Singularity Behaviour of the Density, Information, and Entropy Functions Defining a Uniform Non-stationary Stochastic Process

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Abstract: - Precise definitions and derivatives of the time-dependent continuous and discrete uniform probability density functions and related information and entropy functions are investigated. A stochastic system is formed that can represent a uniform noise source having a time-dependent variance and forming a uniform non-stationary stochastic process. The information and entropy function of the system are defined, and their properties are investigated in the time domain, including the limit cases defined for infinite and zero values of the time-dependent variance. In particular, the singularity properties of the entropy function will be investigated when the time-dependent variance reaches infinity. Like in thermodynamics, where the physical entropy of a system increases all the time, the information entropy of the stochastic system in information theory is also expected to increase towards infinity when the variance increases. All investigations are conducted for both the continuous and discrete random variables and their density functions. The presented theory is of particular interest in analyzing the Gaussian density function having infinite variance and tending to a uniform density function.

Key-Words: - Uniform probability density function, information function, information entropy, continuous and discrete density, limits of information functions.

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1 Introduction

The theory of stochastic processes having a probability density function (pdf) that is a function of time is important in the analysis of non-stationary stochastic processes. In this paper, this theory is further extended to the information theory by defining and deriving the time-dependent information and entropy functions. It is assumed that the pdf function of a stochastic process is uniform and its variance linearly depends on time. The process is analyzed starting with the results in information theory published in Shannon's seminal paper [1]. According to the presented theory, the randomness in the system generating the process will increase in time, and the average information per a random event, or the system entropy, will tend to infinity. Furthermore, it will be shown that the probability of all random events will tend to zero when the variance tends to infinity. We say the probabilities of events reach equilibrium when all probability values reach theoretical zero carrying the information contents that tend to infinity.

This theory is analogous to thermodynamics theory. Namely, the second law of thermodynamics states that the physical entropy of the enclosed system always increases and has an identical expression as the information entropy defined in this paper, as noted, for example, in Glattfelder's paper [2]. Furthermore, the results of this paper will show that the system entropy is finite and tends to infinity when the probability of each random event tends to zero carrying the information content that tends to infinity.

Having in mind the thermodynamics theory, the presented system operates irreversibly out of the thermodynamic equilibrium with the ability to reach the thermodynamic equilibrium. An analysis of a system operating out of the thermodynamic equilibrium is presented in Nicholson's paper [3].

Inside the system, generating a uniform non-stationary stochastic process, the entropy function would be singular when the time-dependent variance reaches infinity, i.e., the entropy will suddenly change from infinity to zero, even though that contradicts Leibniz's famous statement *Natura non facit saltus* (nature makes no jump) [4]. In addition, the information function gets infinite values due to the related all-zero pdf function.

We will assume that the interval of time between any two consecutive realizations of the process is negligibly small compared to the observed interval of time t_∞, and the amplitude values of the process in that interval of time are mutually independent. Likewise, the entropy of the system increases logarithmically towards infinity, reaches infinity, and then drops down to zero theoretically in infinity, as symbolically presented by a dashed thick line in Fig.1.

Inside the system, four basic stochastic processes can be generated: continuous- and discrete-time and continuous-valued processes, and continuous- and discrete-time and discrete-valued processes. The random variables defining these processes are described by the time-dependent continuous and discrete uniform probability density functions, respectively. Detailed analysis of these pdf functions and related information and entropy functions will be presented in the following sections.

2 Time-dependent uniform pdf of a continuous random variable

2.1 Definition of a time-dependent probability density function

Due to the importance of the probability density function (pdf) for our analysis, we will start with its precise definition which will be consistently used in presenting our theory. The uniform pdf of a continuous random variable X is defined as

$$f_c(x) = \begin{Bmatrix} \dfrac{1}{2X_c} & -X_c \le x \le X_c \\ 0 & \text{otherwise} \end{Bmatrix} \underset{X_c^2=3\sigma_c^2}{=} \begin{Bmatrix} \dfrac{1}{2\sqrt{3\sigma_c^2}} & -X_c \le x \le X_c \\ 0 & \text{otherwise} \end{Bmatrix} \quad (1)$$
$$\underset{\sigma_c^2=\sigma^2 t}{=} \begin{Bmatrix} \dfrac{1}{2\sqrt{3\sigma^2 t}} & -X_c \le x \le X_c \\ 0 & \text{otherwise} \end{Bmatrix}$$

for the positive values $0 \le X_c \le \infty$. Furthermore, we will express the variance as a linear function of time, i.e., $\sigma_c^2 = \sigma^2 t$ for a constant σ^2 defined in the interval $0 \le \sigma^2 \le \infty$. The function is graphically presented in Fig. 1a), for the mean value equal to zero and the varying values of X_c that define the variance $\sigma_c^2 = X_c^2/3$. For our analysis, we could not rely on the definition in Montgomery and Runger's book [5], where zero values of pdf are not considered. The closest and a nearly proper definition is in Papoulis and Pillai book [6], Peebles book [7], Manolakis book [8] which defined the variance but not its limits, even though the limit values of the pdf function that have positive values are not specified. The definition in Gray's book [9], and Proakis's book [10] are incomplete and cannot be used as such to develop our theory.

In our rigorous definition, we specify the interval of limit values to be $0 \le X_c \le \infty$, which also includes the equation sign due to the necessity to explain the behavior of the pdf function and related information function at zero and infinite values of parameters X_c and σ_c, i.e., when these parameters not only tend to infinity but when they reach infinity. It is also strictly specified the interval of uniform density values different from zero as $-X_c \le x \le X_c$, which allows us to define the values of the information function for every x in the interval $-\infty \le x \le \infty$ and calculate the entropy of the information function of the uniformly distributed random variable X. Namely, alongside pdf function, we will investigate the behavior of the information function $I(X)$ and the entropy $H(X)$ of random variable X. These three functions are presented in Fig. 1.

Unlike Shannon, who defines, in his famous paper [1], the entropy of a continuous random variable with the pdf function $f_c(x)$, we will first define and analyze the information function as a log function with the base 2 of the pdf function in equation (1) and express it as

$$I(X) = -\log_2 f_c(x) = \begin{Bmatrix} \log_2 2X_c & -X_c \le x \le X_c \\ -\log_2 0 & \text{otherwise} \end{Bmatrix}$$
$$= \begin{Bmatrix} \log_2 2\sigma_c\sqrt{3} & -X_c \le x \le X_c \\ \infty & \text{otherwise} \end{Bmatrix}, \quad (2)$$
$$\underset{\sigma_c^2=\sigma^2 t}{=} \begin{Bmatrix} \log_2 2\sqrt{3\sigma^2 t} & -X_c \le x \le X_c \\ \infty & \text{otherwise} \end{Bmatrix}$$

which simplifies our understanding of the physical sense of both the information function and the related entropy function that is expressed as the mean value of the information function.

The information function values are increasing inside the interval X_c when the interval width is decreasing as shown in Fig. 1, for the intervals defined by X_c = 0, 1/8, 1/4, 1/2, 1, and 2 with the corresponding values of the pdf function being ∞, 4, 2, 1, 1/2, and 1/4, respectively, which are presented in italic font. If the interval X_c drops to zero, the pdf function becomes the Dirac delta function, represented by an arrow line pointing + ∞ in Fig. 1a). If the interval X_c tends to infinity, the pdf function tends to zero.

If we form a continuous-time i.i.d. stochastic process $X(t)$, defined by the random variable X at each time instant t, we may define the related realization of this process as uniform random signals $x(t)$. The theory of these processes uses the same notation as presented in chapter 19 of Berber's book [11]. Suppose three pdf functions of uniform random variables X are defined by $X_c = 2$, 0, and 1. The three realizations of the related stochastic processes, also called random signals in the presented theory, are presented in Fig. 2a) on the same coordinate system for the sake of simplicity. The first realization, or the random signal $x_1(t)$, takes the values between -2 and 2 in the time interval from 0 to 2. The second realization is a horizontal line overlapping the abscissa defining certain events of generating zero amplitudes at each time instant inside the interval from 2 to 6. The third realization takes the values from -1 to 1 in the interval from 6 to 8.

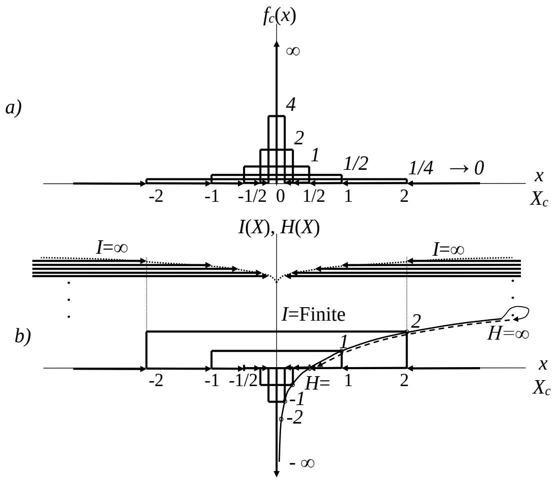

Figure 1 a) Continuous uniform pdf function, and b) related information and entropy functions.

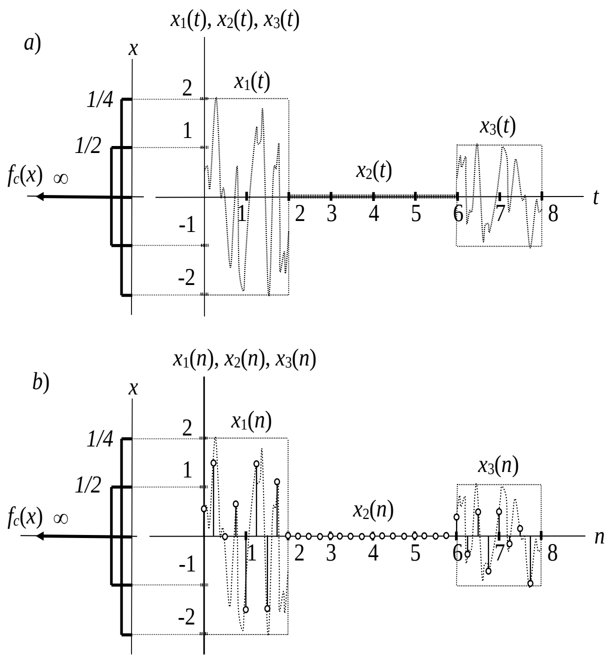

Figure 2 a) Realisations of three continuous-time, and b) three discrete-time stochastic processes.

