0,\quad x\in R,$$

$$H_y f(x) := \frac{1}{\pi} \int_{|x-t|>y} \left\{ \frac{1}{x-t} + \frac{t}{1+t^2} \right\} f(t)dt,$$
$$y>0, \quad x \in R,$$

$$Hf(x) := \lim_{y \to +0} H_y f(x) =$$
$$= \frac{1}{\pi} v.p. \int_R \left\{ \frac{1}{x-t} + \frac{t}{1+t^2} \right\} f(t)dt,$$

where f is a locally integrable function, i.e. $f \in L_{loc}(R)$.

Note that $Hf(x)$ denotes the Hilbert transform (see, for instance, [3]).

Let us check that

$$\frac{1}{\pi i} \left\{ \frac{1}{t-z} - \frac{t}{1+t^2} \right\} =$$
$$= P_y(x-t) + i\{Q_y(x-t) - Q_1(-t)\}, \qquad (1)$$

where $z = x+iy$, $y>0$. We have

$$P_y(x-t) + i\{Q_y(x-t) - Q_1(-t)\} =$$
$$= \frac{1}{\pi} \cdot \frac{y}{(x\text{-}t)^2 + y^2} + i\frac{1}{\pi} \left\{ \frac{x\text{-}t}{(x\text{-}t)^2 + y^2} + \frac{t}{1+t^2} \right\} =$$
$$= \frac{1}{\pi} \left\{ \frac{y}{(x\text{-}t)^2 + y^2} + i\frac{x\text{-}t}{(x\text{-}t)^2 + y^2} + i\frac{t}{1+t^2} \right\} =$$
$$= \frac{1}{\pi} \left\{ \frac{y + i(x+t)}{[(t-x)-iy][(t-x)+iy]} + i\frac{t}{1+t^2} \right\} =$$
$$= \frac{1}{\pi} \left\{ \frac{-i[(t-x)+iy]}{[(t-x)-iy][(t-x)+iy]} + i\frac{t}{1+t^2} \right\} =$$
$$= \frac{1}{\pi} \left\{ \frac{-i}{t-(x+iy)} + i\frac{t}{1+t^2} \right\} = \frac{1}{\pi i} \left\{ \frac{1}{t-z} - \frac{t}{1+t^2} \right\}.$$

If $z = x-iy$, $y>0$, then we get

$$\frac{1}{\pi i} \cdot \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) = \frac{1}{\pi i} \cdot \left(\frac{1}{t-(x-iy)} - \frac{t}{1+t^2} \right) =$$
$$= \frac{1}{\pi i} \cdot \left(\frac{1}{(t-x)+iy} - \frac{t}{1+t^2} \right) =$$
$$= \frac{1}{\pi i} \cdot \left(\frac{(t-x)-iy}{(t-x)^2 + y^2} - \frac{t}{1+t^2} \right) =$$
$$= \frac{1}{\pi i} \cdot \left(\frac{t-x}{(t-x)^2 + y^2} - \frac{t}{1+t^2} - \frac{iy}{(t-x)^2 + y^2} \right) =$$
$$= \frac{1}{\pi i} \cdot \left(\frac{t-x}{(t-x)^2 + y^2} - \frac{t}{1+t^2} \right) - \frac{1}{\pi} \cdot \frac{y}{(t-x)^2 + y^2} =$$
$$= i \cdot \frac{1}{\pi} \cdot \frac{x-t}{(x-t)^2 + y^2} + \frac{t}{1+t^2} -$$
$$- \frac{1}{\pi} \cdot \frac{y}{(x-t)^2 + y^2} =$$
$$= -P_y(x-t) + i\{Q_y(x-t) - Q_1(-t)\},$$

i.e. for $z = x-iy$, $y>0$, the following equality holds

$$\frac{1}{\pi i} \left\{ \frac{1}{t-z} - \frac{t}{1+t^2} \right\} =$$
$$= -P_y(x-t) + i\{Q_y(x-t) - Q_1(-t)\}. \qquad (2)$$

If denote $u(z) = u(x+iy) = P_y * f(x) = P_y f(x)$ and $\tilde{u}(z) = \tilde{u}(x+iy) = Q_y f(x)$, then for $z = x+iy$, $y>0$, from (1) it follows that

$$u(z) + i\tilde{u}(z) =$$
$$= u(x+iy) + i\tilde{u}(x+iy) = P_y f(x) + iQ_y f(x) =$$
$$= \frac{1}{\pi} \int_R \frac{y}{(x-t)^2 + y^2} f(t)dt +$$
$$+ i\frac{1}{\pi} \int_R \left\{ \frac{x-t}{(x-t)^2 + y^2} + \frac{t}{1+t^2} \right\} f(t)dt =$$
$$= \int_R \{P_y(x-t) + i\{Q_y(x-t) - Q_1(-t)\}\} f(t)dt =$$
$$= \frac{1}{\pi i} \int_R \left\{ \frac{1}{t-z} - \frac{t}{1+t^2} \right\} f(t)dt = 2Kf(z).$$

