Fuzzy Estimators of Drain spacing in Subsoil Drainage using Fuzzy Logic and Possibility Theories

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Abstract: - In the permanent flow of subsoil drainage, a lot of equations are used, most of them based on the Dupuit assumption. All related mathematical models present uncertainties and fuzziness, which create problems in the design of drainage networks. Fuzzy Logic deals with this problem and allows the management of uncertain information. This paper presents the solution of the Hooghout equation based on Fuzzy Logic and Possibility theories, using the Reduced Transformation Method for the related numerical calculations. This results in a fuzzy estimator for the drain spacing, whose α-cuts, provide, according to Possibility Theory, the confidence intervals of the drain spacing with a certain strong probability. Results on subsoil drainage in the case of soils with parallel drains located at any position from the impermeable bottom are presented. The possibility theory application enables the engineers and designers of irrigation, drainage, and water resources projects to gain knowledge of hydraulic properties (e.g., water level, outflow volume) and make the right decision for rational and productive engineering studies.

Key-Words: Drainage Networks; Hooghout equation; fuzzy logic; possibility theory; transformation method.

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1 Introduction

In underground drainage, drains are used to control the water level by draining the excess water. In practice, parallel drainage conduits are used which are either ditches or tubular drains. The mathematical description of the underground flow to the drains is achieved by using the following assumptions: a) Two-dimensional flow. This assumption is true for long drains. b) Uniform distribution of rainfall intensity, natural or artificial, c) Homogeneous and isotropic soils.

Many drainage equations are reduced to one-dimensional, accepting parallel and horizontal flow lines. This assumption is valid if the impermeable subsoil stratum is close to the drains. Hooghoudt [1, 2] first used Dupuit [3] assumptions in 1940 and extracted an analytical form also known as the Donnan equation [4]. He then considered the Dupuit assumptions for the horizontal flow section beyond a short distance from the drains and the radial curve assumption for the area near the drains to be valid, and derived a new equation based on equivalent depth. He also presented tables for the determination of the equivalent depth.

Many related solutions consider the case of stratified soil with two permeable layers, and the parallel drains to be located at any position from the interface of the two layers (Toksöz and Kirkham [5, 6], Ernst [7, 8], Wesseling [9], Terzidis and Karamouzis [10], Terzidis [11], Tzimopoulos [12], Kirkham [13]. All these researchers presented a series-type solution, based on two-dimensional flow and potential theory, by solving the Laplace equation. Ernst [7] provided an approximate method of solving the problem, which is an extension of Hooghoudt's [1, 2] method for drainage of homogeneous soils, which is mainly suitable for stratified soils with certain limitations. Dagan [14] considered a radial flow near the drains and a horizontal flow quite far from them and presented an approximate solution. The Toksöz-Kirkham [5, 6] method is an extension of Kirkham's method for draining homogeneous soils, considering the flow to be two-dimensional, and solving the corresponding Laplace equation. Walczak et al. [15] presented an algorithm based on the Kirkham equation. Van der Molen and Wesseling [16] presented a closed form of the Hooghoudt equivalent depth equation with great accuracy. According to Lovel and Youngs [17], and Ritzema [18], from the above-mentioned solutions, the Hooghoudt equation in combination with the simplistic solution of Van der Molen & Wesseling gives the best results without any restrictions. More recently, Mishra and Singh [19], modified Hooghoudt's method and improved the free surface in the area near the drains. Afruzi et al [20] have also presented a solution of the two-dimensional Laplace equation for the flow into the drains using the Schwarz – Christoffel transform in conformal representation.