The theory presented in this section can be applied to discrete-time stochastic processes. A realization of this process is expressed as a random function of the discrete-time variable n instead of t, as shown in Fig. 2b). In explaining and using these processes we will follow the notation and theory presented in chapter 4 of Berber's book [11].

Contrary to the system with a fixed statistical property of the random experiment, we can imagine a process that is generated in time according to the varying distributions presented in Fig. 3, starting with an all-zero pdf function producing an undefined random signal. This signal is followed by a random signal defined by the Dirac pdf delta function and finishes with a distribution defined by $X_c \rightarrow \infty$, producing a random signal with possible $\pm\infty$ amplitudes. Let us analyze the limit cases when the pdf function parameter σ_c or X_c tends to infinity and zero, which are essential for understanding the properties of a non-stationary process having a time-dependent variance.

Parameter σ_c or X_c tends to infinity. In this case, when the variance of the pdf function tends to infinity causing the defined positive pdf function values to tend to zero, we may have

$$\lim_{\substack{X_c \to \infty \\ \text{or } \sigma_c \to \infty}} f_c(x) = \lim_{X_c \to \infty} \frac{1}{2X_c} = \lim_{\sigma_c \to \infty} \frac{1}{2\sqrt{3\sigma_c^2}} \underset{\sigma_c^2 = \sigma^2 t}{=} \lim_{t \to \infty} \frac{1}{2\sqrt{3\sigma^2 t}} = 0 \quad (3)$$

as is notified in Fig. 3 by a horizontal line on the left graph. All random values x are spread in the infinite interval from $-\infty$ to $+\infty$ and occur with the probability that tends to zero (dashed arrow line) and reaches infinity (bold arrow line). When the variance tends to infinity, the probabilities are infinitesimally small, and the process still has its realizations defined on an infinite interval of possible values.

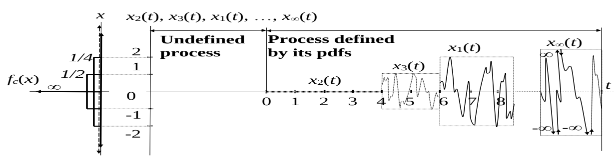

Figure 3 System represented by hypothetical realizations of one undefined process and four continuous-time random processes characterized by four uniform pdf functions for $X_c = 0$, 1, 2, and ∞.

Therefore, when X_c reaches infinity, the random signal values are generated with the probability of zero causing the process and its realization to vanish. We can say that the process is presented by an empty graph. Someone can argue that these random values cannot exist due to the zero probability of their generation and can be ignored and, consequently, the defined zero values of the limiting pdf function can be pointless. However, we will show that this function has meaning in the physical world from the information function point of view and cannot be ignored.

Parameter σ_c or X_c tends to zero. If one of the parameters, σ_c, X_c, or t tends to zero, the pdf function becomes the Dirac delta function according to

$$\lim_{\substack{X_c \to 0 \\ \sigma_c \to 0}} f_c(x) = \lim_{\sigma_c \to 0} \frac{1}{2\sigma_c \sqrt{3}} = \lim_{t \to 0} \frac{1}{2\sqrt{3\sigma^2 t}} = \delta(x) = \begin{Bmatrix} \infty & x = 0 \\ 0 & \text{otherwise} \end{Bmatrix} \quad (4)$$

having the infinite value at $x = 0$. A realization of a stochastic process, which is defined by this limit uniform pdf function that is represented by the Dirac delta function, is shown in Fig. 4. All amplitude values are zero because the probability of their generation is one.

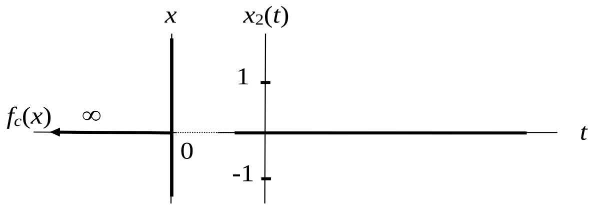

Figure 4 A realization of a stochastic process defined by random variable X that is defined by the uniform pdf function as a Dirac delta function at the point $x = 0$.

2.2 Information function

Due to its importance for our analysis, we will define and investigate the properties of the information function expressed y eq. (2) for two limit cases when σ_c or X_c tends to infinity or zero.

Parameters σ_c or X_c tends to infinity. For the first case, the information function is

$$I_\infty(X) = \lim_{\substack{X_c \to \infty \\ \sigma_c \to \infty}} I(X) = \lim_{\substack{X_c \to \infty \\ \sigma_c \to \infty}} \log_2 \frac{1}{f_X(x)}$$
$$= \lim_{X_c \to \infty} \begin{Bmatrix} \log_2 2X_c & -X_c \le x < X_c \\ \infty & \text{otherwise} \end{Bmatrix}, \quad (5)$$
$$= \lim_{\sigma_c \to \infty} \begin{Bmatrix} \log_2 2\sigma_c \sqrt{3} & -X_c \le x < X_c \\ \infty & \text{otherwise} \end{Bmatrix}$$
$$= \lim_{t \to \infty} \begin{Bmatrix} \log_2 2\sqrt{3\sigma^2 t} & -X_c \le x < X_c \\ \infty & \text{otherwise} \end{Bmatrix} = \infty$$

for all x values, $-\infty \le x \le \infty$. We can understand this function in the following way. When the interval $\pm X_c$ of random variable X stretches to infinity, all values of the random variable will exist and appear with the infinitesimally small probability nearly equal to zero. Thus, having these small probabilities of appearance, the information content of all of them will be close to infinity. The random events persist to exist, and they can happen with a probability close to zero.

In infinity (i.e., when the interval X_c reaches infinity) the pdf function values become zero, thus the probability of any event becomes zero, and a realization of that stochastic process is an empty graph. The random events persist to potentially exist, and they can happen with the probability of zero. From a theoretically strict point of view, the random signal does not exist in time, i.e, there are no changes in signal values, or there are no changes in time. The pdf function is zero in the entire interval of x values, as defined in (3) and shown in Fig. 5a), the information function is ∞ for all x values, as shown in Fig. 5b). A realization $x(t)$ of the related stochastic process $X(t)$ is an empty coordinate system.

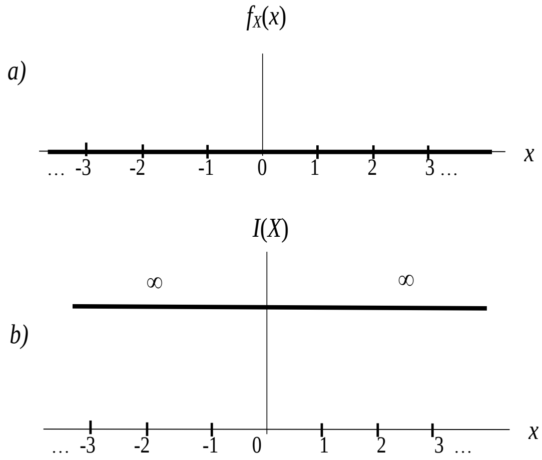

Figure 5 Continuous uniform pdf function and related information function defined for the infinite variance value.

Parameter σ_c or t tends to zero. In this case, the information content is

$$I_0(X) = \lim_{\sigma_c \to 0} I(X) = \lim_{\sigma_c \to 0} \log_2 \frac{1}{f_c(x)}$$
$$= \lim_{\sigma_c \to 0} \begin{Bmatrix} \log_2 2\sigma_c \sqrt{3} & -X_c \le x \le X_c \\ \log_2 (1/0) & \text{otherwise} \end{Bmatrix} \cdot \qquad (6)$$
$$= \begin{Bmatrix} -\infty & x = 0 \\ \infty & \text{otherwise} \end{Bmatrix}$$

or

$$I_0(X) = \lim_{t \to 0} \begin{Bmatrix} \log_2 2\sqrt{3\sigma^2 t} & -X_c \le x \le X_c \\ \log_2 (1/0) & \text{otherwise} \end{Bmatrix} \qquad (7)$$
$$= \begin{Bmatrix} -\infty & x = 0 \\ \infty & \text{otherwise} \end{Bmatrix}$$

which is presented in Fig. 6b) alongside the corresponding pdf function shown in Fig. 6a).

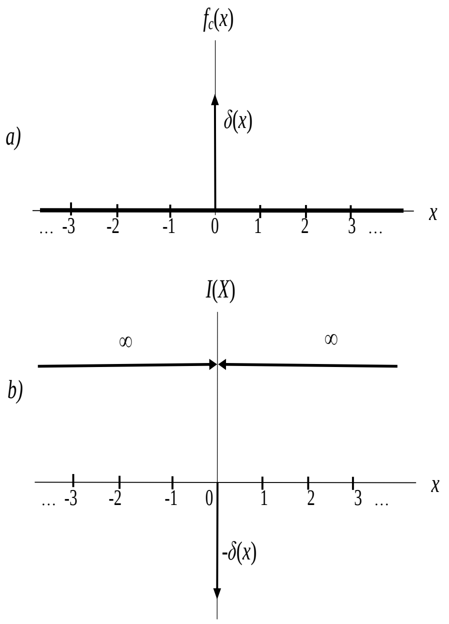

Figure 6 Continuous uniform pdf function and related information function defined for the zero-variance value.

Therefore, in the case when the pdf function is a delta function, the minimum information content is $-\infty$ and is represented by the inverted Dirac delta function. For this case, we are certain that any realization x of random variable X will be zero and there is no uncertainty (information) about the value of this realization, i.e., the information takes the minimum value which is $-\infty$. However, the information content takes and remains of the $+\infty$ value everywhere else on the x-axis where the pdf function of X has zero values. A realization of the related stochastic process is a horizontal line having an amplitude of zero as shown in Fig. 4. Zero x value occurs for sure, that is a certain event. Therefore, there is a substantial difference between the graphs in Fig. 5 and 6.

The same behavior of the information function can be observed if interval X_c tends to zero, as presented in Fig. 7. The function values are increasing from minus infinity, for zero value $2X_c$, to infinity for the infinite value of $2X_c$, which complies with the findings in equations (5) and (7).