Consequently, if $z = x+iy$, $y>0$, then

$$Kf(z) = \frac{1}{2}(u(z) + i\tilde{u}(z)) =$$
$$= \frac{1}{2}(u(x+iy) + i\tilde{u}(x+iy)) =$$

$$=\frac{1}{2}\left(P_y f(x)+iQ_y f(x)\right). \qquad (3)$$

Analogously, if $z=x-iy$, $y>0$, then by means of equality (2) we receive that

$$Kf(z)=\frac{1}{2}\left(u(z)+i\tilde{u}(z)\right)=$$

$$=\frac{1}{2}\left(u(x-iy)+i\tilde{u}(x-iy)\right)=$$

$$=\frac{1}{2}\left(-P_y f(x)+iQ_y f(x)\right). \qquad (4)$$

Let f be a function locally integrable in R, that is $f\in L_{loc}(R)$, $I(x,r):=[x-r,\ x+r]$, where $x\in R$, $r>0$, $|I(x,r)|$ be the length of the segment $I(x,r)$, and let

$$f(I(x,r)):=\frac{1}{|I(x,r)|}\int_{I(x,r)} f(t)dt,$$

$$\Omega(f,I(x,r))=\frac{1}{|I(x,r)|}\int_{I(x,r)}|f(t)-f(I(x,r))|dt.$$

$\Omega(f,I(x,r))$ is called the mean oscillation of the function f on the segment $I(x,r)$. Let $x_0\in R$ be a fixed point. Let's designate

$$m_f(x_0;\delta)=\sup\{\Omega(f,I(x_0,r)):\ r\le\delta\},\quad \delta>0,$$

$$M_f(\delta):=\sup\{m_f(x;\delta):\ x\in R\},\quad \delta>0.$$

$M_f(\delta)$ is called the modulus of the mean oscillation of the function f. Note that the function $M_f(\delta)$ was first introduced in [12], the function $m_f(x_0;\delta)$, apparently, in [8]. By $BMO_\varphi=BMO_\varphi(R)$ we denote the set of all functions $f\in L_{loc}(R)$ satisfying the condition

$$\|f\|_{BMO_\varphi}:=\sup\left\{\frac{M_f(\delta)}{\varphi(\delta)}:\ \delta>0\right\}<+\infty,$$

where $\varphi(\delta)$ is a positive monotonically increasing function on $(0;+\infty)$. In other words,

$$BMO_\varphi=\{f\in L_{loc}(R^n):\ M_f(\delta)=O(\varphi(\delta)),\ \delta>0\}.$$

BMO_φ will be regarded as space of functions modulo constants and is Banach space. If $\varphi(t)\equiv 1$, then BMO_φ is the usual BMO of John and Nirenberg [5].

By direct testing it can be shown that the following propositions hold.

Lemma 2.1. If $f(x)\equiv 1=:\mathbf{1}(x)$, then

$$Q_y f(x)=Q_y\mathbf{1}(x)=$$

$$=\frac{1}{\pi}\int_{-\infty}^{+\infty}\left\{\frac{x-t}{(x-t)^2+y^2}+\frac{t}{1+t^2}\right\}dt\equiv 0,$$

$$x\in R,\quad y>0.$$

Lemma 2.2. If $f(x)\equiv 1=:\mathbf{1}(x)$, then

$$H_y f(x)=H_y\mathbf{1}(x)=$$

$$=\frac{1}{\pi}\int_{|x-t|>y}\left\{\frac{1}{x-t}+\frac{t}{1+t^2}\right\}dt=$$

$$=\frac{1}{2\pi}\ln\frac{1+(x-y)^2}{1+(x+y)^2},\quad x\in R,\quad y>0.$$

In particular, from lemma 2.2 it follows that $H\mathbf{1}(x)=\lim_{y\to+0}H_y\mathbf{1}(x)\equiv 0$, $x\in R$.

Lemma 2.3. ([8]). Let $f\in L_{loc}(R)$, $x_0\in R$. Then for $0<\eta<\xi<+\infty$ the following inequality holds

$$|f(I(x_0,\xi))-f(I(x_0,\eta))|\le$$

$$\le\frac{2}{\ln 2}\left(m_f(x_0;\xi)+\int_\eta^\xi t^{-1}m_f(x_0;t)dt\right).$$

If the finite limit $\lim_{\varepsilon\to 0} f(I(x_0,\varepsilon))=:s_f(x_0)$ exists, then the point $x_0\in R$ is called the d-point for $f\in L_{loc}(R)$. The set of all d-points of the function f we denote by $D(f)$.

The point $x_0\in R$ is called the l-point for the function $f\in L_{loc}(R)$ if there exists a finite number $l_f(x_0)$ such that

$$\lim_{\varepsilon\to 0}\frac{1}{|I(x_0,\varepsilon)|}\int_{I(x_0,\varepsilon)}|f(t)-l_f(x_0)|dt=0.$$

The set of all l-points of f we denote by $L(f)$. It is known that if $f\in L_{loc}(R)$, then almost all points

$x \in R$ are l -points of f and the equality $l_f(x) = f(x)$ holds almost everywhere.

If $\lim_{\varepsilon \to 0} m_f(x_0; \varepsilon) = 0$, then the point $x_0 \in R$ is called the m-point for $f \in L_{loc}(R)$ (see [8]). The set of all m-points for f we denote by $M(f)$.

Theorem 2.1. ([8]). A point $x_0 \in R$ is l-point for the function $f \in L_{loc}(R)$ if and only if it is simultaneously d-point and m-point for f, in other words $L(f) = D(f) \cap M(f)$.

3 Estimation of $\left|P_y f(x) - f(I(x,y))\right|$

Theorem 3.1. Let $x \in R$ and $f \in L_{loc}(R)$. Then the following estimation holds

$$\left|P_y f(x) - f(I(x,y))\right| \le$$

$$\le cy \int_y^\infty \frac{m_f(x;t)}{t^2} dt, \quad y > 0, \tag{5}$$

where c is a positive absolute constant.

