Hyers-Ulam Stability of Quantum Logic Fuzzy Implication

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Abstract: - There are four different types of fuzzy implications in fuzzy logic, referred to as (S, N)-Implications, R-Implications, QL-Implications, and D-Implications. Only one research work on the implications of two functional equations for the Hyers-Ulam stability for (S, N) has been published recently. The Hyers-Ulam stability of two functional equations for QL-Implication is being examined in this study.

Keywords: - Fuzzy implication, *QL*-Implication, Stability.

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1 Introduction

In 1964, the problem of the stability of the functional equations has been presented to Ulam. He asked if it is potential to find near a function that satisfies the equation precisely (S. M. Ulam, 1964)). In 1941, Gerhard Hyers presented a partial solution, also in 1978, Song and Rassias established the Ulam-Hyers-Rassias stability problem of the functional equation. This problem was named as Ulam-Hyers-Rassias stability problem. Many studies have looked into the problems of stability of several equations. The stability of classical functional equations including the additive mappings, quadratic equations, cubic equations in the fuzzy normed space, and various differential fuzzy equations was also investigated. In these equations, however, there is no fuzzy implication, [5], [8], [10], [11]

The first study on the stability of two functional equations for fuzzy implication, [13], was finished in the year 2020, although this study only looks at the first type of fuzzy implication (S, N)-implication. Studying this equation's stability for the QL-Implication problem that this research is trying to solve is fascinating, [7].

This study examines fuzzy implications that come near to, but don't completely, satisfy these equations. The Hyers-Ulam stability of the QL-implication iterative functional equations is then established. But, if there is a fuzzy negation N, t-norm, and t-conorm, we declare that the implication is QL-Implication as follows: $I(X,Y) = S(N(X)\,,T(X,Y))$, [3], [4].

We would examine the Hyers-Ulam stability of the following two functional equations for QL-Implication in our work:

1. Study the stability of Hyers-Ulam of the derived Boolean law which formulated in fuzzy logic $(\alpha,\beta) = I\big(\alpha, I(\alpha,\beta)\big)$, for QL-implications.
2. Study the stability of Hyers-Ulam of the importation law which is formulated in fuzzy logic $I(T(\alpha,\beta),\gamma) = I\big(\alpha, I(\beta,\gamma)\big)$, for QL-implications.

2 Preliminaries

Definition 2.1 [2]**,** a function $T:[0,1]^2 \to [0,1]$ which forms a triangular norm (in short, t-norm), if T is commutative, increasing, associative, and has 1 as identity.

Definition 2.2 [2]**,** a function $S:[0,1]^2 \to [0,1]$ which forms a triangular conorm (in short, t-conorm), if S is commutative, increasing, associative, and 0 is its identity.

Definition 2.3. [12]**,** a function $N:[0,1] \to [0,1]$ which is defined as a negation function, if:

1. $N(0) = 1,\ N(1) = 0$;
2. $N(p) \le N(q)$, if $p \ge q.\ \forall p,q \in [0,1]$.

Proposition 2.1. [12]**,** for: t-norm T, t-conorm S and strong negation N so, S is N-dual of T when:

$$S(p,q) = N(T(N(p),N(q))),\ \forall p,q \in [0,1],$$

And T is N-dual of S when $T(p,q) = N(S(N(p),N(q))), \forall p,q \in [0,1]$.

Definition 2.4. [1], [6], a function $I:[0,1]^2 \to [0,1]$ is a fuzzy implication if $\forall\, p,q,r \in [0,1]$, the following conditions were fulfilled:

($I1$) $I(1,1) = I(0,1) = I(0,0) = 1$ and $I(1,0) = 0$.
($I2$) $I(p,q) \geq I(r,q)$ if $p \leq r$.
($I3$) $I(p,q) \leq I(p,r)$ if $q \leq r$.
In fuzzy logic, where quantum mechanics is employed as a detriment to conventional reasoning, QL-implications are presented by analogy. The extension of the quantum logic implication is now referred to as a QL operation.
The abbreviation for a quantum logic fuzzy implication is QL-implication. Using inspiration from the classical logic equivalence, fuzzy negation, a t-norm, and a t-conorm provide a QL-implication.
$p \Rightarrow q \equiv \neg p \vee (p \wedge q), \forall p,q \in [0,1]$.
Definition 2.5. [6], let T is a t-norm, S is a t-conorm and N is fuzzy negation. The QL-implication can be defined as:
$I_{T,S,N}(p,q) = S\big(N(p), T(p,q)\big), \forall p,q \in [0,1]$.
QL-implication created by t-norm$T$, a t-conorm S and fuzzy negation N can be denoted by $I_{T,S,N}$.