2.3 Entropy

By following Shannon's theory presented in his famous paper[1], the entropy of a continuous random variable with the defined pdf function $f_c(x)$ can be expressed as

$$H(X) = -\int_{-\infty}^{\infty} f_c(x) \log_2 f_c(x) dx , \qquad (8)$$

and calculated as

$$H(X) = -\int_{-X_c}^{X_c} \frac{1}{2X_c} \log_2 \frac{1}{2X_c} dx$$
$$-\int_{-\infty}^{-X_c} 0 \cdot \log_2 0 dx - \int_{X_c}^{\infty} 0 \cdot \log_2 0 dx = \log_2 2X_c \cdot \qquad (9)$$
$$= \log_2 2\sqrt{3\sigma_c^2} \underset{\sigma_c^2 = \sigma^2 t}{=} \log_2 2\sqrt{3\sigma^2 t}$$

We say that the entropy is defined as the mean value of the information function (2), and represents the information (uncertainty) content per a random value x of random variable X. The contribution to entropy value is zero for all x values outside of the $2X_c$ interval where the pdf function is zero. The entropy values are numbers that can be positive, $X_c > 1/2$, zero for $X_c = 1/2$, and negative for $X_c < 1/2$. The positive values of the entropy are increasing inside the interval X_c when the interval width is increasing as shown in Fig. 1b), for the intervals defined with $X_c = 1$ and 2 with the corresponding values of the function being 1 and 2, which are presented in italic font. For $X_c < 1/2$, the entropy is negative and increases in absolute value.

One note more on the entropy: In this theoretical analysis, we accept Shannon's definition of entropy in contrast to its definition as a differential entropy which can be found in some books, for example, in Haykin's book [12]. We consider it unnecessary to introduce the differential entropy due to the continuity of the random variable

X and the strict definition of entropy as an integral transform, which makes the presented theory to be consistent. Let us analyze the limit cases for the entropy when the pdf function parameter σ_c, t, or X_c tends to infinity or zero.

Parameter σ_c, t, or X_c tends to infinity. If the interval X_c tends to be infinite, someone can calculate mistakenly the entropy using expression (9) as

$$H_\infty(X) = \lim_{X_c\to\infty} \log_2 H(X) = \lim_{X_c\to\infty} \log_2 2X_c = \lim_{\sigma_c\to\infty} \log_2 2\sigma_c\sqrt{3} \underset{\sigma_c^2=\sigma^2 t}{=} \lim_{t\to\infty} \log_2 2\sqrt{3\sigma^2 t} = \infty , \quad (10)$$

which is specifically valid for the case when $X_c \to\infty$, but not for the case when $X_c = \infty$, i.e., when X_c reaches infinity. However, following (10), the influence of zero values of the pdf function in infinity is not considered. If the interval X_c reaches infinity the entropy should be calculated using its definition, which will include the zero-valued pdf function, i.e.,

$$H_\infty(X) = \lim_{\substack{X_c\to\infty\\ \sigma_c\to\infty}} \left(-\int_{-\infty}^{\infty} f_c(x)\log_2 f_c(x)dx \right) = \left(-\int_{-\infty}^{\infty} 0\log_2 0dx \right) = 0$$

or

$$H_\infty(X) = \lim_{t\to\infty} \left(-\int_{-\infty}^{\infty} f_c(x)\log_2 f_c(x)dx \right) = \left(-\int_{-\infty}^{\infty} 0\log_2 0dx \right) = 0 \quad (11)$$

Another confirmation of validity for (11) can be obtained as follows. When X_c tends to infinity, the random variable takes values in the infinite interval stretching from $-\infty$ to $+\infty$ that occur with the probability of zero. To consider these probability values we can confirm (11) by calculating the entropy as

$$H_\infty(X) = \lim_{\sigma_c\to\infty} \int_{-\infty}^{\infty} f_c(x)\log_2 \frac{1}{f_c(x)}dx = \lim_{\substack{X_c\to\infty\\ \sigma_c\to\infty}} \int_{-\infty}^{\infty} \frac{1}{2\sigma_c\sqrt{3}_c}\log_2 2\sigma_c\sqrt{3}dx$$
$$= \frac{1}{2\sqrt{3}} \lim_{\sigma_c\to\infty} \int_{-\infty}^{\infty} \frac{1}{\sigma_c}(\log_2 2\sqrt{3} + \log_2 \sigma_c)dx$$
$$= \frac{1}{2\sqrt{3}} \lim_{\sigma_c\to\infty} \int_{-\infty}^{\infty} \frac{1}{\sigma_c}(\log_2 2\sqrt{3})dx + \frac{1}{2\sqrt{3}} \lim_{\sigma_c\to\infty} \int_{-\infty}^{\infty} \frac{1}{\sigma_c}(\log_2 \sigma_c)dx$$
$$\underset{\sigma_c^2=\sigma^2 t}{=} \frac{1}{2\sqrt{3}} \lim_{t\to\infty} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{t}}(\log_2 2\sqrt{3})dx + \frac{1}{2\sqrt{3}} \lim_{t\to\infty} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{t}}(\log_2 \sigma\sqrt{t})dx$$

If the limit of the integral is equal to the integral of the limits, we may have

$$H_\infty(X) = \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty} \lim_{\sigma_c\to\infty} \frac{1}{\sigma_c}(\log_2 2\sqrt{3})dx + \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty} \lim_{\sigma_c\to\infty} \frac{1}{\sigma_c}(\log_2 \sigma_c)dx$$
$$\underset{\sigma_c^2=\sigma^2 t}{=} \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty} \lim_{t\to\infty} \frac{1}{\sigma\sqrt{t}}(\log_2 2\sqrt{3})dx + \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty} \lim_{t\to\infty} \frac{1}{\sigma\sqrt{t}}(\log_2 \sigma\sqrt{t})dx$$

The first limit is zero and the second is of an indeterminate form. Applying the L'Hopital's rule, the second integral is also zero, i.e.,

$$H_\infty(X) = \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty} \lim_{\sigma_c\to\infty} \frac{d\log_2 \sigma_c / d\sigma_c}{d\sigma_c / d\sigma_c}dx = \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty} \lim_{\sigma_c\to\infty} \frac{(\log_2 e)/\sigma_c}{1}dx = \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty} 0\cdot dx = 0 \quad .(12)$$

or in respect to the time variable t, we may have

$$H_\infty(X) = 0 + \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty} \lim_{t\to\infty} \frac{1}{\sigma\sqrt{t}}(\log_2 \sigma\sqrt{t})dx = \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty} \lim_{t\to\infty} \frac{d\log_2 \sigma\sqrt{t}/dt}{d\sigma\sqrt{t}/dt}dx$$
$$= \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty} \lim_{t\to\infty} \frac{\log_2 e(1/2)\sigma/\sqrt{t}}{\sigma\sqrt{t}\cdot\sigma(1/2)1/\sqrt{t}}dx = \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty} \lim_{t\to\infty} \frac{\log_2 e}{\sigma\sqrt{t}}dx = \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty} 0\cdot dx = 0 \quad . \quad (13)$$

Therefore, considering the values of the information and corresponding probability values, the entropy, as the measure of the average information content inside the random values x, is zero. These entropy values are presented in Fig. 1b) by cycles connected by a full curve that reaches infinite entropy.

Parameter σ_c or X_c tends to zero. When parameters σ_c or X_c tends to zero, the entropy value can be calculated as follows

$$H_0(X) = \lim_{\substack{T_c\to 0\\ \sigma_c\to 0}} \int_{-\infty}^{\infty} f_c(x)\log_2 \frac{1}{f_c(x)}dx = -\int_{-\infty}^{\infty} \delta(x)\log_2 \delta(x)dx = -\log_2 \delta(0) = -\infty \quad (14)$$

For our analysis, we have separately defined and used the information function $I(X)$ as defined by equation (2), which contains the information content of the random variable X. Therefore, for the uniform density, the information content defined inside the X_c interval is numerically equal to the calculated entropy, as can be seen in Fig. 7.

Precise graphical presentations of relevant functions as shown in Fig. 8. It is important to note the following: While the interval X_c tends to infinity the entropy value tends to infinity. In infinity, the intervals of zero values of entropy disappear and the entropy calculated in the entire infinite interval becomes zero. In contrast to entropy, if the appearance of all values of random variable X is happening with the probability of zero, the information content of all of them is infinite as will be seen from the following analysis.

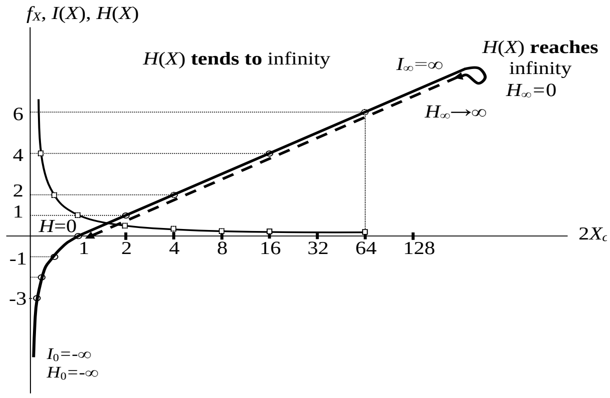

Figure 7 Continuous uniform pdf function, the related information, and entropy, as functions of the size of interval $2X_c$.

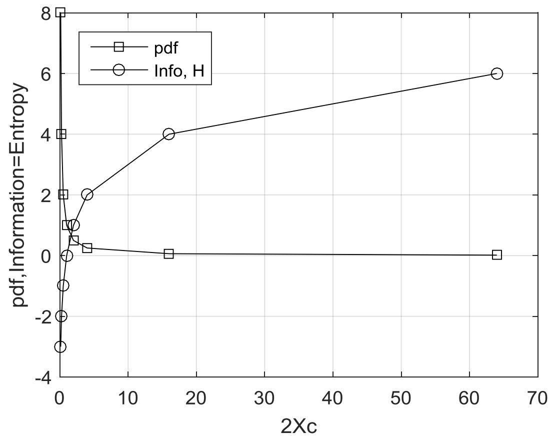

Figure 8 Precise presentation of the continuous uniform pdf function, information, and entropy as functions of the interval $2X_c$.