Proof. Taking into account

$$\int_R P_y(x-t) dt \equiv 1 \quad (x \in R, \ y > 0),$$

we have

$$\left|P_y f(x) - f(I(x,y))\right| \le$$
$$\le \int_R P_y(x-t) \left| f(t) - f(I(x,y)) \right| dt =$$
$$= \int_{|x-t| \le y} P_y(x-t) \left| f(t) - f(I(x,y)) \right| dt +$$
$$+ \sum_{k=0}^\infty \int_{2^k y < |x-t| \le 2^{k+1} y} P_y(x-t) \left| f(t) - f(I(x,y)) \right| dt =$$
$$=: i_1 + \sum_{k=0}^\infty i_{2k} . \tag{6}$$

Let us estimate integrals i_1 and i_{2k}. We have

$$i_1 = \frac{1}{\pi} \int_{|x-t| \le y} \frac{y}{(x-t)^2 + y^2} \left| f(t) - f(I(x,y)) \right| dt \le$$
$$\le \frac{2}{\pi} \cdot \frac{1}{2y} \int_{I(x,y)} \left| f(t) - f(I(x,y)) \right| dt \le$$
$$\le \frac{2}{\pi} m_f(x;y);$$

$$i_{2k} =$$
$$= \frac{1}{\pi} \int_{A_k} \frac{y}{(x-t)^2 + y^2} \left| f(t) - f(I(x,y)) \right| dt \le$$
$$\le \frac{1}{\pi} \int_{A_k} \frac{y}{(x-t)^2 + y^2} \left| f(t) - f(I(x,2^{k+1}y)) \right| dt +$$
$$+ \frac{1}{\pi} \int_{A_k} \frac{y}{(x-t)^2 + y^2} \left| f(I(x,2^{k+1}y)) - f(I(x,y)) \right| dt \le$$
$$\le \frac{1}{\pi} \cdot \frac{y}{(2^k y)^2 + y^2} \int_{A_k} \left| f(t) - f(I(x,2^{k+1}y)) \right| dt +$$
$$+ \frac{1}{\pi} \cdot \frac{y}{(2^k y)^2 + y^2} \left| f(I(x,2^{k+1}y)) - f(I(x,y)) \right| \times$$
$$\times \int_{A_k} dt, \tag{7}$$

where $A_k := \{t : \ 2^k y < |x-t| \le 2^{k+1} y\}$.

Extending integration domain of integrals in the right hand side of the last inequality and applying lemma 2.3, we obtain

$$i_{2k} \le$$
$$\le \frac{1}{\pi} \cdot \frac{y}{\left((2^k)^2 + 1\right) y^2} \int_{|x-t| \le 2^{k+1} y} \left| f(t) - f(I(x,2^{k+1}y)) \right| dt +$$
$$+ \frac{1}{\pi} \cdot \frac{y}{\left((2^k)^2 + 1\right) y^2} \left| f(I(x,2^{k+1}y)) - f(I(x,y)) \right| \times$$
$$\times \int_{|x-t| \le 2^{k+1} y} dt \le$$
$$\le \frac{4}{\pi} \cdot \frac{1}{2^k} \cdot \frac{1}{2^{k+2} y} \int_{I(x,2^{k+1}y)} \left| f(t) - f(I(x,2^{k+1}y)) \right| dt +$$
$$+ \frac{4}{\pi} \cdot \frac{1}{2^k} \cdot \frac{2}{\ln 2} \left(m_f(x;2^{k+1}y) + \int_y^{2^{k+1}y} \frac{m_f(x;t)}{t} dt \right) \le$$

$$\le \frac{4}{\pi}\cdot\frac{1}{2^k}\cdot m_f\left(x;2^{k+1}y\right)+$$

$$+\frac{4}{\pi}\frac{1}{2^k}\frac{2}{\ln 2}\left(m_f\left(x;2^{k+1}y\right)+\int\limits_y^{2^{k+1}y}\frac{m_f(x;t)}{t}dt\right)\le$$

$$\le \frac{11}{\pi\ln 2}\cdot\frac{1}{2^k}\times$$

$$\times\left(m_f\left(x;2^{k+1}y\right)+\int\limits_y^{2^{k+1}y}\frac{m_f(x;t)}{t}dt\right). \qquad (8)$$

On the other hand, if $\varphi(t)$ is a non-negative increasing function on $(0,+\infty)$, then for $y>0$ we get

$$\sum_{k=0}^{\infty}\frac{1}{2^k}\varphi\left(2^{k+1}y\right)\le 8\int\limits_2^{\infty}\frac{\varphi(ty)}{t^2}dt\,,$$

$$\sum_{k=0}^{\infty}\frac{1}{\left(2^k\right)^2}\varphi\left(2^{k+1}y\right)\le 32\int\limits_2^{\infty}\frac{\varphi(ty)}{t^3}dt\,. \qquad (9)$$

