Generating Cyclic Permutations: Insights to the Traveling Salesman Problem

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Abstract: Some results for the traveling salesman problem (TSP) are known for a prime number of cities. In this paper we extend these results to an odd number of cities. For an odd integer n, we show that there is an algorithm that generates $n - 1$ cyclic permutations, called tours for the traveling salesman problem, which cover the distance matrix. The algorithm allows construction of a two-dimensional array of all tours for the TSP on an odd number of cities. The array has the following properties: (i) A tour on a vertical line in the array moves the salesman uniquely compared to all other tours on the vertical line. (ii) The sum of the lengths of all tours on a vertical line is equal to the sum of all non-diagonal elements in the distance matrix for the TSP.

Key Words: cyclic permutation, entangled set of permutations, traveling salesman problem, tour for the salesman, distance matrix, odd number of cities

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1 Introduction

The ideas in this paper extend the results in [1] where it is shown for a prime number of cities n, that the distance matrix for a traveling salesman problem (TSP) is covered by P, P^2, …, P^{n-1} where P is a cyclic permutation of the n cities. Our main result extends this to odd integers. In other words, in [1] each entry in the distance matrix is used exactly once to evaluate P, P^2, …, P^{n-1} when n is a prime number. We generalize this to odd integers, which leads to geometric and analytic properties about the covering of the distance matrix. It is open question to determine if there is an extension to even integers. All integers are assumed to be positive.

This paper is more than nice, simple remarks. It was motivated by the hope of finding structure for the TSP that would yield insight to the P = NP question. This insight is an open area.

A permutation is a one-to-one mapping of a finite set into itself. A cycle is a circular rearrangement of the permuted elements. A cyclic permutation has only one cycle which is called a tour for the TSP. We arrange permutations so that object 1 is in position 1. For example, (1324) is a cyclic permutation on four objects that maps 1->3, 3->2, 2->4 and 4->1. If P is a permutation and k is a positive integer, then P^k signifies P being applied k times and is called the kth power of P. The square of the previous example is 1->2, 2->1, 3->4 and 4->3 and is (12)(34) in cycle notation. It has two cycles and is not a cyclic permutation.

Given n cities and the distance between each pair of cities, the TSP asks for a shortest route that visits each city once and returns to the starting city. If d_{ij} is the distance from city i to city j, the TSP asks for a cyclic permutation μ of {1, 2, …, n} such that $\sum_{i=1}^{n} d_{i\mu(i)}$ is a minimum over all cyclic permutations of {1, 2, …, n}. The distance matrix for a TSP on n cities is the n x n matrix with entries d_{ij}. The cities are designated 1, 2, …, n. A tour for a TSP on n cities is a cyclic permutation of {1, 2, …, n}. A covering of an n x n distance matrix is a set of tours such that if i and j are in {1, 2, …, n} and $i \neq j$, then there is a tour μ in the set such that $\mu(i) = j$. If P = (1 p_1 p_2 … p_{n-1}) and T = (1 t_1 t_2 … t_{n-1}) are tours, we say that P moves the salesman uniquely compared to T if $p_j \neq t_j$ for each j in $\{2, \ldots, n\}$.

Reference books [2–7] and papers [8-10] testify to the importance and many applications of the TSP.

Distance matrices characterize the TSP. Some properties of distance matrices are in [2, Chapter 4]. The work in [11, Section 2] has a classification of distance matrices. In [12, Section 4] a canonical form is given for distance matrices and it is shown how to transform a given distance matrix to its canonical form in polynomial time. Reference [13] contains basic work about permutations, some of which applies to the TSP. Reference [14] is an excellent paper about generating permutations that is distinct from our results.

Section 2 of this paper contains results about entwined sets of permutations. In Section 3 we show that the set of powers of a cyclic permutation has five properties when the number of elements in the permutation is a prime. In Section 4 the properties are used to find an algorithm that constructs $n - 1$ cyclic permutations that cover the distance matrix when n is odd and n is the number of cities. Geometric and analytic applications to the TSP are developed in Section 5 for an odd number of cities. Section 6 concludes with an open question about an algorithm for an even number of cities and several results for the even case.

2 Entwined Set of Permutations

Entwined sets of permutations, all on the same set of objects, will be useful to describe our work. We will define this concept and prove some features about it.

Let Ψ be a set of permutations on the same objects. We define Ψ to be entwined if there is an object j and there is a pair of permutations A and B in Ψ such that $A(j) = B(j)$. Otherwise, we say that Ψ is non-entwined.

Theorem 2.1. A set of permutations on $\{1, 2, \ldots, n\}$ is non-entwined if and only if each of its subsets of order 2 is non-entwined.