In the infinity, the events, the realizations of random variable X, potentially exist and they can happen with the probability of zero, i.e., theoretically never. Consequently, from the generation of the random signal $x(t)$ strict point of view, which is a realization of the stochastic process $X(t)$, there are no changes in the appearance of these random values $x(t)$ at time instants t. However, the information content of all possible random values is infinite.

3 Discrete uniform random variables

3.1 Probability density function

It is important to note that we will distinguish and use two types of the discrete uniform pdf functions: a pdf function expressed in terms of the Dirac delta functions, and a pdf function expressed in terms of the Kronecker delta functions. Even though these two types can be used to represent the same pdf function, they are different in practical applications and have different meanings in defining related information functions.

When the Dirac delta functions are used, the pdf function will be expressed as a function of a continuous random variable value x. On the infinite uncountable set of real values x, we will define an infinite countable number of discrete points for integer values $x = s$, where the pdf function has either values different from zero or zero values that are defined by the final weights of Dirac delta functions. Consequently, the intervals between any two adjacent delta functions contain continuous values of x with the pdf function of zero value, as shown in Fig. 9a). When Kronecker delta functions are used, the pdf function will be expressed as a function of discrete random variable values $x = s$ having amplitudes defined by the weights of the Kronecker functions. Consequently, the pdf values inside intervals between any two adjacent delta functions will be undefined, because they are not zero, or we say that these intervals contain nothing instead of the pdf values.

We must use these presentations of pdf functions to derive appropriate expressions for the information function and entropy and understand their properties. These presentations are consistent with the theoretical explanation presented by Berber [14], for the case when the Dirac delta functions are used and Kronecker delta functions are assumed as an additional possible solution. Delta functions are used to present the discrete pdf functions in Papoulis and Pillai's book [6], even though the type of the delta function is not specified. In the same book, a primitive definition of the uniform discrete pdf is presented on p. 98, which cannot be considered as a precise one to be used in our theoretical developments. A presentation of the pdf function in terms of the impulse delta function is given in Peebles book [7], where a detailed analysis of the Dirac delta function (called the unit-impulse function) is presented. The uniform discrete pdf function can be expressed in terms of Dirac delta functions as

$$f_d(x) = \begin{Bmatrix} \sum_{s=-S}^{s=S} \frac{1}{2S+1} \cdot \delta(x-s) & -S \le x = s \le S \\ 0 & \text{otherwise} \end{Bmatrix}$$

$$= \begin{Bmatrix} \sum_{s=-S}^{s=S} \frac{1}{2S+1} \cdot \delta(x-s) & -S \le x = s \le S \\ \sum_{s<-S,\, s>S} 0 \cdot \delta(x-s) & x = s < -S \text{ and } x = s > S \\ 0 & x \ne s \end{Bmatrix}, \quad (15)$$

One example of this function is graphically presented in Fig. 9a) for the size of the discrete interval defined by $S = 2$. It is important to note that the pdf function is defined on a continuous interval $-\infty < x < +\infty$ of possible random variable values, having the values $f_d(x) \ge 0$ at a set of countable finite discrete instants of random values x, and zeros everywhere else.

We can also use here the Kronecker delta functions to express the discrete pdf function in this form

$$f_d(x) = \begin{Bmatrix} \sum_{s=-S}^{s=S} \frac{1}{2S+1} \cdot \delta(x-s) & -S \le x = s \le S \\ \sum_{s<-S,\, s>S} 0 \cdot \delta(x-s) & x = s < -S \text{ and } x = s > S \\ \text{undefined} & x \ne s \end{Bmatrix} \quad (16)$$

This density is graphically presented in Fig. 9b) for the discrete interval $S = 2$, which defines the variance $\sigma_d^2 = S(S+1)/3$ =2. The pdf function can be time-dependent if we just express S as a function of time. Because this change of variables will not change the generality of the explanation, it will be avoided here.

In summary, we can say that the interval of the random variable values x is continuous if the pdf function values are zero everywhere else except at points s, which are defined by the Dirac delta functions. The Dirac delta function can be replaced by the Kronecker delta function assuming that the variable x is a discrete random variable having the integer values from $-\infty$ to $+\infty$. In this case, the values of the pdf function are defined at discrete instants s and are not defined between them. We say that the values of the pdf function do not exist between points s. The pdf functions based on Dirac’s and Kronecker's presentation are shown in Fig. 9a) and 9b), respectively, with the related random signals generated according to these pdf functions and presented in Fig. 9c) and 9d), respectively.

The random signal in Fig. 9c) is a continuous-time discrete-valued signal. Any amplitude of the signal $x(n)$ is generated with the related probability $f_d(x)$ and preserves that value until the next time instant (n+1) because the probability of generating amplitudes between $x(n)$ and $x(n+1)$ is zero. For that reason, we can express this signal as a function of continuous-time t, as in Fig. 9c). In contrast to this signal, the signal in Fig. 9d) is a discrete-time discrete-valued signal. These two signals combined with the signals presented in section 1.1 complete the set of basic signals that can be generated in the system. Therefore, the use of the Dirac and Kronecker delta functions have some differences and will be separately analyzed.

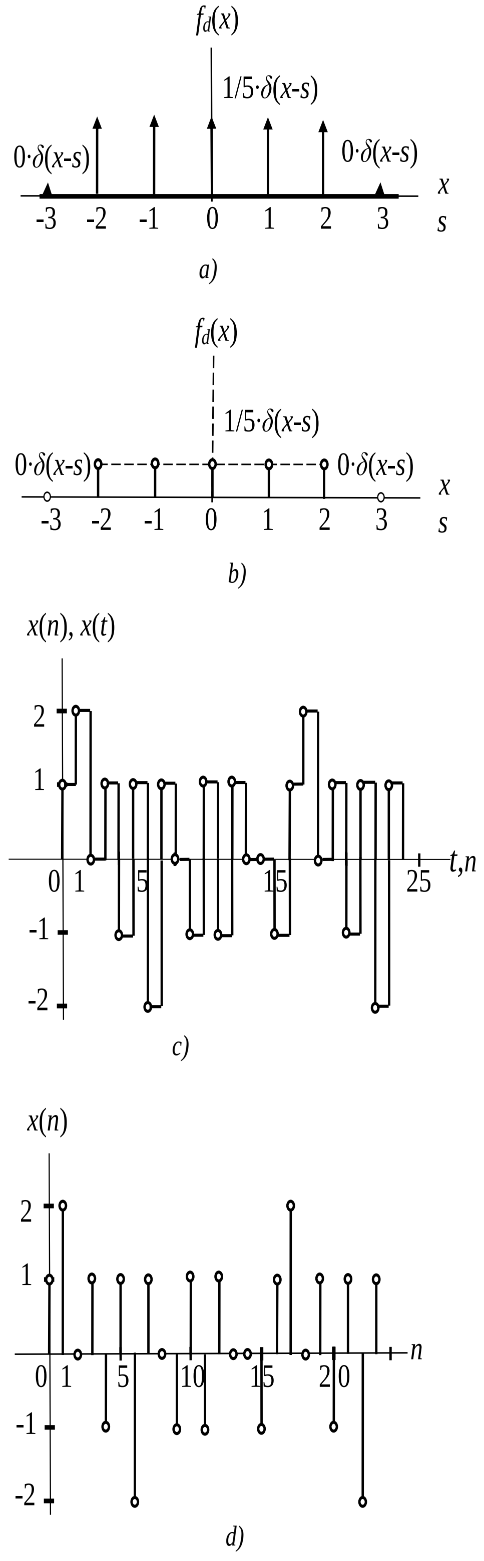

Figure 9 Discrete uniform pdf function represented by a) Dirac and b) Kronecker delta functions and the realizations (random signals) of related stochastic processes c) and d).

Let us analyze the limit cases when the density parameters σ_d or S tend to infinity and zero. We are to notify the differences in presenting the pdf function by Dirac and Kronecker functions.

Parameter σ_d or S tends to infinity. The limits of the pdf function, expressed by the Dirac delta functions, when the variance tends to infinity, can be obtained as follows

$$\lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} f_d(x) = \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} \sum_{s=-S}^{s=S} \frac{1}{2S+1}\cdot\delta(x-s) = 0\,. \quad (17)$$

Precisely calculating, the limit values of the pdf function are expressed in terms of Dirac delta functions as

$$\lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} f_d(x) = \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} \sum_{s=-S}^{s=S} \frac{1}{2S+1}\cdot\delta(x-s)$$
$$= \begin{Bmatrix} \sum_{s=-S}^{s=S} \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} \frac{1}{2S+1}\cdot\delta(x-s) & x=s \\ 0 & x\neq s \end{Bmatrix}, \quad (18)$$
$$= \begin{Bmatrix} \sum_{s=-\infty}^{\infty} 0\cdot\delta(x-s) & x=s \\ 0 & x\neq s \end{Bmatrix}$$

because the values $1/(2S+1)$ tend to zero when S tends to infinity. In this case, all random values are distributed in the infinite interval stretching from $-\infty$ to $+\infty$ and occurring with the probability of zero. The pdf function for this case is presented in Fig. 10a), and the related random signal in Fig. 10c).

If the same pdf function is expressed in terms of Kronecker delta functions, the limit values are

$$\lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} f_d(x) = \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} \sum_{s=-S}^{s=S} \frac{1}{2S+1}\cdot\delta(x-s)$$
$$= \begin{Bmatrix} \sum_{s=-\infty}^{\infty} 0\cdot\delta(x-s) & x=s \\ \text{undefined} & x\neq s \end{Bmatrix} \quad (19)$$

and presented in graphical form as in Fig. 10b) with the related random signal in Fig. 10d). In this case, all random values are distributed in the infinite countable interval stretching from $-\infty$ to $+\infty$ and occur with the probability of zero.