There have been more recent contributions to the problem of subsoil drainage: Vlotman et al. [21], provide a resource book for envelope design and research, and collected related data from all over the world. Rimidis and Dierickx [22] proposed a second-degree polynomial, similar to the well-known equation of Hooghoudt, in order to express the relationship between hydraulic head loss and discharge for each of the plots during each of the measuring seasons. Skaggs et al. [23], developed design drainage rates for use in the Hooghoudt equation to estimate required drain depth and spacing in the eastern United States. Castanheira and Santos [24], use a two dimensional saturated - unsaturated Galerkin finite element numerical model to predict water table height between parallel drains. The results obtained with this model agree well with Khirkam's and Hooghoudt analytical solution for the distribution of total head in ideal drains and for the total head calculations midway between drains.

Ali [25] describes the hydraulic design of subsurface drains applied in Bangladesh. For steady state problems, Hooghoudt's equation is proposed, based on the Dupuit-Forcheimer assumptions. He also describes the DRAINMOD hydrologic model, the Colorado State University Irrigation and Drainage (CSUID) model, and the EnDrain model.

Valipour [26] carried out a comparison between horizontal and vertical drainage in anisotropic soils. He determined this purpose, using EnDrainWin and WellDrain softwares drain spacing and well spacing, respectively. The results showed that in the same situation, horizontal drainage systems due to the higher spacing between drains were better than vertical drainage systems. Valipour [27] has investigated the effect of drainage parameters change on drain discharge, which is essential in subsurface drainage systems. For this purpose, he used to change all the drainage parameters by EnDrain software and investigated changes of drains' discharge in subsurface drainage systems. Skaggs [28] introduces three coefficients, namely: a) the subsurface drainage coefficient, calling it Kirkham Coefficient (KC). b) The Drainage intensity coefficient (DI), and c) The drainage coefficient (DC), which quantifies the hydraulic capacity of the system and is estimated by the Hooghoudt equation. Inclusion of these three coefficients in the research and design projects would facilitate comparison of results from different soils and drainage systems, and generally, the meta-analysis of data pertaining to drainage studies. The KC, DI, and DC coefficients represent the minimum information needed to characterize a drainage site. Recently, several authors provided useful insights concerning subsurface drainage system solutions (Kacimov and Obnosov, [29], Chahar and Vadoria, [30], Emikh, [31], Baru and Alan, [32], Sarmah and Tiwari, [33], Ren et al, [34], Bao et al., [35], Zhang et al., [36]).

In all above-mentioned models, the variables are the hydraulic conductivity K, the rainfall intensity R which is entirely infiltrated into the subsoil, the water height in the middle distance between the drains h, and the distance of the bottom of the drains from the impermeable subsoil D. The K, R, D variables are measurable, and contain inaccuracies and fuzziness due to human error, due to measurement apparatus, due to the inhomogeneity and the anisotropy of the soils, etc. The above-mentioned uncertainties heavily influence the reduction of precise conclusions and do not allow engineers to take the right decisions in the design of a drainage network.

In classical Logic, the mathematical models must be extremely accurate, avoiding and dismissing inaccuracies. However, the inaccuracy and the fuzziness are very interesting because they contain information concerning real processes especially in cases where the inaccuracy becomes not acceptable. According to Goguen [37], fuzziness is the rule rather than the exception in problems of engineering, and usually there is no well-defined perfect solution. This weakness is covered by Fuzzy Logic which was developed in 1965 from Zadeh [38]. Fuzzy logic introduces fuzzy numbers, and any inaccuracy or fuzziness is represented numerically. At the same time, the models provide fuzzy numerical calculations based on the theory of fuzzy sets, which allows the management of fuzzy information. According to Goualles [39], uncertainty and fuzziness have received acceptance in scientific research and in the scientific consideration of the world in general.

For the realization of the fuzzy numerical calculations, the following procedure is followed:

•Fuzzification of the variables K, S, d (d = equivalent depth). Usually, symmetrical fuzzy numbers of triagonal form are used.

•A Fuzzy model is chosen.

•The method of numerical calculations is chosen.

Because many calculations are involved in the models, they are executed as interval calculations of the cuts of the triangular numbers. To avoid overestimation in the calculation of the intervals when the number of variables is high, the VERTEX method is used (Dong and Shah, [40]), or the corresponding reduced transformation method (Hanss, [41, 42]).