3 Stability

The Boolean law and the law of importation are tautologies in classical logic, but they are two fundamental features of fuzzy logic. Answers to their solution have surfaced in recent years, [9]. The stability of the fuzzy functional equation with fuzzy implications has not yet been discovered, though.
In response to Ulam's query from 1940, researchers are looking at stability issues as they pertain to functional equations (S.M.Ulam, 1964), he suggested the stability question mentioned below:
Let K_1, K_2 are 2 groups, $D(\cdot,\cdot)$ is a metric on K_2. A number $\varepsilon > 0$ is given, and there is a $\delta > 0$ as if a mapping h: $K_1 \rightarrow K_2$ fulfilled
$D(h(\alpha\,\beta), (\,h(\alpha)h(\beta)) < (\varepsilon)$ For all $\alpha, \beta \in K_1$,
So, there is a group homomorphism $H : K_1 \rightarrow K_2$ and:
$D(h(\alpha), H(\alpha)) < \varepsilon$ For all $\alpha \in K_1$.
When the reply is in the affirmative, the equation $h(\alpha\beta) = h(\alpha)\, h(\beta)$ of the homomorphism is then named stable.
In another formulation, the homomorphism equation is said to be stable if and only if all the approximations can be made using this equation's solution.
Hyers originally proposed a solution to the Ulams puzzle in 1941, and he established the following theorem:
Theorem 3.1 [13], let $h: K_1 \rightarrow K_2$ is a function between the 2 Banach spaces K_1 and K_2 as
$|h(\alpha + \beta) - h(\alpha) - h(\beta)| \leq \varepsilon$, for some $\varepsilon \geq 0$,
for every $\alpha, \beta \in K_1$. There is only a unique function $H: K_1 \rightarrow K_2$ satisfying $|h(\alpha) - H(\alpha)| \leq \varepsilon$, and $H(\alpha + \beta) = H(\alpha) + H(\beta)$, for any $\alpha, \beta \in K_1$.
Because of the Ulam question and answer of Hyers, that type of stability is named Hyers-Ulam stability.
In 2020, [13], explored the law of importation for (S, N)-an implication that is the first kind of fuzzy implication and Hyers Ulam stability for Boolean law, [13]. They look at hazy implications that, although not quite fitting these equations, come near.

3.1 The Study of Hyers-Ulam Stability for Quantum Logic

There are four different kinds of fuzzy implication (S, N), as well as R, QL, D-implication, and functional equations as attributes. Recently, the stability of these functional equations for (S, N) implication was explored. We attempt to prove the Hyers-Ulam stability of two functional equations for QL-implication, a different kind of fuzzy implication, in the present section, [13].
Fuzzy implication properties come in many different forms, including identification, importation law, exchange principle, and others. In fuzzy thinking, these qualities are crucial, [4]. Earlier works on functional equations have always been solutions-oriented.

$$I\big(p, I(p,q)\big) = I(p,q)\,, \tag{1}$$

I is referred to as a fuzzy implication in what is known as derived Boolean law. There have been several types of research regarding the solution of this functional equation for various implications since Shi and his associates identified the solution for equation (1) for various types of fuzzy implications, [4].

$$I(T(p,q),\sigma) = I(p, I(q,\sigma)). \tag{2}$$

In equation (2), which is frequently referred to as the importation law, I stands for a fuzzy implication and T stands for the t-norm. Jayaram clarified the importation law's resolution for several murky consequences, [3]. Many investigations that followed concentrated on solving equation (2) with various implications.
Theorem 3.2 [13], let $l: F_1 \rightarrow F_2$ is a function between the two Banach spaces F_1, F_2 as:

$$|\,l(p+q) - l(p) - l(q)| \leq \varepsilon \tag{3}$$

For some of $\varepsilon \geq 0$, all $p, q \in F_1$, there is only a unique function $L: F_1 \rightarrow F_2$ satisfying

$$\big|\,l(p) - L(p)\big| \leq \varepsilon\,, \tag{4}$$

$$L(p+q) = L(p) + L(q), \tag{5}$$

There are numerous requirements for the stability of several functional equations, even though difficulties with the stability of the functional equations with fuzzy implications have received little attention.