Parameter σ_d or S tends to zero. In the second case, parameters σ_d tends to zero and the pdf function can be expressed by the Dirac delta function at the zero point, i.e.,

$$\lim_{\substack{S\to 0 \\ \sigma_d\to 0}} f_d(x) = \lim_{\substack{S\to 0 \\ \sigma_d\to 0}} \sum_{s=-S}^{s=S} \frac{1}{2S+1}\cdot\delta(x-s) = \delta(x) = \begin{Bmatrix} 1\cdot\delta(0) & x=s=0 \\ 0 & \text{otherwise} \end{Bmatrix}$$

or in a precise form as

$$\lim_{\substack{S\to 0\\ \sigma_d\to 0}} f_d(x) = \lim_{\substack{S\to 0\\ \sigma_d\to 0}} \sum_{s=-S}^{s=S} \frac{1}{2S+1}\cdot\delta(x-s)$$

$$= \left\{ \begin{matrix} 1\cdot\delta(x-s) & x=s=0 \\ \sum_{s=-\infty, s\neq 0}^{\infty} 0\cdot\delta(x-s) & x=s\neq 0 \\ 0 & \text{other } x \end{matrix} \right\}. \quad (20)$$

$$= \left\{ \begin{matrix} 1\cdot\delta(0) & x=s=0 \\ \sum_{s=-\infty, s\neq 0}^{\infty} 0\cdot\delta(x-s) & x=s\neq 0 \\ 0 & \text{other } x \end{matrix} \right\}$$

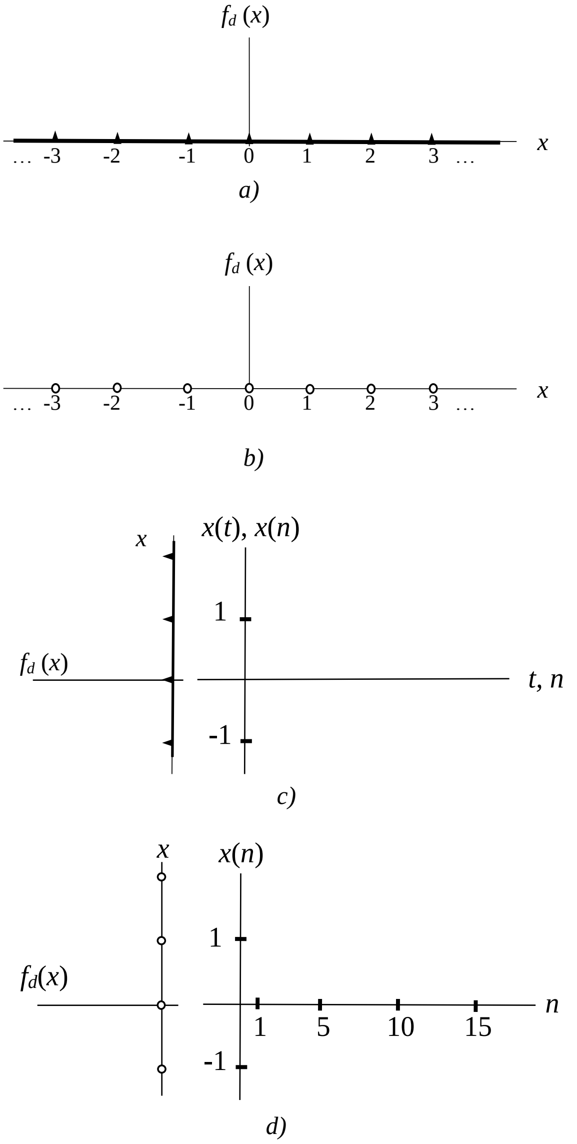

Figure 10 Discrete uniform pdf function represented by a) Dirac and b) Kronecker delta functions when the variance tends to infinity, and realizations of the related stochastic processes, c) and d), respectively.

Therefore, the pdf function is defined as the Dirac delta at x = 0 having the weight one, and by a stream of delta functions of zero weights for all the other values s, as presented in Fig. 11a). In between these delta impulses, the pdf function values are zero. Therefore, the zero-weight delta functions fill in the x-axis and make it to be continuous.

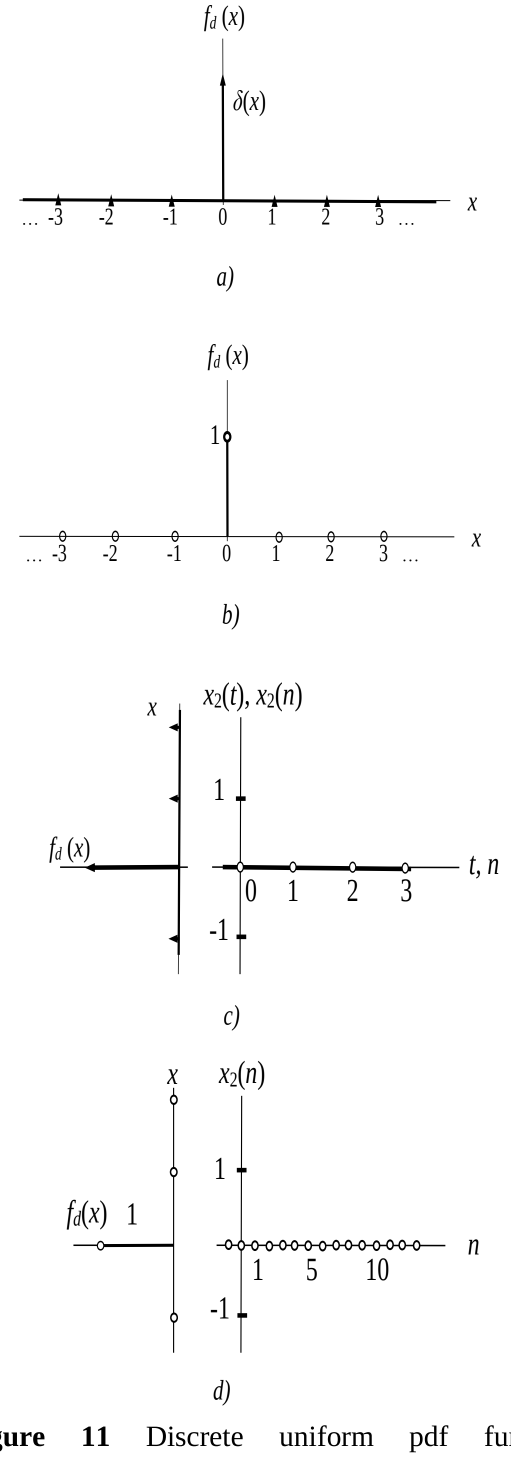

Figure 11 Discrete uniform pdf function represented by a) Dirac and b) Kronecker delta functions when the variance tends to zero, and

realizations of the related stochastic processes, c) and d)), respectively.
If the pdf function is expressed in terms of Kronecker delta functions and the interval S tends to zero, the pdf function is

$$\lim_{\substack{S\to 0\\ \sigma_d\to 0}} f_d(x)=\lim_{\substack{S\to 0\\ \sigma_d\to 0}}\sum_{s=-S}^{s=S}\frac{1}{2S+1}\cdot\delta(x-s)$$
$$=\begin{cases}\lim\limits_{\substack{S\to 0\\ \sigma_d\to 0}}\sum\limits_{s=-S}^{s=S}\frac{1}{2S+1}\cdot\delta(x-s) & -S\le x=s\le S\\ \lim\limits_{\substack{S\to 0\\ \sigma_d\to 0}}\sum\limits_{s<-S,\,s>S}0\cdot\delta(x-s) & x=s<-S \text{ and } x=s>S\\ \text{undefined} & x\ne s\end{cases}$$
$$=\begin{cases}1\cdot\delta(x-s) & x=s=0\\ \sum\limits_{s<-S,\,s>S}0\cdot\delta(x-s) & x=s\ne 0\\ \text{undefined} & \text{other } x\ne s\end{cases}. \quad (21)$$
$$=\begin{cases}1 & x=s=0\\ \sum\limits_{s<-S,\,s>S}0\cdot\delta(x-s) & x=s\ne 0\\ \text{undefined} & \text{other } x\ne s\end{cases}$$

This pdf function can be understood as the Kronecker delta at x = 0 and by a stream of delta functions of zero weights for all other $x = s$ discrete values. In between these delta functions, the pdf function values are not defined. This function is presented in Fig. 11b), and the related all-zero discrete-time random signal in Fig. 11d).

3.2 Information function

Assuming that the pdf function is expressed in terms of Dirac delta functions, we will separately define and use the information function $I(X)$ that contains the information content, or the information, of the random variable X. Having in mind the properties of the delta function, the information function can be derived in this form

$$I(X)=-\log_2 f_d(x)=$$
$$\begin{cases}-\log_2\sum\limits_{s=-S}^{s=S}\frac{1}{2S+1}\cdot\delta(x-s) & -S\le x=s\le S\\ -\log_2\sum\limits_{s<-S,\,s>S}0\cdot\delta(x-s) & x=s<-S \text{ and } x=s>S\\ -\log_2 0 & x\ne s\end{cases}$$
$$=\begin{cases}\sum\limits_{s=-S}^{s=S}\log_2(2S+1)\cdot\delta(x-s) & -S\le x=s\le S\\ \sum\limits_{s<-S,\,s>S}\infty\cdot\delta(x-s) & x=s<-S \text{ and } x=s>S\\ \infty & x\ne s\end{cases} \quad (22)$$

Similarly, if the pdf function is expressed in terms of Kronecker delta functions, the information function is

$$I(X)=-\log_2 f_d(x)$$
$$=\begin{cases}\sum\limits_{s=-S}^{s=S}-\log_2\frac{1}{2S+1}\cdot\delta(x-s) & -S\le x=s\le S\\ \sum\limits_{s<-S,\,s>S}-\log_2 0\cdot\delta(x-s) & x=s<-S \text{ and } x=s>S\\ \text{undefined} & x\ne s\end{cases}$$
$$=\begin{cases}\sum\limits_{s=-S}^{s=S}\log_2(2S+1)\cdot\delta(x-s) & -S\le x=s\le S\\ \sum\limits_{s<-S,\,s>S}\infty\cdot\delta(x-s) & x=s<-S \text{ and } x=s>S\\ \text{undefined} & x\ne s\end{cases} \quad (23)$$

due to the definition of the Kronecker delta function. For example, if S = 2 we may have

$$I(X)=\begin{cases}\sum\limits_{s=-2}^{s=2}2.322\cdot\delta(x-s) & -2\le x=s\le 2\\ \sum\limits_{s<-2,\,s>2}\infty\cdot\delta(x-s) & x=s<-2 \text{ and } x=s>2\\ \text{undefined} & x\ne s\end{cases} \quad (24)$$

The graphs of both information functions, for Dirac and Kronecker delta functions, are presented in Fig. 12a) and 12b), respectively.