Indeed,

$$\int\limits_2^{\infty}\frac{\varphi(ty)}{t^2}dt=\sum_{k=0}^{\infty}\int\limits_{2^{k+1}}^{2^{k+2}}\frac{\varphi(ty)}{t^2}dt\ge$$

$$\ge\sum_{k=0}^{\infty}\varphi\left(2^{k+1}y\right)\frac{1}{\left(2^{k+2}\right)^2}\left(2^{k+2}-2^{k+1}\right)=$$

$$=\frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{2^k}\varphi\left(2^{k+1}y\right);$$

$$\int\limits_2^{\infty}\frac{\varphi(ty)}{t^3}dt=\sum_{k=0}^{\infty}\int\limits_{2^{k+1}}^{2^{k+2}}\frac{\varphi(ty)}{t^3}dt\ge$$

$$\ge\sum_{k=0}^{\infty}\varphi\left(2^{k+1}y\right)\frac{1}{\left(2^{k+2}\right)^3}\left(2^{k+2}-2^{k+1}\right)=$$

$$=\frac{1}{32}\sum_{k=0}^{\infty}\frac{1}{\left(2^k\right)^2}\varphi\left(2^{k+1}y\right).$$

Taking into account (6), (7), (8) and (9) we obtain

$$\left|P_y f(x)-f\left(I(x,y)\right)\right|\le\frac{2}{\pi}m_f(x;y)+\frac{11}{\pi\ln 2}\times$$

$$\times\sum_{k=0}^{\infty}\left\{\frac{1}{2^k}\left(m_f\left(x;2^{k+1}y\right)+\int\limits_y^{2^{k+1}y}\frac{m_f(x;t)}{t}dt\right)\right\}\le$$

$$\le\frac{11}{\pi\ln 2}\left\{m_f(x;y)+\sum_{k=0}^{\infty}\frac{1}{2^k}m_f\left(x;2^{k+1}y\right)+\right.$$

$$\left.+\sum_{k=0}^{\infty}\frac{1}{2^k}\int\limits_y^{2^{k+1}y}\frac{m_f(x;t)}{t}dt\right\}\le$$

$$\le\frac{11\cdot 32}{\pi\ln 2}\cdot\left\{m_f(x;y)+\int\limits_2^{\infty}\frac{m_f(x;ty)}{t^2}dt+\right.$$

$$\left.+\int\limits_2^{\infty}\frac{1}{t^2}\left(\int\limits_y^{ty}\frac{m_f(x;\tau)}{\tau}d\tau\right)dt\right\}. \qquad (10)$$

Since

$$\int\limits_2^{\infty}\frac{m_f(x;ty)}{t^2}dt\ge m_f(x;2y)\int\limits_2^{\infty}t^{-2}dt\ge\frac{1}{2}m_f(x;y),$$

then by inequality (10) we get

$$\left|P_y f(x)-f\left(I(x,y)\right)\right|\le c_1\left\{\int\limits_2^{\infty}\frac{m_f(x;ty)}{t^2}dt+\right.$$

$$\left.+\int\limits_2^{\infty}\frac{1}{t^2}\left(\int\limits_y^{ty}\frac{m_f(x;\tau)}{\tau}d\tau\right)dt\right\}, \qquad (11)$$

where $c_1>0$ is an absolute constant. Using the change of variables $ty=\tau$ in the first integral of the inequality (11), we obtain

$$\int\limits_2^{\infty}\frac{m_f(x;ty)}{t^2}dt=\int\limits_{2y}^{\infty}\frac{m_f(x;\tau)}{(\tau/y)^2}\cdot\frac{1}{y}d\tau=$$

$$=y\int\limits_{2y}^{\infty}\frac{m_f(x;\tau)}{\tau^2}d\tau\le y\int\limits_y^{\infty}\frac{m_f(x;\tau)}{\tau^2}d\tau\,. \qquad (12)$$

Besides,

$$\int\limits_2^{\infty}\frac{1}{t^2}\left(\int\limits_y^{ty}\frac{m_f(x;\tau)}{\tau}d\tau\right)dt=\int\limits_y^{2y}\frac{m_f(x;\tau)}{\tau}\left(\int\limits_2^{\infty}t^{-2}dt\right)d\tau+$$

$$+\int_{2y}^{\infty}\frac{m_f(x;\tau)}{\tau}\left(\int_{\tau/y}^{\infty}t^{-2}dt\right)d\tau=$$

$$=\frac{1}{2}\int_{y}^{2y}\frac{m_f(x;\tau)}{\tau}d\tau+y\int_{2y}^{\infty}\frac{m_f(x;\tau)}{\tau^2}d\tau\le$$

$$\le\frac{1}{2}\ln 2\cdot m_f(x;2y)+y\int_{2y}^{\infty}\frac{m_f(x;\tau)}{\tau^2}d\tau\le$$

$$\le c_2 y\int_{y}^{\infty}\frac{m_f(x;\tau)}{\tau^2}d\tau, \qquad (13)$$

where $c_2>0$ is an absolute constant.

Inequality (5) follows from inequality (11) by inequalities (12) and (13). The theorem is proved.

Taking into account the inequality $m_f(x;\delta)\le M_f(\delta)$ $(x\in R,\ \ \delta>0)$, from theorem 3.1 we obtain the following facts.

Corollary 3.1. Let $x\in R$ and f be a locally integrable function on R. Then the following estimate

$$\left|P_y f(x)-f(I(x,y))\right|\le cy\int_{y}^{\infty}\frac{M_f(t)}{t^2}dt,\qquad y>0,$$

holds, where c is a positive absolute constant.

Corollary 3.2. Let $x\in R$, $f\in L_{loc}(R)$. If

$$\int_{1}^{\infty}\frac{m_f(x;t)}{t^2}dt<+\infty$$

and $x\in L(f)$, then $\lim_{y\to 0}P_y f(x)=l_f(x)$.