The current portion introduces the study of Hyers-Ulam stability for equations (1) and (2) for QL-implication.

Many studies on stability exist, including one that examines the stability of conventional functional equations, [9].

Yet, these equations do not have any fuzzy implications. 2020 saw the study of Hyers-Ulam stability for two functional equations, [13].

Finding a stable QL-implication is what we're aiming for, so in other expressions when we take a fuzzy implication I satisfying the inequality.

$\left| I\left(p, I(p,q)\right) - I(p,q) \right| \leq \varepsilon.$ (6)

We try finding a mapping $Q\colon [0,1]^2 \to [0,1]$ fulfilling

(1) Q is known as a QL-implication.

(2) $Q\left(p, Q(p,q)\right) = Q(p,q)$.

(3) $|Q(p,q) - I(p,q)| \leq \delta,\ \forall p,q \in [0,1]$.

(4) Q Is the unique QL-implication satisfying (2), (3), δ is very small.

In this work, we use the minimum t-norm $T_M(p,q) = min(p,q)$, t-conorm $S(p,q) = max(p,q)$.

Minimum T_M is a t-norm that is the strongest and $S(p,q) = max(p,q)$ is the weakest t-conorm $\forall p,q \in [0,1]$.

Definition 3.1 [12], for a t-norm T, t-conorm S and strong negation N then S is N-dual of T when

$$S(x,y) = N(T(N(x),N(y))),$$

and T is N-dual of S,

If $T(x,y) = N(S(N(x),N(y))), \forall x,y \in [0,1]$.

Definition 3.2 [1], let T be a t-norm, S is a t-conorm and N is a fuzzy negation. QL-implication can be defined by:

$$I_{T,S,N}(x,y) = S(N(x),T(x,y)), \forall x,y \in [0,1].$$

Lemma 3.1 When N is defined as a continuous fuzzy negation. [13], for all $\varepsilon > 0$, $N_1(x) = 1 + \varepsilon$ and 1 as If any function fulfilling the functional equation is very close to a real solution for the functional equations, it is defined as stable. For QL-implication, we will discuss two functional equations and their Hyers-Ulam stability.

3.2 Stability of Boolean Law $I\left(p, I(p,q)\right) = I(p,q)$

We provide our latest finding about the stability of two functional equations for QL-implication throughout the following section.

Theorem 3.3 [4], let $I \in \mathbb{I}$ (the family of fuzzy implication) be an $QL - implication$ defined using a continuous negation N, a t-norm T, and a t-conorm, then I fulfills derived Boolean law only if $T = T_M$ for any $p,q \in [0,1]$. Here, we can introduce the solution to the problem of stability of iterative functional equations

$$I(p,q) = I\left(P, I(p,q)\right),$$

for QL-implication.

Theorem 3.4 Let $I \in \mathbb{I}$ be an $QL - implication$ defined using a continuous negation N and at-norm T and a t-conorm S when for some $\varepsilon > 0$, I fulfills the inequality (6), there is a $QL - implication\, Q$ satisfying Equation 1 and

$|Q(p,q) - I(p,q)| \leq \varepsilon$, for all $p,q \in [0,1]$. (7)

Proof

(1) Let $Q(p,q) = max(N(p), T_M(p,q))$, then Q is a QL-implication satisfying Equation 1.

(2) Now we prove the Inequality (7). Let $q = 0$, then

$I(p,q) = I(p,0) = S\left(N(p), T_M(p,0)\right) = N(p)$,

and $(p\, I(p,0)) = S(N(p), T_M\left(p, N(p)\right))$, so

$|S(N(p), T_M(p, N(p)) - N(p)| \leq \varepsilon$, (8)

$\forall p \in [0,1]$.