• A fuzzy number that represents the length of the drain spacing L is obtained, after an iteration process, as the final result.

•The α-cuts of this fuzzy number, represent the confidence intervals of L with probability $P \geq 1-\alpha$, according to the Possibility Theory (Dubois and Prade [43, 44], Dubois et al. [45, 46, 47], Mylonas [48]).

In the present paper, we present the fuzzification of the solution of Hooghoudt in the case of the equivalent length d, to show the difference in the drain spacing, and to estimate the related fuzziness. The fuzzy estimation of variables K, S, R, was obtained using the theory of non-asymptotic fuzzy estimators (Sfiris and, Papadopoulos, [49]), while d is estimated with the closed solution of Van der Molen and Wesseling [16]. The choice of this model was made because it was considered by Ritzema [50] to provide the best results, without restrictions. As a result, we obtain the estimation of drain spacing L as a fuzzy number whose α-cuts according to the possibility theory define the strong probability of the confidence intervals of L. Consequently, the possibility theory application, enables the engineers and designers of irrigation, drainage, and water resources projects to gain knowledge of hydraulic properties and take the right decision for rational and productive engineering studies.

2 Problem Formulation

In the present section, definitions of Fuzzy numbers, Fuzzy sets and Possibility Theory are provided, in paragraph 2.1. The presentation of the classic Hooghout equation is reminded in paragraph 2.2. The Fuzzy form of the equation is presented in paragraph 2.3, and the transformation method is presented in paragraph 2.4, more specifically the decomposition of the fuzzy numbers of the variables involved, and the transformation of their intervals.

2.1 Fuzzy numbers

To facilitate the readers not familiar with the fuzzy theory, some definitions are provided here, concerning some preliminaries in Fuzzy Logic theory and Possibility theory.

Definition 1. A fuzzy number is a fuzzy set $\tilde{u}$: $R^1 \rightarrow I=[0,1]$ with the following properties: (i) $\tilde{u}$ is upper semicontinuous, (ii) $\tilde{u}(x)=0$ outside of some interval [c, d], (iii) there are real numbers a and b, c≤a≤b≤d such that $\tilde{u}$ is increasing (non-decreasing) on [c, a], decreasing (non-increasing) on [b, d] and $\tilde{u}(x)=1$ for each x∈ [a,b], (iv) $\tilde{u}(\lambda x+(1+\lambda)x) \geq \min\{\tilde{u}(\lambda x), \tilde{u}((1+\lambda)x)\}, \lambda \in [0,1]$, $\tilde{u}$ is convex, (v) This fuzzy number has a membership function, denoting the degree of set membership. The membership function of a fuzzy set $\tilde{u}$ is denoted *by* $\mu_{\tilde{u}}(x)$ or by $\tilde{u}(x)$.

Definition 2. Define $L_\alpha(\tilde{u})$ by:

$$L_\alpha(\tilde{u}) = \begin{cases} \{(x,\alpha) | \tilde{u}(x) \geq \alpha\} if 0 < \alpha \leq 1, \\ \overline{[\tilde{u}]_0} \; if \alpha = 0, \end{cases}$$

where $\overline{[\tilde{u}]_0}$ *denotes the closure of the support of* $\tilde{u}$. *Then it is easily established that* $\tilde{u}$ *is a fuzzy number if and only if: (i)* $L_\alpha(\tilde{u})$ *is a closed and bounded interval for each* $\alpha | \alpha \in [0,1]$, *, and (ii)* $L_{\alpha=1}(\tilde{u}) \neq 0$. *The* $L_\alpha(\tilde{u})$ *is called* $\alpha-$*level set of* $\tilde{u}$.

Definition 3. Let K(X) the family of all nonempty compact convex subsets of a Banach space. A fuzzy set on X is called compact if $[\tilde{u}]_\alpha \in K(X)$. The space of all compact and convex fuzzy sets on X is denoted as Ƒ (X).