Corollary 3.3. Let $f\in BMO_\varphi$, $\varphi(0)=0$ and

$$\int_{1}^{\infty}\frac{\varphi(t)}{t^2}dt<+\infty.$$

Then at the points $x\in L(f)=D(f)$ the equality $\lim_{y\to 0}P_y f(x)=l_f(x)$ holds.

Corollary 3.4. Let $f\in BMO$. Then at each point $x\in L(f)$ the equality $\lim_{y\to 0}P_y f(x)=l_f(x)$ holds.

4 Estimation of $\left|Q_y f(x)-H_y f(x)\right|$

Theorem 4.1. Let $f\in L_{loc}(R)$. Then for $0<y\le 1$ and $x\in R$ the following estimate

$$\left|Q_y f(x)-H_y f(x)\right|\le C\left(y^2\int_{y}^{\infty}\frac{m_f(x;t)}{t^3}dt+\right.$$

$$\left.+\left|f(I(x,y))\right|\left|\ln\frac{1+(x-y)^2}{1+(x+y)^2}\right|\right), \qquad (14)$$

holds, where C is a positive absolute constant.

Proof. Let $x\in R$ and $f\in L_{loc}(R)$. Then, applying lemma 2.1 and 2.2, we get

$$Q_y f(x)-H_y f(x)=$$

$$=\frac{1}{\pi}\int_R\left\{\frac{x-t}{(x-t)^2+y^2}+\frac{t}{1+t^2}\right\}f(t)dt-$$

$$-\frac{1}{\pi}\int_{|x-t|>y}\left\{\frac{1}{x-t}+\frac{t}{1+t^2}\right\}f(t)dt=$$

$$=\frac{1}{\pi}\int_R\left\{\frac{x-t}{(x-t)^2+y^2}+\frac{t}{1+t^2}\right\}\left[f(t)-f(I(x,r))\right]dt-$$

$$-\frac{1}{\pi}\int_{|x-t|>y}\left\{\frac{1}{x-t}+\frac{t}{1+t^2}\right\}\left[f(t)-f(I(x,r))\right]dt-$$

$$-f(I(x,r))\cdot\frac{1}{\pi}\int_{|x-t|>y}\left\{\frac{1}{x-t}+\frac{t}{1+t^2}\right\}dt=$$

$$=\frac{1}{\pi}\int_{|x-t|\le y}\left\{\frac{x-t}{(x-t)^2+y^2}+\frac{t}{1+t^2}\right\}\left[f(t)-f(I(x,r))\right]dt+$$

$$+\frac{1}{\pi}\int_{|x-t|>y}\left\{\frac{x-t}{(x-t)^2+y^2}+\frac{t}{1+t^2}\right\}\left[f(t)-f(I(x,r))\right]dt-$$

$$-\frac{1}{\pi}\int_{|x-t|>y}\left\{\frac{1}{x-t}+\frac{t}{1+t^2}\right\}\left[f(t)-f(I(x,r))\right]dt-$$

$$-f(I(x,r))\cdot\frac{1}{2\pi}\ln\frac{1+(x-y)^2}{1+(x+y)^2}=$$

$$=\frac{1}{\pi}\int_{|x-t|\le y}\frac{x-t}{(x-t)^2+y^2}\left[f(t)-f\left(I(x,r)\right)\right]dt+$$

$$+\frac{1}{\pi}\int_{|x-t|\le y}\frac{t}{1+t^2}\left[f(t)-f\left(I(x,r)\right)\right]dt+$$

$$+\frac{1}{\pi}\int_{|x-t|>y}\left\{\frac{x-t}{(x-t)^2+y^2}-\frac{1}{x-t}\right\}\left[f(t)-f\left(I(x,r)\right)\right]dt-$$

$$-f\left(I(x,r)\right)\cdot\frac{1}{2\pi}\ln\frac{1+(x-y)^2}{1+(x+y)^2}.$$

Then

$$\left|Q_y f(x)-H_y f(x)\right|\le$$

$$\le\frac{1}{\pi}\int_{I(x,y)}\frac{|x-t|}{(x-t)^2+y^2}\left|f(t)-f(I(x,y))\right|dt+$$

$$+\frac{1}{\pi}\int_{I(x,y)}\frac{|t|}{1+t^2}\left|f(t)-f(I(x,y))\right|dt+$$

$$+\sum_{k=0}^{\infty}\frac{1}{\pi}\int_{2^k y<|x-t|\le 2^{k+1}y}\left|\frac{x-t}{(x-t)^2+y^2}-\frac{1}{x-t}\right|\left|f(t)-f(I(x,y))\right|dt+$$

$$+\left|\frac{1}{2\pi}f(I(x,y))\ln\frac{1+(x+y)^2}{1+(x-y)^2}\right|=$$

$$=:i_1+i_2+\sum_{k=0}^{\infty}i_{3k}+i_4. \qquad (15)$$

Let us estimate summands i_1, i_2, i_{3k} $(k=0,1,2,...)$ and i_4 separately for $0<y\le 1$. We have:

$$i_1=\frac{1}{\pi}\int_{I(x,y)}\frac{|x-t|}{(x-t)^2+y^2}\left|f(t)-f(I(x,y))\right|dt\le$$

$$\le\frac{1}{\pi}\int_{I(x,y)}\frac{|x-t|}{y^2}\left|f(t)-f(I(x,y))\right|dt\le$$

$$\le\frac{2}{\pi}\cdot\frac{1}{2y}\int_{I(x,y)}\left|f(t)-f(I(x,y))\right|dt\le$$

$$\le\frac{2}{\pi}m_f(x;y). \qquad (16)$$

In order to estimate i_2 we show that $\frac{|t|}{1+t^2}\le\frac{1}{2}$.