For any $p, q \in [0,1]$, if $N(p) \leq N(q)$, then by Using Eq. (8)

$$S(N(p), T_M(q, N(q)) \leq S(N(q), T_M\left(q, N(q)\right),$$
$$= max(N(q), T_M\left(q, N(q)\right),$$

If $N(q) \leq N(p)$, then

$$T_M(q, N(q)) \leq T_M(p, N(p)).$$

$S(N(p), T_M(q, N(q)) \leq S(N(P), T_M\left(P, N(P)\right),$

$\leq (N(p) + \varepsilon = max(N(p), T_M(q, N(q))$.

Thus we have

$$|max(N(p), T_M(q, N(q)) - S(N(p), T_M(q, N(q))| \leq \varepsilon,$$
$$|max(N(p), \curlyvee) - S(N(p), \curlyvee)| \leq \varepsilon,$$

for any $p, \curlyvee \in [0,1]$. Moreover

$|Q(p, \curlyvee) - I(p, \curlyvee)| \leq \varepsilon,$ (9)

for any $p,q \in [0,1]$. As N is defined as a continuous negation, the N range becomes [0,1]. So, the equation mentioned above can be rewritten in the following form:

$|max(N(p), \curlyvee) - S(N(p), \curlyvee)| \leq \varepsilon, \forall p, \curlyvee \in [0,1]$. (10)

So, $|Q(p, \curlyvee)) - I(p, \curlyvee)| \leq \varepsilon$ for any $p, \gamma \in [0,1]$.

(1) However, $QL - implication$ isn't unique. Let:

$N_1(p) = (1+\varepsilon).\, N(p) \wedge 1$.

For all $p \in [0,1]$, $N_1(p)$ remains a continuous negation by previous lemma. Clearly,

$$Q_1(p,q) = max\left(N_1(p), T_M(p,q)\right),$$

Satisfies Equation (1).

Also, we have

$Q_1(p,q) \geq Q(p,q)$ From $N_1(p) \geq N(p)$,

and

$$\begin{aligned} Q_1(p,q) - Q(p,q) &= max(N_1(p), T_M(p,q)) \\ &\quad - max(N(p), T_M(p,q)) \\ &\le N_1(p) - N(p) \\ &= (1+\varepsilon).N(p) \wedge 1 - N(p) \\ &= (1+\varepsilon)N(p) - N(p) \\ &= \varepsilon.N(p) \\ &\le \varepsilon. \end{aligned}$$

So,
$0 \le Q_1(p,q) - Q(p,q) \le \varepsilon$, for any $p, q \in [0,1]$.
Combined with
$0 \le I(p,q) - Q(p,q) \le \varepsilon$ for any $p, q \in [0,1]$.
We obtain
$0 \le Q_1(p,q)) - I(p,q)) \le \varepsilon$ for any $p, q \in [0,1]$.
Thus
$| Q_1(p,q)) - I(p,q)| \le \varepsilon$, for any $p, q \in [0,1]$.

3.3 Stability of Law of Importation $\boldsymbol{I(T(p,q),\sigma) = I(p, I(q,\sigma))}$

To find the stability of the law of importation $I(T(p,q),\sigma) = I(p, I(q,\sigma))$**,** we will count the problem mentioned below for the case of minimum t-norm, i.e.

$$I(T_M(p,q),\sigma) = I(p, I(q,\sigma)). \quad (11)$$

Same to Ulam's question, here we get the problem below:
A fuzzy implication I is given which fulfills inequality (11)

$$\left|I(T_M(p,q),\sigma) - I\big(p, I(q,\sigma)\big)\right| \le \varepsilon, \quad (12)$$

for all $p, q, \sigma \in [0, 1]$, we try finding a mapping Q: $[0, 1]^2 \to [0, 1]$ fulfilling:

(1) Q is a$QL-implication$;
(2) $Q(T_M(p,q),\sigma) = Q(p, Q(q,\sigma))$;
(3) $|Q(p,q) - I(p,q)| \le \delta, \forall p, q \in [0,1]$;
(4) Q is the unique $QL-implication$ satisfying (2) and (3).