Definition 4. Let $\tilde{u}$ ∈Ƒ (R). The α-cuts of $\tilde{u}$, are $[\tilde{u}]^\alpha = [u_\alpha^-, u_\alpha^+]$¨. According to representation theorem of Negoita and Ralescu [51] and the theorem of Goetschel and Voxman [52], the membership function and the α-cut form of a fuzzy number $\tilde{u}$, are equivalent and in particular the α-cuts uniquely represent $\tilde{u}$, provided that the two functions are monotonic (u_α^- increasing, u_α^+ decreasing) and $u_{\alpha=1}^- \leq u_{\alpha=1}^+$.

The arithmetic operations of α-cuts are the same as for a set of classical interval numbers (Moore [53], Moore et al. [54]). The arithmetic operations for interval numbers have the properties of associativity and commutativity. However, distributivity does not always hold, which implies that after distributing, the interval would probably be widened. The cause of failure of distributivity is due to the treatment of two occurrences of identical interval numbers as two independent interval numbers. To prevent the widening, additional methods can be used: a) Vertex method (Dong and Shah, [40]), or the Reduced Transformation Method (Hanss, [42, 55]).

Definition 5. Possibility theory (Dubois et al. [45]). Let $\tilde{F}$ be a fuzzy subset of referential set Ω, viewed as the set of admissible, mutually exclusive values of a variable x. Let $\widetilde{A}$ be another subset of Ω; one may evaluate to what extent $\widetilde{A}$ intersects $\tilde{F}$ (possibility of event $\widetilde{A}$) and to what extent $\widetilde{A}$ contains $\tilde{F}$ certainty of event $\widetilde{A}$):

(i) Possibility of $\widetilde{A}$: $\Pi(\widetilde{A}) = \sup\{\alpha | \widetilde{A} \cap \tilde{F}_\alpha \neq 0\}$,

(ii) Necessity (certainty) of $\widetilde{A}$:

$$\mathrm{Ness}(\widetilde{A}) = 1 - \sup\{\alpha \,|\, \bar{\widetilde{A}} \cap \widetilde{F}_\alpha \neq 0\}, \bar{\widetilde{A}} = \neg\widetilde{A}.$$

The above relation (ii) means that $\mathrm{Ness}(\bar{\widetilde{A}}) = 1 - \Pi(\bar{\widetilde{A}})$, i.e. the certainty of $\widetilde{A}$ reflects the impossibility of its complement $\bar{\widetilde{A}}$.

Definition 6. A degree of necessity NessX on a set X (e.g., a set of reels) is characterized by the non-possibility (one minus possibility) of A complement (A^C)

$$\forall A \subseteq X, NessX(A) = 1 - \Pi(A^C)$$

Definition 7. A probability distribution p and a possibility distribution π are said to be consistent only if $\pi(u) \geq p(u), \forall u$ (Dubois et al., [45], Mylonas, [48]).

Definition 8. Two possibility distributions π_x, π'_x are consistent with the probability distribution p_x. The π_x distribution is more specific than π'_x, if it is $\pi_x < \pi'_x$. A possibility distribution π'_x consistent with the probability distribution p_x is called maximal specificity, if it is more specific that each other possibility distribution:

$$\pi_x: \pi'_x(x) < \pi_x(x), \forall \pi$$

Definition 9. For a number Y with a known and continuous probability distribution function p, the fuzzy number $\widetilde{Y}$, which has a possibility measure $\Pi(\widetilde{Y}) = \mu_{\widetilde{Y}}$ is the fuzzy estimator of Y and has an α-cut of $\Pi_{\widetilde{Y}}(\alpha) = \widetilde{Y}(\alpha)$. This fuzzy number satisfies the consistency principle and verifies $\Pi_{\widetilde{Y}}(\alpha) = \mathrm{Ness}\widetilde{Y}(\alpha) = 1 - \alpha$, so that the probability of the possibility α- cut is equal to 1−α. The α- cuts $\widetilde{Y}(\alpha)$ are the confidence intervals of P, and the confidence level is α.