Indeed, $(|t|-1)^2\ge 0$. Therefore, $|t|^2-2|t|+1\ge 0$ and $t^2+1\ge 2|t|$. Thus, $\frac{|t|}{1+t^2}\le\frac{1}{2}$.

Using the last inequality in view of $0<y\le 1$, we obtain

$$i_2=\frac{1}{\pi}\int_{I(x,y)}\frac{|t|}{1+t^2}\left|f(t)-f(I(x,y))\right|dt\le$$

$$\le\frac{1}{\pi}\cdot\frac{1}{2}\int_{I(x,y)}\left|f(t)-f(I(x,y))\right|dt\le$$

$$\le\frac{1}{\pi}\cdot\frac{1}{2y}\int_{I(x,y)}\left|f(t)-f(I(x,y))\right|dt\le$$

$$\le\frac{1}{\pi}m_f(x;y). \qquad (17)$$

Now we estimate i_{3k} $(k=0,1,2,...)$ for $0<y\le 1$. We have

$$i_{3k}=\frac{1}{\pi}\int_{2^k y<|x-t|\le 2^{k+1}y}\frac{\left|(x-t)^2-(x-t)^2-y^2\right|}{\left((x-t)^2+y^2\right)|x-t|}\left|f(t)-f(I(x,y))\right|dt\le$$

$$\le\frac{1}{\pi}\int_{2^k y<|x-t|\le 2^{k+1}y}\frac{y^2}{\left((2^k y)^2+y^2\right)2^k y}\left|f(t)-f(I(x,y))\right|dt\le$$

$$\le\frac{1}{\pi}\cdot\frac{1}{\left((2^k)^2+1\right)2^k}\cdot\frac{1}{y}\int_{I(x,2^{k+1}y)}\left|f(t)-f(I(x,y))\right|dt\le$$

$$\le\frac{1}{\pi(2^k)^2}\cdot\frac{4}{2^{k+2}y}\int_{I(x,2^{k+1}y)}\left|f(t)-f(I(x,2^{k+1}y))\right|dt+$$

$$+\frac{1}{\pi(2^k)^2}\cdot\frac{4}{2^{k+2}y}\int_{I(x,2^{k+1}y)}\left|f(I(x,y))-f(I(x,2^{k+1}y))\right|dt\le$$

$$\le\frac{4}{\pi(2^k)^2}\cdot\frac{1}{\left|I(x,2^{k+1}y)\right|}\int_{I(x,2^{k+1}y)}\left|f(t)-f(I(x,2^{k+1}y))\right|dt+$$

$$+\frac{4}{\pi(2^k)^2}\left|f(I(x,y))-f(I(x,2^{k+1}y))\right|.$$

Applying lemma 2.3, from here we obtain

$$i_{3k} \le \frac{4}{\pi\left(2^k\right)^2} m_f(x;2^{k+1}y) +$$

$$+\frac{4}{\pi\left(2^k\right)^2}\frac{2}{\ln 2}\left[m_f(x;2^{k+1}y) + \int_y^{2^{k+1}y}\frac{m_f(x;t)}{t}dt\right] \le$$

$$\le \frac{8}{\pi\ln 2}\cdot\frac{1}{(2^k)^2}\left[m_f(x;2^{k+1}y) + \right.$$

$$\left. + \int_y^{2^{k+1}y}\frac{m_f(x;t)}{t}dt\right]. \qquad (18)$$

Taking into account

$$i_4 = \frac{1}{2\pi}\left|f(I(x,y))\right|\left|\ln\frac{1+(x-y)^2}{1+(x+y)^2}\right|,$$

by inequalities (15), (16), (17) and (18), for $0 < y \le 1$ we get

$$\left|Q_y f(x) - H_y f(x)\right| \le$$

$$\le \frac{3}{\pi}m_f(x;y) + \frac{8}{\pi\ln 2}\sum_{k=0}^{\infty}\left\{\frac{1}{\left(2^k\right)^2}\left(m_f(x;2^{k+1}y) + \right.\right.$$

$$\left.\left. + \int_y^{2^{k+1}y}\frac{m_f(x;t)}{t}dt\right)\right\} +$$

$$+\frac{1}{2\pi}\left|f(I(x,y))\right|\left|\ln\frac{1+(x+y)^2}{1+(x-y)^2}\right| \le$$

$$\le \frac{8}{\pi\ln 2}\left\{m_f(x;y) + \sum_{k=0}^{\infty}\frac{1}{(2^k)^2}m_f(x;2^{k+1}y) + \right.$$

$$+\sum_{k=0}^{\infty}\frac{1}{(2^k)^2}\int_y^{2^{k+1}y}\frac{m_f(x;t)}{t}dt +$$

$$\left. +\left|f(I(x,y))\right|\left|\ln\frac{1+(x+y)^2}{1+(x-y)^2}\right|\right\}.$$