The error δ is defined as a real positive number. It has to be very small.

Theorem 3.5. Let $I \in \mathbb{I}$ is a *QL*-implication which is defined by the strong negation *N* and a t-conorm S, so it fulfills Eq. (11) with t-norm *T* only when $T = T_M$

Theorem 3.6 Let $I \in \mathbb{I}$ is a *QL*-implication which is defined using the strong negation *N* and a t-norm *T* and a t-conorm. when for some $\varepsilon > 0$, *I* satisfy inequality (12), so there is a *QL*-implication *Q* fulfilling Eq. (11) and

$$\left|Q(p,q)\right) - I\big((p,q)\big| \le \varepsilon, \text{ for any } p, q \in [0,1]. \quad (13)$$

Proof.

(i) Let $Q(p,q) = max(N(p), T_M(p,q))$, then Q is a *QL*-implication satisfying Equation (11)
(ii) Now we prove Equation (13). Let $\sigma = 0$, and $N(q) \le p$.

$I(T_M(p,q),\sigma) = I(T_M(p,q),0)=$
$S(N(T_M(p,q)), T_M(T_M(p,q),0) = N(p \wedge q)$
$= N(p) \vee N(q) = N(p) \vee T_M(p, N(q))$.
Thus,
$S(N(p), T_M(p, N(q)) \ge max(N(p), T_M(p, N(q)))$.
We have

$$\begin{aligned} N(p) \vee T_M(p, N(q)) &\le S(N(p), T_M(p, N(q)) \\ &\le N(p) \vee T_M(p, N(q)) + \varepsilon. \end{aligned}$$

As N is continuous, the range of N remains [0,1]. After that, the equation mentioned above can be rewritten as follows:
$N(p) \vee \Upsilon \le S(N(p), \Upsilon) \le N(p) \vee \Upsilon) + \varepsilon$, for any $p, \Upsilon \in [0,1]$.
So, $0 \le I(p,\Upsilon) - Q(p,\Upsilon) \le \varepsilon$, then we have
$|I(p,\Upsilon) - Q(p,\Upsilon)| \le \varepsilon$, For any $p, \Upsilon \in [0,1]$.
For all the strong negation *N*, there is a new strong negation N_1 as:
$0 \le N_1(p) - N(p) \le \varepsilon$ for all $p \in [0, 1]$.
So, the (*S*, *N*)-implication in the above theorem isn't unique.
Let $Q_1(p,q) = max(N_1(p), T_M(p,q))$, then we have, for any $p \in [0, 1]$.

$$0 \le Q_1(p,q) - Q(p,q) =$$
$$max(N_1(p), T_M(p,q)) - max\big(N(p), T_M(p,q)\big),$$
$$0 \le N_1(p) - N(p) \le \varepsilon,$$

and we have

$$-\varepsilon \le Q(p,q) - I(p,q) \le 0,$$

from $0 \le I(p,q) - Q(p,q) \le \varepsilon$. Then we obtain

$$\begin{aligned} -\varepsilon + 0 \le Q_1(p,q) - Q(p,q) + Q(p,q) - I(p,q) \\ \le 0 + \varepsilon. \end{aligned}$$

$-\varepsilon \le Q_1(p,q) - I(p,q) \le \varepsilon$, for any $p \in [0,1]$.
Then $Q_1(p,q) = \max\big(N_1(p), T_M(p,q)\big)$, is a new *QL*-implication fulfilling Eq. (11) and (13) and this shows uniqueness.

4 Conclusion and Future Work

In this work, we attempted to demonstrate the Hyers-Ulam stability of two functional equations for QL-implications with N strong continuous negation, T the lowest t-norm, and S the maximum t-conorm. The other QL-implication Q is close to I with a tiny inaccuracy and satisfies these equations if the two functional equations hold. This indicates that there exists a solution for equations (1) and (2) under sufficient constraints on the functions involved in "near" any solution of the inequality (6). (2). There are more functional equations and fuzzy implications for future investigation. It would be necessary to provide more information and talk about the stability issue with those equations.

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