The pdf function and information function are precisely calculated and presented in Fig. 17a) and 17b), for S = 0, 1, 2, and 4 defining the interval of pdf function values that are different from zero. The values of the defined information functions are presented for all values of the independent variable x from $-\infty$ to $+\infty$. This presentation is important to be understood because we will investigate the behavior of these functions when the variance, or the interval S of these functions, tends to infinity and reaches the infinite value.

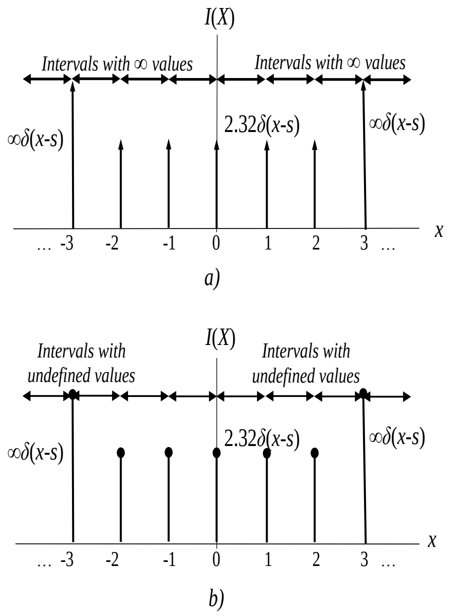

Figure 12 Information functions represented by a) Dirac and b) Kronecker delta functions for S = 2.

Parameter σ_d or S tends to infinity. We will investigate the information contents of the random variable X for two limit cases when σ_d or S tends to infinity or zero. For the first case, if Dirac delta functions are used, the information can be calculated using properties of the impulse function as

$$I_\infty(X) = \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} I(X) = \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} \log_2 \frac{1}{f_X(x)}$$
$$= \begin{cases} \lim\limits_{\substack{S\to\infty \\ \sigma_d\to\infty}} \sum\limits_{s=-S}^{s=S} \log_2(2S+1)\cdot\delta(x-s) & -S \le x = s \le S \\ \lim\limits_{\substack{S\to\infty \\ \sigma_d\to\infty}} \sum\limits_{s<-S,\, s>S} \infty\cdot\delta(x-s) & x = s < -S \text{ and } x = s > S \\ \infty & x \ne s \end{cases}$$
$$= \begin{cases} \sum\limits_{s=-S}^{s=S} \infty\cdot\delta(x-s) & -S \le x = s \le S \\ \sum\limits_{s<-S,\, s>S} \infty\cdot\delta(x-s) & x = s < -S \text{ and } x = s > S \\ \infty & x \ne s \end{cases} = \infty \quad (25)$$

for all x values $-\infty \le x \le \infty$, or in a simplified form as

$$I_\infty(X) = \begin{cases} \sum\limits_{s=-\infty}^{\infty} \infty\cdot\delta(x-s) & x = s \\ \infty & x \ne s \end{cases} = \infty \ . \quad (26)$$

for all x values $-\infty \le x \le \infty$. We can understand this function in the following way. When the interval S of random variable values tends to infinity, all values of the random variable X will potentially exist and appear with an infinitesimally small probability. Thus, having these small probabilities of appearance, the information content of all of them will tend to infinity. This limit information function is shown in Fig. 13a).

If Kronecker delta functions are used, the information function is

$$I_\infty(X) = \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} I(X) = \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} \log_2 \frac{1}{f_X(x)}$$
$$= \begin{cases} \lim\limits_{\substack{S\to\infty \\ \sigma_d\to\infty}} \sum\limits_{s=-S}^{s=S} \log_2(2S+1)\cdot\delta(x-s) & -S \le x = s \le S \\ \lim\limits_{\substack{S\to\infty \\ \sigma_d\to\infty}} \sum\limits_{s<-S,\, s>S} \infty\cdot\delta(x-s) & x = s < -S \text{ and } x = s > S \\ \text{undefined} & x \ne s \end{cases}$$
$$= \begin{cases} \sum\limits_{s=-S}^{s=S} \infty\cdot\delta(x-s) & -S \le x = s \le S \\ \sum\limits_{s<-S,\, s>S} \infty\cdot\delta(x-s) & x = s < -S \text{ and } x = s > S \\ \text{undefined} & x \ne s \end{cases} \quad (27)$$

or in simplified forms as

$$I_\infty(X) = \begin{cases} \sum\limits_{s=-\infty}^{\infty} \infty\cdot\delta(x-s) & x = s \\ \text{undefined} & x \ne s \end{cases} . \quad (28)$$
$$= \begin{cases} \infty & x = s \\ \text{undefined} & x \ne s \end{cases}$$

This function is presented in Fig. 13b). We can understand this function in the following way. When the interval S of random variable values tends to infinity, all values of the random variable at discrete instants $x = s$ will potentially exist and appear with the infinitesimally small probability nearly equal to zero. Thus, having these small probabilities of appearance, the information content of all of them will be close to infinity. Between any two neighboring discrete instants, $x = s$, the information function does not exist, i.e., it is not defined, as the pdf function values in these intervals are not defined as shown in Fig. 10b).

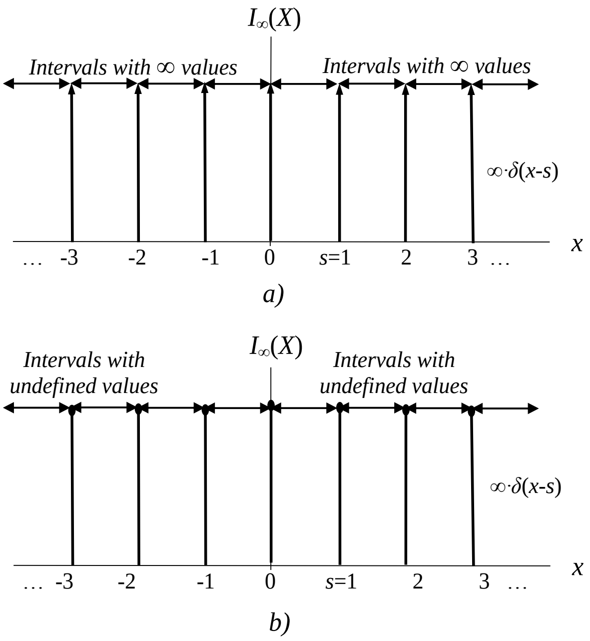

Figure 13 Discrete uniform information function represented by a) Dirac and b) Kronecker delta functions when the variance tends to infinity.

Parameter σ_d or S tends to zero. The information function, for Dirac delta functions, can be expressed as

$$I_0(X) = \lim_{\substack{S\to 0\\ \sigma_d\to 0}} I(X) \tag{29}$$

$$= \begin{cases} \sum_{s=-S}^{s=S} \lim\limits_{\substack{S\to 0\\ \sigma_d\to 0}} \log_2(2S+1)\cdot\delta(x-s) & -S \le x = s \le S \\ \sum_{s<-S,\, s>S} \infty\cdot\delta(x-s) & x = s < -S \text{ and } x = s > S \\ \infty & x \ne s \end{cases}$$

and calculated in this form

$$I_0(X) = \begin{cases} 0\cdot\delta(x-s) & x = s = 0 \\ \sum_{s<0,\, s>0} \infty\cdot\delta(x-s) & x = s \ne 0 \\ \infty & x \ne s \end{cases} \tag{30}$$

or, in its simplified form as

$$I_0(X) = \lim_{\substack{S\to 0\\ \sigma_d\to 0}} I(X)$$

$$= \begin{cases} \lim\limits_{\substack{S\to 0\\ \sigma_d\to 0}} \log_2(2S+1) & -S \le x \le S \\ -\log_2 0 & \text{otherwise} \end{cases}. \tag{31}$$

$$= \begin{cases} 0 & x = 0 \\ \infty & \text{otherwise} \end{cases}$$

The information function is presented in Fig. 14a). In this case, the minimum information content is again 0 for $x = 0$, as shown in Fig. 14a). For this discrete case, we are certain that any realization x of the random variable X will be zero and there is no uncertainty (information) about the value of this realization, i.e., the information takes the minimum value which is 0. The other discrete events of X different from zero, and defined at the instants $x = s$, are occurring with the probability of 0. We are certain these events will not happen, thus their information content is infinite, as shown in Fig. 14a) by arrow lines pointing to the infinity. The infinite values for the information are symbolically presented by Dirac delta functions defined at instants $x = s$. The probability of random values $x \ne s$ is zero and their information content is infinite, which is presented by the left-right arrows at the top in Fig. 14a. In this case, a realization of a stochastic process $X(n)$, defined by X at any discrete time-instant n, is a discrete process having amplitudes of zero values, as shown in Fig. 15. Zero sample values (outcomes) occur for sure, they are certain events at all time instants n. Between the time instants n the random signal $x_2(n)$ preserves zero values because of the zero probability of events that occur between discrete realizations defined by any $x = s$, as shown in Fig. 11a).

If the σ_d or S tends to zero, the information function, for Kronecker delta functions, is

$$I_0(X) = \lim_{\substack{S\to 0\\ \sigma_d\to 0}} I(X)$$

$$= \begin{cases} \lim\limits_{\substack{S\to 0\\ \sigma_d\to 0}} \sum_{s=-S}^{s=S} \log_2(2S+1)\cdot\delta(x-s) & -S \le x = s \le S \\ \lim\limits_{\substack{S\to 0\\ \sigma_d\to 0}} \sum_{s<-S,\, s>S} \infty\cdot\delta(x-s) & x = s < -S \text{ and } x = s > S \\ \text{undefined} & x \ne s \end{cases}$$

$$= \begin{Bmatrix} 0\,\delta(x-s) & x=s=0 \\ \sum_{s<0,\, s>0} \infty \cdot \delta(x-s) & x=s\neq 0 \\ \text{undefined} & x\neq s \end{Bmatrix}. \qquad (32)$$

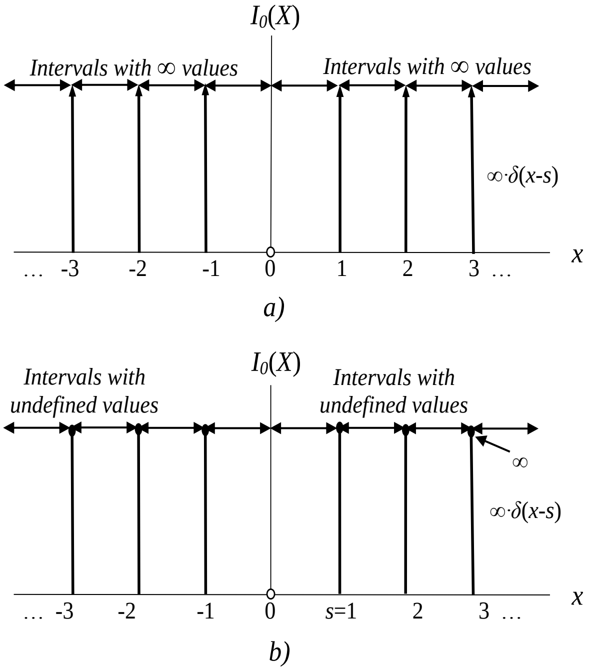

Figure 14 Discrete uniform information functions represented by a) Dirac and b) Kronecker delta functions when the variance tends to zero.