Proof. The theorem is easily verified in both directions by the definition of entwined set.

Theorem 2.2. Let T and U be permutations on n objects. If $\{T, U\}$ is a non-entwined set, then $\{T^{-1}, U^{-1}\}$ is non-entwined.

Proof. Suppose $\{T^{-1}, U^{-1}\}$ is entwined. Then for some x in $\{1, 2, \ldots n\}$, it follows that $T^{-1}(x) = U^{-1}(x)$. Then $T(T^{-1}(x)) = x = U(U^{-1}(x))$, which means that $\{T, U\}$ is entwined.

Corollary 2.1. Let £ be a set of permutations on n objects. If £ is a non-entwined set, then the set of inverses of permutations in £ is non-entwined.

Theorem 2.3. Let T be a cyclic permutation on $\{1, 2, \ldots, n\}$. The set $\{T, T^{-1}\}$ is non-entwined if and only if $n > 2$.

Proof. If $n > 2$ and x is in $\{1, 2, \ldots, n\}$, then there are unique w and y in $\{1, 2, \ldots, n\}$ such that $T = (1\ldots wxy\ldots)$. Thus $T(x) = y \neq w = T^{-1}(x)$. On the other hand, if $n \leq 2$, then $T = T^{-1}$.

Theorem 2.4. Let T and U be cyclic permutations on n objects for $n > 2$. If $\{T, T^{-1}, U\}$ is a non-entwined set, then $\{T, T^{-1}, U^{-1}\}$ is a non-entwined set.

Proof. We will show that $\{T, U^{-1}\}$ and $\{T^{-1}, U^{-1}\}$ are non-entwined. Since $\{T, T^{-1}, U\}$ is non-entwined, $\{T, U\}$ is non-entwined. By Theorem 2.2 $\{T^{-1}, U^{-1}\}$ is non-entwined. On the other hand, since $\{T, T^{-1}, U\}$ is non-entwined, $\{T^{-1}, U\}$ is non-entwined. By Theorem 2.2 $\{T, U^{-1}\}$ is non-entwined.

Theorem 2.5. Let T and U be cyclic permutations on n objects for $n > 2$. If $\{T, T^{-1}, U\}$ is a non-entwined set, then $\{T, T^{-1}, U, U^{-1}\}$ is a non-entwined set.

Proof. By Theorem 2.3 $\{T, T^{-1}\}$ and $\{U, U^{-1}\}$ are non-entwined. By Theorem 2.4, $\{T, T^{-1}, U^{-1}\}$ is non-entwined, which implies that $\{T, U^{-1}\}$ and $\{T^{-1}, U^{-1}\}$ are non-entwined. By assumption $\{T, T^{-1}, U\}$ is non-entwined, which means that $\{T, U\}$ and $\{T^{-1}, U\}$ are non-entwined. We have checked that all two-element subsets of $\{T, T^{-1}, U, U^{-1}\}$ are non-entwined, therefore $\{T, T^{-1}, U, U^{-1}\}$ is a non-entwined set.

Theorem 2.6. Let T_j for $j = 1, 2, \ldots, k$ and U be cyclic permutations on n objects for $n > 2$. If $\{T_j, T_j^{-1}, U : j = 1, 2, \ldots, k\}$ is a non-entwined set, then $\{T_j\ T_j^{-1}, U, U^{-1} : j = 1, 2, \ldots, k\}$ is a non-entwined set.

Proof. By induction on k. By Theorem 2.5 the result is true for k = 1. We assume that the result is true for $k - 1$. For each j we have $\{T_j^{-1}, U^{-1}\}$ is non-entwined. Since $\{T_j\ T_j^{-1}\}$ is non-entwined for each j by Theorem 2.3 and likewise $\{U, U^{-1}\}$, it follows that all the pieces are in place for the result. The pieces are the non-entwined subsets $\{T_j\ T_j^{-1}\}$, $\{T_j\ U\}$, $\{T_j\ U^{-1}\}$, $\{T_j^{-1}\ U\}$, $\{T_j^{-1}\ U^{-1}\}$, $\{U, U^{-1}\}$.

3 Properties of Sets of Cyclic Permutations for a Prime

A cyclic permutation is characterized by having exactly one cycle. It is well-known that if P is a cyclic permutation on n objects, then P^2, P^3, …, P^{n-1} are cyclic permutations if and only if n is a prime number. By relabeling, we may assume that P is the shift permutation S = (123…n).