Applying inequality (9), from here we have

$$\left|Q_y f(x) - H_y f(x)\right| \le \frac{256}{\pi\cdot\ln 2}\left(m_f(x;y) + \right.$$

$$+\int_2^{\infty}\frac{m_f(x;ty)}{t^3}dt + \int_2^{\infty}\frac{1}{t^3}\left(\int_y^{ty}\frac{m_f(x;\tau)}{\tau}d\tau\right)dt +$$

$$\left. +\left|f(I(x,y))\right|\left|\ln\frac{1+(x-y)^2}{1+(x+y)^2}\right|\right). \qquad (19)$$

Since

$$\int_2^{\infty}\frac{m_f(x;ty)}{t^3}dt \ge m_f(x;2y)\cdot\int_2^{\infty}t^{-3}dt \ge \frac{1}{8}m_f(x;y),$$

then from (19) it follows that for $0 < y \le 1$

$$\left|Q_y f(x) - H_y f(x)\right| \le$$

$$\le c_1\left(\int_2^{\infty}\frac{m_f(x;ty)}{t^3}dt + \int_2^{\infty}\frac{1}{t^3}\left(\int_y^{ty}\frac{m_f(x;\tau)}{\tau}d\tau\right)dt + \right.$$

$$\left. +\left|f(I(x,y))\right|\left|\ln\frac{1+(x-y)^2}{1+(x+y)^2}\right|\right) \qquad (20)$$

holds, where $c_1 > 0$ is an absolute constant.

Changing variables $ty = \tau$ in the first integral and changing order of integration in the second integral of inequality (20), we obtain

$$\int_2^{\infty}\frac{m_f(x;ty)}{t^3}dt = \int_{2y}^{\infty}\frac{m_f(x;\tau)}{(\tau/y)^3}\cdot\frac{1}{y}d\tau =$$

$$= y^2\int_{2y}^{\infty}\frac{m_f(x;\tau)}{\tau^3}d\tau \le y^2\int_y^{\infty}\frac{m_f(x;\tau)}{\tau^3}d\tau;$$

$$\int_2^{\infty}\frac{1}{t^3}\left(\int_y^{ty}\frac{m_f(x;\tau)}{\tau}d\tau\right)dt = \int_y^{2y}\frac{m_f(x;\tau)}{\tau}\left(\int_2^{\infty}t^{-3}dt\right)d\tau +$$

$$+\int_{2y}^{\infty}\frac{m_f(x;\tau)}{\tau}\left(\int_{\tau/y}^{\infty}t^{-3}dt\right)d\tau =$$

$$= \frac{1}{8}\int_y^{2y}\frac{m_f(x;\tau)}{\tau}d\tau + \frac{1}{2}y^2\int_{2y}^{\infty}\frac{m_f(x;\tau)}{\tau^3}d\tau \le$$

$$\le \frac{1}{8}\ln 2\cdot m_f(x;2y) + \frac{1}{2}y^2\int_{2y}^{\infty}\frac{m_f(x;\tau)}{\tau^3}d\tau \le$$

$$\le c_2 y^2\int_y^{\infty}\frac{m_f(x;\tau)}{\tau^3}d\tau,$$

where $c_2 > 0$ is an absolute constant. Therefore, in view of inequality (20), we get for $0 < y \le 1$, $x \in R$ that the following inequality

$$\left|Q_y f(x) - H_y f(x)\right| \le$$
$$\le c_3\left(y^2 \int_y^{\infty} \frac{m_f(x;t)}{t^3} dt + \left| f(I(x,y))\right| \left|\ln \frac{1+(x-y)^2}{1+(x+y)^2}\right|\right)$$

holds, where $c_3 > 0$ is an absolute constant. The theorem is proved.

Corollary 4.1. Let f be a locally integrable function on R. Then for $0 < y \le 1$ and $x \in R$ the estimation

$$\left|Q_y f(x) - H_y f(x)\right| \le$$
$$\le C\left(y^2 \int_y^{\infty} \frac{m_f(x;t)}{t^3} dt + \left| f(I(x,y))\right| h(x,y)\right)$$

holds, where C is an absolute constant and

$$h(x,y) := \begin{cases} |x| y, & \text{if } |x| \le 1, \\ \frac{1}{|x|} y, & \text{if } |x| > 1. \end{cases}$$

Corollary 4.2. Let f be a locally summable function on R. Then for $0 < y \le 1$ and $x \in R$ the following estimation

$$\left|Q_y f(x) - H_y f(x)\right| \le$$
$$\le C\left(y^2 \int_y^{\infty} \frac{M_f(t)}{t^3} dt + \left| f(I(x,y))\right| \left|\ln \frac{1+(x-y)^2}{1+(x+y)^2}\right|\right),$$

holds, where $C > 0$ is an absolute constant.

Corollary 4.3. Let $f \in L_{loc}(R)$, $x \in M(f)$ and

$$\int_1^{\infty} \frac{m_f(x;t)}{t^3} dt < +\infty,$$
$$\sup_{0<y\le 1} \frac{1}{|I(x,y)|} \int_{I(x,y)} |f(t)| dt < +\infty.$$

If one of limits $\lim_{y \to 0} Q_y f(x)$ and $\lim_{y \to 0} H_y f(x)$ exists, then other one exists also, moreover they are equal.

Corollary 4.4. Let $f \in BMO_\varphi$ and

$$\int_1^{\infty} \frac{\varphi(t)}{t^2} dt < +\infty.$$

Then the limit $\lim_{y \to 0} Q_y f(x) = Hf(x)$ exists almost everywhere.