Definition 10. Conjecture (Mylonas [48]). For a function Y = Y ($X_1, X_2, \ldots X_n$) with unknown probability distribution function, a fuzzy number may be constructed $\widetilde{Y}^* = \widetilde{Y}(\widetilde{X}_1^*, \widetilde{X}_2^*, \ldots, \widetilde{X}_n^*)$ and the α- cut is equal to the following:

$$\widetilde{Y}^*(\alpha) = \widetilde{Y}(\widetilde{X}_1^*(\alpha), \widetilde{X}_2^*(\alpha), \ldots, \widetilde{X}_n^*(\alpha)).$$

In this case, the fuzzy number $\widetilde{Y}^*$ is the fuzzy estimator of Y and verifies the following:

$$P(\widetilde{Y}^*(\alpha)) \geq \mathrm{Ness}\widetilde{Y}^*(\alpha) = 1 - \alpha,$$

so that the probability of the possibility α- cut is greater than 1-α.

2.2 Hooghoudt classic function

Hooghoudt [1] presented a formula for the drain spacing accepting parallel and horizontal streamlines, and Hooghoudt [2] considered more practical to have a formula accounting for the extra resistance caused by the radial flow, and he introduced a reduction of the depth D to a smaller equivalent depth d. Finally, his formula is:

$$L^2 = \frac{8Kdh + 4Kh^2}{R} \quad (1)$$

where: K =the hydraulic conductivity (m/d), R=recharge rate per unit surface area (m/d), h=height above the drain level, midway between two drains (m), D = the actual thickness of the aquifer between the drains and the impervious bottom (m), d = the equivalent depth (m). Hooghoudt [2], presented tables with values for the equivalent depth d, for different values of L (5 to 250 m), D (0.5 to 60m), and radius drain r_0=0.1m.

Moody [56], proposed the following iterative formula for the equivalent depth d, which is quite accurate:

$$d = \frac{D}{1 + \frac{D}{L}[\frac{8}{\pi} ln(\frac{D}{r_0}) - 3.4]}, 0 < \frac{D}{L} \leq 0.3 \quad (2)$$

$$d = \frac{L}{\frac{8}{\pi}[ln(\frac{D}{r_0}) - 1.15]}, \frac{D}{L} > 0.3 \quad (3)$$

Van der Molen and Wesseling [16], proposed the following formula for the equivalent depth d:

$$d = \frac{\frac{\pi L}{8}}{ln(\frac{L}{\pi r_0}) + F(x)}. \quad (4)$$

Equation 4 is a combination between the Hooghout's [2] and Kirkham's [13] equations.

$$F(x) = \sum_{n=1}^{\infty} \frac{4e^{-2nx}}{n(1-e^{-2nx})} (n = 1,3,5\ldots), x = \frac{2\pi D}{L} \quad (5),$$

The above series converges rapidly for x> 0.5 with a mean reduced error less than 0.15 %. In this case it takes the form:

$$F(x) = \frac{4e^{-2x}}{1-e^{-2x}}. \quad (6)$$

For x << 0.5 convergence is slow, but for this case, Van der Molen and Wesseling compared it with Dagan's formula and proposed the following closed approximation:

$$F(x) = \frac{\pi^2}{4x} + ln\frac{x}{2\pi} \quad (7)$$

Equation 7 converges conveniently with the above series with a mean reduced error equal to 0.000129 for $x \leq 1$. Figure 1 illustrates the series with a line and the closed solution with black squares.