A cycle in a permutation is a closed loop. We will use cycle notation to specify a permutation, e.g., the permutation (123)(45) maps 1->2, 2->3, 3->1, 4->5 and 5->4. It has two cycles.

Let n be a prime number. Let S = (123…n) be the shift permutation. We define $\Gamma = \{S, S^2, S^3, \ldots, S^{n-1}\}$. Next we will list five properties of Γ. After sketching proofs of the properties, we will discuss uses of these properties.

1. Each member of Γ is a cyclic permutation.
2. Γ is a non-entwined set of permutations.
3. Γ is a maximal, non-entwined set of permutations, in the sense that if P is a permutation on n objects, is not in Γ and is not the identity, then Γ ∪ {P} is entwined.
4. The inverse of each permutation in Γ is in Γ.
5. Let k be in $\{1, 2, \ldots, n - 1\}$. If we write $S^k = (1\ t_1\ t_2 \ldots t_{n-1})$, then $t_j + t_{n-j} = n + 2$ for each j in $\{1, 2, \ldots, n - 1\}$.

Proof of Property 1. n is a prime number.
Proof of Property 2. $S^j(k) = (k + j - 1)(\text{mod } n) + 1$.
Proof of Property 3. Since permutation P is not the identity mapping, there is k in $\{1, 2, \ldots, n\}$ such that $P(k) \neq k$. By Property 2 $\{S^j(k) : j = 1, 2, \ldots, n - 1\} = \{1, 2, \ldots, n\}\backslash\{k\}$, which contains P($k$). Thus, Γ ∪ {P} is entwined.
Proof of Property 4. $S^{-j} = S^{n-j}$ for j in $\{1, 2, \ldots, n - 1\}$.
Proof of Property 5. $t_j = (jk)(\text{mod } n) + 1$ and $t_{n-j} = (n - jk)(\text{mod } n) + 1$.

Property 1 assures that each member of Γ is a tour for the TSP. We will use Properties 2 and 3 in Section 5 to provide a geometric decomposition of the family of all tours for the TSP. The decomposition has useful analytical qualities. Properties 4 and 5 add structure that we will use to generalize Γ in Section 4. Property 5 requires that n be an odd integer.

4 Sets of Cyclic Permutations for Odd Integers

In this section we will describe an algorithm that constructs a set Θ of $n - 1$ permutations on n objects that satisfies Properties 1 – 5 in Section 3 when n is an odd integer. In addition, we want the shift permutation S = (123…n) to be in Θ. We require that $\Theta \neq \Gamma$ because Γ is not a set of cyclic permutations when n is not a prime.

Example 4.1. Consider {S, T, U} where S = (1234567), T = (1352746) and U = (1426375). Then $\{S, T, U, S^{-1}, T^{-1}, U^{-1}\}$ satisfies all the requirements for Θ in the first paragraph of Section 4. This is the only such set for $n = 7$. We note that $T = S^2$ with objects 2 and 7 exchanged and $U = S^3$ with objects 2 and 7 exchanged and objects 3 and 6 exchanged.

Example 4.2. Consider the following set of cyclic permutations on 9 objects: {S, X, Y, Z} where S = (123456789), X = (136947258), Y = (142683597) and Z = (157392846). We observe that T and U in Example 4.1 and X and Y in 4.2 have different patterns that require different algorithms to generate them. However, the result is similar, since $\{S, X, Y, Z, S^{-1}, X^{-1}, Y^{-1}, Z^{-1}\}$ satisfies all the requirements for Θ in the first paragraph of this Section.

Given odd integer n and shift permutation S, how do we generate $((n - 1)/2) - 1$ cyclic permutations on n such that the set of these permutations and their inverses satisfy Θ in the first paragraph?

We observe that if $n < 7$, there are no sets Θ that differ from Γ and satisfy Properties 1 – 5 of Section 3. First we observe that Property 5 implies that n is an odd integer. We work with $n = 5$ and renumber the objects so that S = (12345) is in Θ. Then we enumerate the six cyclic permutations T_j = (13abc). Lastly we consider the six possible sets $\Theta = \{S, S^{-1}, T_j \text{ and } T_j^{-1}\}$. None of the sets Θ is different from Γ and satisfies Properties 1 – 5. In conclusion, we may assume that n is an odd integer and $n \geq 7$.

Let S = (123…n) be the shift permutation on n objects. We would like cyclic permutations $T_1, T_2, \ldots, T_{(n-3)/2}$ based on $S^2, S^3, \ldots, S^{(n-1)/2}$ so that $\{S, T_1, T_2, \ldots, T_{(n-3)/2}, S^{-1}, T_1^{-1}, T_2^{-1}, \ldots, T_{(n-3)/2}^{-1}\}$ satisfies all the requirements for Θ in the first paragraph of this Section.