In this case, the minimum information content is 0 at the origin, as shown in Fig. 14b), unlike for the continuous random variable when the minimum information content is –∞ and represented by an inverted Dirac delta function, as shown in Fig. 6b). For this case, we are certain that any realization of the random variable X will be zero and there is no uncertainty (information) about the value of this realization, i.e., the information takes the minimum value which is 0. The information content remains of the +∞ value everywhere else on the x-axis where the discrete pdf function of X has zero value, i.e., for $x = s$. Let us explain the behavior of the information function for these two presentations. A realization of the related stochastic process $X(n)$, defined by random variable X at any time-instant n, is a discrete-time process having amplitude zero at points in time n and is not defined between these points, as shown in Fig. 16 for a hypothetical random signal $x(n)$. Zero random values (outcomes) occur for sure; they are certain events at all time instants n and carry no information. Because a zero outcome occurs with the probability of one for every n, the information content is zero, as shown in Fig. 14b). The other events of X at the instants $x = s \neq 0$ are occurring with the probability of 0, therefore their information content is infinite, as shown in Fig. 14b). For that reason, the random signal in Fig. 16 does not have any amplitudes greater than zero. The dependence of the pdf and information function on the interval S is presented in Fig. 17.

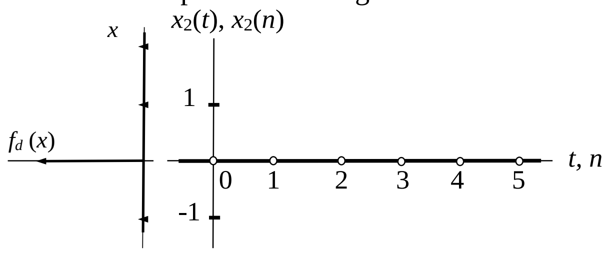

Figure 15 Realizations of a continuous-time and discrete-valued stochastic process defined by random variable X having the uniform pdf function expressed by the Dirac delta function for S = 0.

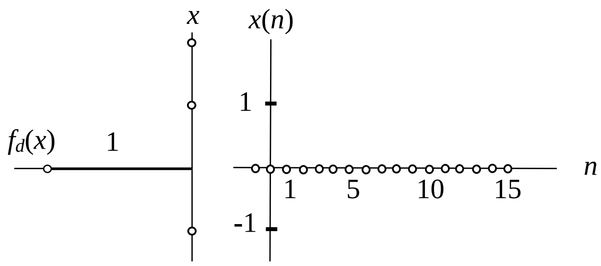

Figure 16 A realization of discrete-time and discrete-valued stochastic processes defined by random variable X having the uniform pdf function expressed by a Kronecker delta function at S = 0.

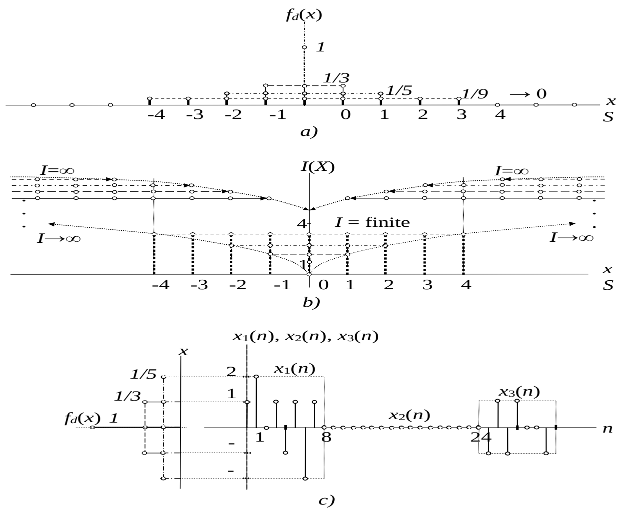

Figure 17 a) Discrete uniform pdf function presented by Kronecker delta functions, b) related information and entropy functions, and c) an exemplary realization of the related stochastic process.

The pdf function is expressed in terms of Kronecker functions in Fig. 17a) for the mean value equal to zero and varying values of discrete interval S = 0, 1, 2, and 4. The related dependence of the information function on the interval S is presented in Fig. 17b). The information function values (information contents) of random values x are increasing when the values of the pdf function are decreasing. For the zero-values of the pdf function, the information function is of the infinite value, which is symbolically presented by non-overlapping dashed and dashed-dot straight arrow lines in Fig. 17b). When the interval S of the pdf function tends to infinity, the discrete information function values tend to infinity. When S reaches infinity, all the information values are equal to infinity corresponding to the pdf function with all zero values.

If we form a discrete-time stochastic process defined by the realizations of random variable X at each discrete-time instant n, we may represent the related realizations of three processes for pdf functions of X defined by S = 2, 0, and 1, as shown in Fig. 17c). These three random signals are presented on the same graph for the sake of explanation. The first realization $x_1(n)$ takes the whole values between 2 and -2 in the time interval 0 to 7. The second realization $x_2(n)$ is represented by zero amplitudes at each discrete-time instant inside the interval from 8 to 23. The third realization $x_3(n)$ takes the integer values from -1 to 1 in the interval from 24 to 31. The presented processes are discrete-time and discrete-valued processes. The presented analysis will be valid for the case when we use Dirac delta functions to represent the pdf function, which will produce a continuous-time discrete-valued process. Precise exemplary calculations for the discrete pdf function and related information function are presented in Fig. 18. When the interval S increases from 0 to 20, the information values are increasing from 0 to the value above 5, as shown by cycles in Fig. 18. For the same S values, the pdf function values are increasing from 1/4 to one, as shown by squares in Fig. 18.

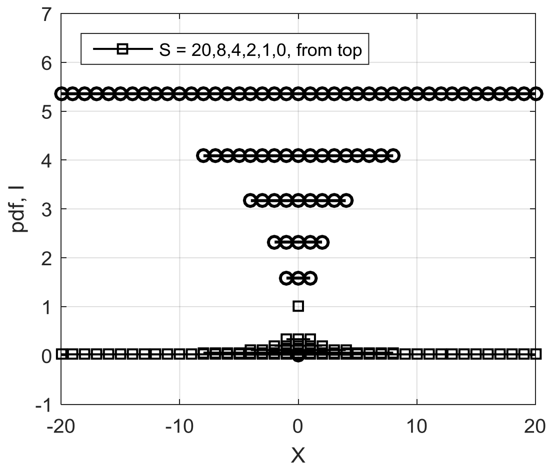

Figure 18 Discrete information functions (circles), and uniform pdf functions (squares) for the interval S as a parameter.

3.3 Entropy

Dirac delta function. By following Shannon's theory [1], the entropy can be calculated as the mean value of the information function using the integral transform. If the pdf function is expressed in terms of Dirac delta functions, as in Berber's paper [13], the entropy function is expressed as follows

$$\begin{aligned} H(X) &= -\int_{-\infty}^{\infty} f_d(x)\log_2 f_d(x)dx \\ &= -\int_{x=-\infty}^{\infty}\sum_{s=-S}^{s=S}\left[\frac{1}{2S+1}\cdot\delta(x-s)\right]\left[\log_2\frac{1}{2S+1}\sum_{s=-S}^{s=S}\delta(x-s)\right]dx+0 \\ &= -\frac{1}{2S+1}\sum_{s=-S}^{s=S}\int_{x=-\infty}^{\infty}\delta(x-s)\cdot\log_2\frac{1}{2S+1}\sum_{s=-S}^{s=S}\delta(x-s)dx \\ &= -\frac{1}{2S+1}\sum_{s=-S}^{s=S}\int_{x=-\infty}^{\infty}\delta(x-s)\cdot\log_2\frac{1}{2S+1}\left[\delta(x+S)+...+\delta(x-S)\right]dx \end{aligned}.$$

The solution of the integral is

$$\begin{aligned} H(X) &\underset{x=s}{=} -\frac{1}{2S+1}\sum_{s=-S}^{s=S}\log_2\frac{1}{2S+1}\left[\delta(s+S)+...+\delta(s-S)\right] \\ &= -\frac{1}{2S+1}\left[\log_2\frac{1}{2S+1}\left[\delta(-S+S)+...+\delta(-S-S)\right]\right]_{s=-S} - \\ &\quad ...-\frac{1}{2S+1}\left[\log_2\frac{1}{2S+1}\left[\delta(S+S)+...+\delta(S-S)\right]\right]_{s=S} \end{aligned}$$

$$= -\frac{1}{2S+1}\left[\log_2 \frac{1}{2S+1}\left[1+0+...+0\right]\right]_{s=-S} - ... - \frac{1}{2S+1}\left[\log_2 \frac{1}{2S+1}\left[0+0+...+1\right]\right]_{s=S} = -\frac{1}{2S+1}\left[\log_2 \frac{1}{2S+1} + ... + \log_2 \frac{1}{2S+1}\right] = -\frac{1}{2S+1}\cdot \sum_{s=-S}^{s=S} \log_2 \frac{1}{2S+1} = \log_2(2S+1) \quad (33)$$

Kronecker delta function. The same result can be obtained if we express the pdf function in terms of Kronecker delta functions. In this case, we will express the entropy as a sum

$$H(X) = E\{I(X)\} = -\sum_{x=-\infty}^{\infty} f_d(x)\cdot \log_2 f_d(x) = -\sum_{x=-S}^{S} f_d(x)\cdot \log_2 f_d(x) + 0 = -\sum_{x=-S}^{S}\left[\frac{1}{2S+1}\sum_{s=-S}^{s=S}\delta(x-n)\right]\left[\log_2 \frac{1}{2S+1}\sum_{s=-S}^{s=S}\delta(x-n)\right] = -\frac{1}{2S+1}\sum_{x=-S}^{S}\left[\delta(x+S)+...+\delta(x-S)\right]\left[\log_2 \frac{1}{2S+1}\sum_{s=-S}^{s=S}\delta(x-n)\right].$$