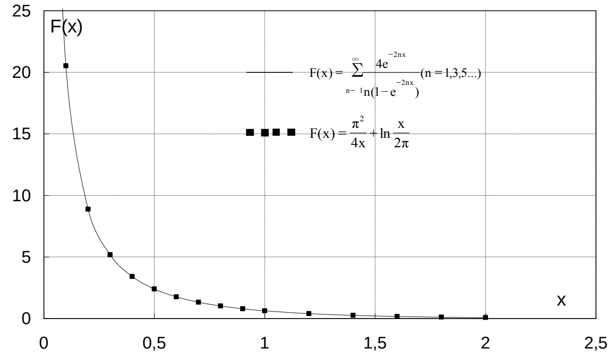


Figure 1. F(x) series and F(x) exact formula.

2.3 Hooghoudt fuzzy function

For the case where the variables $\tilde{K},\tilde{R},\tilde{d}$ are fuzzy numbers of triangular form, Equation 1 becomes:

$$\tilde{L}=\sqrt{\frac{8\tilde{K}\tilde{d}\square+4\tilde{K}\square^{2}}{\tilde{R}}}. \quad (8)$$

The fuzzy variables $\tilde{K},\tilde{R}$ are triangular form fuzzy estimators of K, R and $\tilde{d}$ is a triangular form fuzzy estimator of d with his form:

$$\tilde{d}=\frac{\frac{\pi\tilde{L}}{8}}{ln(\frac{\tilde{L}}{\pi r_0})+\tilde{F}(x)} \quad (9)$$

For practical reasons it is posed:

$\tilde{L}=\tilde{Y},\tilde{K}=\tilde{X}_1,\tilde{R}=\tilde{X}_2,\tilde{d}=\tilde{X}_3,\tilde{X}_4=\tilde{D}$: ,

and Equations 8 and 9 become:

$$\tilde{Y}=\sqrt{\frac{8\tilde{X}_1\tilde{X}_3\square+4\tilde{X}_1\square^{2}}{\tilde{X}_2}}=\tilde{F}_1(\tilde{X}_1,\tilde{X}_2,\tilde{X}_3)(a)$$
$$\tilde{d}=\tilde{X}_3=\tilde{X}_3(\tilde{Y},\tilde{X}_4)(b) \quad (10)$$

According to Nguyen theorem [57], if:

$$x_2\in\tilde{X}_1,x_2\in\tilde{X}_2,x_3\in\tilde{X}_3,\tilde{Y}:\tilde{X}_1\times\tilde{X}_2\times\tilde{X}_3\rightarrow\tilde{Z} \quad (11)$$

then, a sufficient and necessary condition for obtaining the following equality,

$$[\tilde{F}_1(\tilde{X}_1,\tilde{X}_2,\tilde{X}_3)]_\alpha=\tilde{F}_1([\tilde{X}_1]_\alpha,[\tilde{X}_2]_\alpha,[\tilde{X}_3]_\alpha) \quad (12)$$

is that the function is continuous, and the following relation is achieved:

$$\forall z\in Z, sup_{(x_1,x_2,x_3)\in\tilde{F}^{-1}(z)}[\,\mu_{\tilde{X}_1}(x_1)\wedge\mu_{\tilde{X}_2}(x_2)\wedge\mu_{\tilde{X}_3}(x_3)]. \quad (13)$$

Applying Equation 10, each fuzzy number is decomposed into as set of (m+1) intervals X(j) (j=0,1, 2, … m)

$$F_1=\{F_1{}^{(0)},F_1{}^{(1)},\ldots\ldots,F_1{}^{(m)}\}, \quad (14)$$

with

$$F_1{}^{(j)}=[a^{(j)},b^{(j)}]=[\tilde{F}_1]_{\mu_j}, j=1,2,\ldots m$$
$$\mu_1=0.1,\mu_2=0.2,\ldots\ldots,\mu_m=1$$
$$F^{(0)}=[a^{(0)},b^{(0)}],]a^{(0)},b^{(0)}[=sup\,p\,[\tilde{F}]_\alpha, \quad (15)$$