We found a two-step algorithm that will generate a set Θ for an odd integer n. Step 1 finds a non-entwined set composed of $(n - 1)/2$ cyclic permutations on n objects. Step 1 is awkward and unwieldy involving modifications to objects of permutations based on odd and even subscripts. Therefore, we do not present Step 1 here. Step 2 inserts the inverse of the permutations in Step 1. The result is a set containing cyclic permutations that is non-entwined by Theorem 2.6.

In summary, we have shown there are sets of cyclic permutations on n objects for $n \geq 7$ and n odd, such that if a and b are distinct objects then there is exactly one permutation in the set that maps a to b. A possible limitation is the difficulty to generate these sets.

5 Application to the Traveling Salesman Problem

We will form an $(n - 1) \times (n - 2)!$ array in the first quadrant of the plane that contains all tours for a TSP on n cities. The array will have the property that any vertical line on the array has exactly one tour T such that $T(j) = k$ for distinct j and k in the set of cities. We assume that n is odd and $n \geq 7$.

We place S = (123…n) at the origin of the array. On the positive y-axis we place the $n - 2$ members of $\Theta \backslash \{S\}$ one unit apart. Recall that Θ is described in the first paragraph of Section 4.

On the positive x-axis we place a tour that is not already in the array by interchanging two cities in the last entry on the x-axis. There are several ways to do this, some of which are described in [15, Chapter 7]. We fill in the tours above the x-axis by making the same exchange of two cities as was done on the x-axis.

Theorem 5.1. Let n be the number of cities in a TSP. We assume that n is odd and $n \geq 7$. Each vertical line in the array has the property that if j and k are distinct cities, then there is precisely one tour T on the line such that $T(j) = k$.

Proof. For $n = 7$ we demonstrated this result for Θ in Example 4.1 and extended it to larger n in Section 4. When two cities are permutated in a tour, the result continues to be valid.

Corollary 5.1. Let d_{jk} where $j \neq k$ be a distance entry in the distance matrix. Then there is precisely one tour on each vertical line which uses d_{jk}.

Corollary 5.2. The sum of all the distance entries in the distance matrix equals the sum of all the distances for the tours on a vertical line. This is valid for each vertical line in the array.

Recall that the diagonal entries in the distance matrix have no role for the TSP.

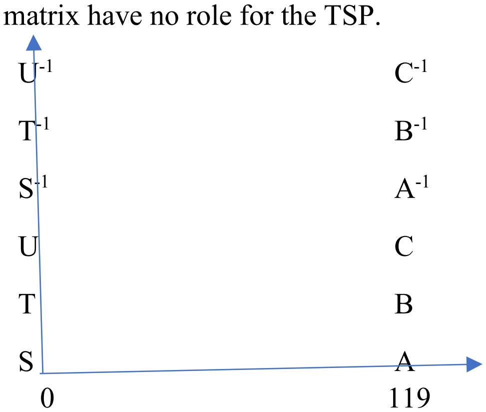

Figure 5.1 First and Last Vertical Line of Tours for 7 Cities

In Figure 5.1 S, T and U are cyclic permutations from Example 4.1. The shift permutation S = (123…7) is at the origin. The others from Example 4.1 are arranged one unit apart on the vertical axis. A new cyclic permutation for the horizontal axis can be obtained by interchanging two cities in the previous entry on the horizontal axis. We designate the last one of these as A.

Entries on the vertical line containing A are obtained by making the same exchange on the corresponding entry in the previous column.

More precisely, let Ψ be a mapping of {1, 2, 3, …, 7} onto itself. For a cyclic permutation Z = ($s_1 s_2 s_3 \ldots s_7$), we define Ψ(Z) to be the cyclic permutation ($\Psi(s_1)\Psi(s_2)\ldots\Psi(s_7)$). Returning to Figure 5.1, let Ψ be such a mapping on S. There are 119 = 5! – 1 such mappings on S for $n = 7$. We designate one of these mappings as Ψ(S) = A and place A on the horizontal axis as the base for a new vertical line. Then Ψ(T) = B and Ψ(U) = C.

6 Generalization to the Even Case

It is an open question whether there is algorithm that generates a non-entwined set of $n – 1$ cyclic permutations when n is an even integer. The smallest even integer for which there is a set of $n – 1$ cyclic permutations that is non-entwined is $n = 8$. These sets also exist for $n = 10$. We conjecture that they exist for all even integers greater than 7.

Software: No commercial software was used for the technical features of this paper.

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