Using the sifting property of the delta function, we may solve the sum for x as follows

$$H(X) = -\frac{1}{2S+1}\sum_{x=-S}^{S}\left[\delta(x+S)\right]\left[\log_2 \frac{1}{2S+1}\sum_{s=-S}^{s=S}\delta(x-n)\right], -... - \frac{1}{2S+1}\sum_{x=-S}^{S}\left[\delta(x-S)\right]\left[\log_2 \frac{1}{2S+1}\sum_{s=-S}^{s=S}\delta(x-n)\right]$$

and then each sum is calculated as follows

$$H(X) = -\frac{1}{2S+1}\left[\log_2 \frac{1}{2S+1}\sum_{s=-S}^{s=S}\delta(-S-s)\right] - ... - \frac{1}{2S+1}\left[\log_2 \frac{1}{2S+1}\sum_{s=-S}^{s=S}\delta(S-s)\right] = -\frac{1}{2S+1}\left[\log_2 \frac{1}{2S+1}\cdot\delta(0)\right] - ... - \frac{1}{2S+1}\left[\log_2 \frac{1}{2S+1}\cdot\delta(0)\right] = -\frac{1}{2S+1}\left[\sum_{s=-S}^{s=S}\log_2 \frac{1}{2S+1}\cdot\delta(0)\right] = -\frac{1}{2S+1}\left[\sum_{s=-S}^{s=S}\log_2 \frac{1}{2S+1}\cdot 1\right] = \log_2(2S+1) \quad (34)$$

The entropy values are numbers that are always zero or positive for the discrete case, and represent the average value of information (uncertainty) per random value x of X. Due to our intention to investigate the entropy as a function of the variance, $\sigma_d^2 = S(S+1)/3$ or as a function of the width of interval S, we denote the entropy as $H(X)$. The positive values of the entropy are increasing inside the interval S when the interval width is increasing as shown in Fig. 19 for the intervals defined with S = 0, 1, 2, and 4 with the corresponding values of the entropy that are presented in italic font.

Parameter σ_d or S tends to infinity. ***Kronecker delta function.*** If the interval S tends to infinity someone can calculate mistakenly the entropy in infinity using expression (34) as

$$H_\infty(X) = \lim_{\substack{S\to\infty \\ \sigma d\to\infty}} \log_2(2S+1) = \infty \quad (35)$$

However, in this case, the influence of the zero values of the pdf function in infinity is not taken into account. When the interval S reaches infinity, the entropy should be calculated using its definition, which will include the zero-valued probabilities in the pdf function, i.e.,

$$H_\infty(X) = \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}}\left(-\sum_{s=-\infty}^{s=\infty} f_d(x)\log_2 f_d(x)\right) = \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}}\left(-\sum_{s=-\infty}^{s=\infty} 0\log_2 0\right) = 0. \quad (36)$$

In this case, all random variable values x are in the infinite interval stretching from $-\infty$ to $+\infty$ and occur with the probability of zero. To consider these probability values we can confirm (36) by calculating the entropy using a mid-term derivative of entropy (34) as follows. The limit is

$$H_\infty(X) = \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} \sum_{x=-\infty}^{x=\infty} f_d(x)\log_2 \frac{1}{f_d(x)} = \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} \sum_{x=-S}^{x=S} \frac{1}{2S+1}\cdot\log(2S+1)$$

Then, following the rule that the limit of the sum is equal to the sum of the limits, we are getting an indeterminate case $\infty\cdot 0$ that requires us to apply the L'Hopital's rule to get

$$H_\infty(X) = \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} \sum_{x=-S}^{x=S} \frac{1}{2S+1} \cdot \log(2S+1)$$
$$= \sum_{x=-S}^{x=S} \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} \frac{1}{2S+1} \cdot \log(2S+1) \underset{L'H}{=} \sum_{x=-S}^{x=S} \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} \frac{2/(2S+1)\log_e 2}{2}$$
$$= \sum_{n=-S}^{n=S} \lim_{\substack{S\to\infty \\ \sigma_d\to\infty}} \frac{1}{(2S+1)\log_e 2} = 0 \qquad (37)$$

Therefore, considering the values of the information and corresponding probability values, the entropy, as the measure of the average information contents inside the random values *x*, is zero. These entropy values are presented in Fig. 19 by cycles connected by a full curve that reaches infinite entropy, which then drops back to zero entropy as symbolically presented by a dashed curve alongside the full curve.

It is important to note the following two properties of entropy, its infinite value when *S* tends to infinity, and zero value when *S* reaches infinity. While the discrete interval *S* tends to infinity the calculated entropy value in the interval *S* tends to infinity and is zero beyond the *S* interval. In infinity, the intervals of random variable *x* with zero values of entropy disappear, and in the entire infinite interval, the entropy becomes zero because the *S* value reaches infinity. The appearance of any value *x* of the random variable *X* is happening with the probability of zero and its information content is infinite, as will be seen from the following analysis.

Dirac delta function. If the pdf function is expressed in terms of Dirac delta functions, then the pdf function in infinity will have all zero values and the entropy value is zero calculated as follows

$$H_\infty(X) = -\lim_{\sigma_d\to\infty} \int_{-\infty}^{\infty} f_d(x)\log_2 f_d(x)dx$$
$$= \left(-\int_{-\infty}^{\infty} 0\log_2 0dx\right) = 0 \qquad (38)$$

Therefore, considering the values of the information and corresponding probability values, the entropy, as the measure of the average information content inside the random values *x*, is zero when the interval *S* reaches infinity.

Parameter σ_d or *S* tends to zero. *Kronecker delta function*. For the Kronecker delta function presentation and the case when parameter σ_d or *S* tends to zero, the entropy value can be calculated as follows

$$H_0(X) = \lim_{\substack{S\to 0 \\ \sigma_d\to 0}} \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \log(2S+1) \qquad (39)$$
$$= -\lim_{S\to 0} \log_2(2S+1) = 0$$

The dependence of the pdf function values, information, and entropy on the size of interval *S* is presented in Fig. 19.

Dirac delta function. If the information is expressed in terms of Dirac delta functions, we may have

$$H_0(X) = -\lim_{S\to 0} H(X) = -\lim_{S\to 0} \log_2(2S+1) = 0 \qquad (40)$$

The calculated entropy values are increasing towards infinity when *S* tends to infinity. When *S* reaches infinity, the entropy drops down from the infinite value to zero, as shown in eq. (36), which is presented by a full curve with a loop at the end (in infinity) and a dashed arrow curve showing symbolically the return of the entropy to zero.

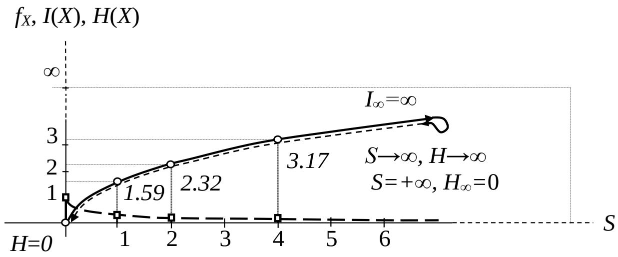

Figure 19 Discrete uniform pdf function and related information and entropy as functions of the size of interval *S*.

4 Conclusions

The presented theory has shown that the entropy of the uniform stochastic process, having a time-dependent variance, increases in time, reaches infinity, and then drops to zero showing the singularity property in infinity. In contrast to the entropy function of the process, all information function values tend to infinity when the variance tends to infinite and attains infinite values for the infinite variance. The paper presented precise definitions and related derivatives of the information and entropy functions both for the continuous and discrete uniform random variables assuming that the variance can have any value between zero and infinity. In particular, a uniform density function with the variance that linearly depends on time is defined and derived, and related non-stationary processes are formed. This function takes all zero values in infinity causing singularity behavior of the entropy function. Due to the possible existence of the continuous and discrete processes in a stochastic system, the time-dependent continuous and discrete probability density functions and related information and entropy functions are derived.

References:

[1]. Shannon, C. E. A Mathematical Theory of Communication. The Bell System Technical Journal, Vol. 27, pp. 379–423, 623–656, July, October, (1948).
[2]. Glattfelder, J. B. *Information–Consciousness–Reality*, (Springer Nature Switzerland AG, 2019).
[3]. Nicholson, B. S. at all. Time-information uncertainty relations in thermodynamics, Nature Physics, VOL 16, Dec. 2020, p. 1211-1215.
[4]. Bell, J. L. *Continuity and infinitesimals*, In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy, Summer 2017 Edition.
[5]. Montgomery, D.C., Runger, G. C. *Applied Statistics and Probability for Engineers*, (John Wiley and Sons, New York, Fourth Edition, 1994).
[6]. Papoulis, A., Pillai, S. U. *Probability, Random Variables and Stochastic Processes*, (McGraw-Hill, New York, Fourth Edition, 2002).
[7]. Peebles. P. Z. *Probability, Random Variables, and Random Signal Principles*, (McGraw-Hill, New York, Fourth Edition, 2001).
[8]. Manolakis, D. G., Ingle, V. K., Kogan, S. M. *Statistical and Adaptive Signal Processing*, (Artech House. INC., 2005).
[9]. Gray, M. R. Davisson, L. D. *An Introduction to Statistical Signal Processing*, (Cambridge University Press, 2004).
[10]. Proakis, G. J. *Digital Communications*, (McGraw-Hill, New York, Fourth Edition, 2001).
[11]. Berber, S. *Discrete Communication Systems*, (Oxford University Press, 2021).
[12]. Haykin, S. *Communication Systems*, (John Wiley & Sons Ltd., Fourth Edition, 2001).
[13]. Berber, S. The Exponential, Gaussian and Uniform Truncated Discrete Density Functions for Discrete-Time Systems Analysis, WSEAS Transactions on Mathematics, ISSN / E-ISSN: 1109-2769 / 2224-2880, Volume 16, Art. #26, 2017, pp. 226-238.