5 Radial boundary values of Cauchy type integral

If the integral expressing $Kf(z)$ converges, then the function $Kf(z)$ is analytical both in upper and lower half-planes. This proposition holds when $f \in L^\infty(R)$, $f \in BMO(R)$ and etc., for instance.

By means of inequalities (5) and (14), for $z = x + iy$, $y > 0$ we obtain

$$\left| Kf(z) - \frac{1}{2}\left(f(I(x,y)) + iH_y f(x)\right)\right| =$$
$$= \frac{1}{2}\left| u(x+iy) + i\tilde{u}(x+iy) - f(I(x,y)) - iH_y f(x)\right| \le$$
$$\le \frac{1}{2}\left\{ \left| u(x+iy) - f(I(x,y))\right| + \left|\tilde{u}(x+iy) - H_y f(x)\right|\right\} =$$
$$= \frac{1}{2}\left\{ \left| P_y f(x) - f(I(x,y))\right| + \left| Q_y f(x) - H_y f(x)\right|\right\} \le$$
$$\le c_1\left(y\int_y^{\infty} \frac{m_f(x;t)}{t^2} dt + y^2 \int_y^{\infty} \frac{m_f(x;t)}{t^3} dt + \right.$$
$$\left. + \left| f(I(x,y)) \ln \frac{1+(x-y)^2}{1+(x+y)^2}\right|\right) \le$$
$$\le c\left(y\int_y^{\infty} \frac{m_f(x;t)}{t^2} dt + \left| f(I(x,y)) \ln \frac{1+(x-y)^2}{1+(x+y)^2}\right|\right).$$

Thus, for $0 < y \le 1$, $z = x + iy$, the inequality

$$\left| Kf(z) - \frac{1}{2}\left(i\frac{1}{\pi} \int_{|x-y|>y} \left\{ \frac{1}{x-t} + \frac{t}{1+t^2}\right\} f(t) dt + \right.\right.$$
$$\left.\left. + f(I(x,y))\right)\right| \le c\left(y\int_y^{\infty} \frac{m_f(x;t)}{t^2} dt + \right.$$

$$+\left|f(I(x,y))\ln\frac{1+(x-y)^2}{1+(x+y)^2}\right|\Bigg) \qquad (21)$$

holds, where c is a positive absolute constant.

If $z = x - iy$, $0 < y \le 1$, then we get analogously

$$\left|Kf(z)-\frac{1}{2}\left(iH_y f(x)-f\left(I(x,y)\right)\right)\right|=$$

$$=\frac{1}{2}\left|-u(x-iy)+i\tilde{u}(x-iy)-iH_y f(x)+f(I(x,y))\right|\le$$

$$\le\frac{1}{2}\left\{\left|u(x-iy)-f(I(x,y))\right|+\left|\tilde{u}(x-iy)-H_y f(x)\right|\right\}=$$

$$=\frac{1}{2}\left\{\left|P_y f(x)-f(I(x,y))\right|+\left|Q_y f(x)-H_y f(x)\right|\right\}\le$$

$$\le c_1'\left(y\int_y^\infty\frac{m_f(x;t)}{t^2}dt+y^2\int_y^\infty\frac{m_f(x;t)}{t^3}dt+\right.$$

$$\left.+\left|f(I(x,y))\ln\frac{1+(x-y)^2}{1+(x+y)^2}\right|\right)\le$$

$$\le c'\left(y\int_y^\infty\frac{m_f(x;t)}{t^2}dt+\right.$$

$$\left.+\left|f(I(x,y))\ln\frac{1+(x-y)^2}{1+(x+y)^2}\right|\right), \qquad (22)$$

where c' is a positive absolute constant.

Inequalities (21) and (22) imply the following theorem.

Theorem 5.1. If $f \in L_{loc}(R)$, $x \in L(f)$, singular integral $Hf(x)$ exists and

$$\int_1^\infty \frac{m_f(x;t)}{t^2}dt < +\infty, \qquad (23)$$

then the formulas

$$(Kf)^+(x) := \lim_{y\to+0} Kf(x+iy) =$$

$$=\frac{1}{2}\left(i\frac{1}{\pi}v.p.\int_R\left\{\frac{1}{x-t}+\frac{t}{1+t^2}\right\}f(t)dt+l_f(x)\right), \qquad (24)$$

$$(Kf)^-(x) := \lim_{y\to+0} Kf(x-iy) = .$$

$$=\frac{1}{2}\left(i\frac{1}{\pi}v.p.\int_R\left\{\frac{1}{x-t}+\frac{t}{1+t^2}\right\}f(t)dt-l_f(x)\right) \qquad (25)$$

are valid.

Equalities (24) and (25) are analogues of Yu. B. Sokhotskii's [11] formulas for radial boundary values of Cauchy type integral when the functions f satisfy condition (23) (if $f \in BMO_\varphi$ and

$$\int_1^\infty t^{-2}\varphi(t)dt < +\infty,$$

then the condition (23) is satisfied at each point $x \in R$ and $Hf(x)$ exists a. e. in R [4], [7]).

It is well known that if

$$\delta\int_\delta^\infty t^{-2}\varphi(t)dt = O(\varphi(\delta)), \quad \delta > 0, \qquad (26)$$

then the singular integral operator H is bounded in BMO_φ ([4], [7]). Consequently, if the function φ satisfies condition (26) and $f \in BMO_\varphi$, then $(Kf)^+ \in BMO_\varphi$, $(Kf)^- \in BMO_\varphi$.

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