The arithmetic operations of α-cuts are made applying the Reduced Transformation Form, and finally the following fuzzy number arises (Fig.2):

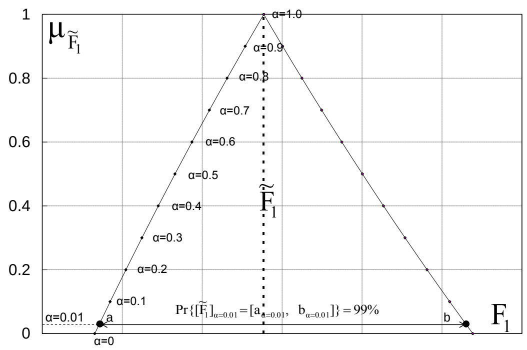


Figure 2. Fuzzy number and fuzzy intervals using possibility theory.

The fuzzy number $\tilde{F}_1$is an estimator of the crisp number F_1 and according to the Possibility theory (Dubois and Prade [44], Dubois et al. [46], Dubois et al. [47], Mylonas [48]), there is a Possibility distribution function, and the following relationship is valid:

$$P([\tilde{F}_1]_\alpha)\geq 1-\alpha \quad (16)$$

Additionally, for the α-cut=0.01, it is valid:

$$P([\tilde{F}_1]_{\alpha=0.01})\geq 99\% \quad (17)$$

2.4 Transformation Method

2.4.1 Decomposition of fuzzy numbers

The Transformation method can be divided into two forms: a General and a Reduced Form, Hanss ([41], [42], [55]). The Reduced Form is used for cases in which there is a function with n independent parameters, assumed to be uncertain. In addition, the function is monotonic with respect to each variable, without local extrema. In the opposite case, the general transformation method can be applied for

complex, non-monotonic problems. In this article the Reduced Transformation Form is used and the function $\tilde{Y}=\tilde{F}_1$ (Equation 10) is monotonic, nonlinear and has three fuzzy variables.
The fuzzy number can be decomposed into m several intervals, j=0, 1, ..., m, given by the α-cuts at the α-levels μ_j

$$\mu_j=\frac{j}{m}, j=0,1,\ldots\ldots,m. \tag{18}$$

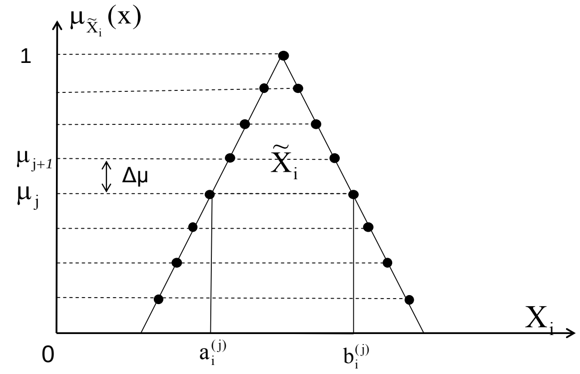


Figure 3. Decomposition of the fuzzy number into intervals.

The fuzzy numbers of Equation 12 (n=3 in this case) can be decomposed into a set of (m + 1) intervals, j=0, 1, …, m, of the form (decomposition principle, Zadeh [38])

$$[\tilde{X}_i]^m=\{X_i^{(0)},X_i^{(1)},\ldots\ldots,X_i^{(m)}\}, i=1,2,\ldots.,n \tag{19}$$

with

$$X_i^{(j)}=[a_i^{(j)},b_i^{(j)}]=cut_{\mu_j}(\tilde{X}_i), a_i^{(j)}\le b_i^{(j)}, j=0,1,2,\ldots.,m,$$
$$X_i^{(0)}=[a_i^{(0)},b_i^{(0)}]=[w_{\ell i},w_{ri}] and]w_{\ell i},w_{ri}[=sup\,p\,(\tilde{P}_i). \tag{20}$$

The set $W=[w_{\ell i},w_{ri}]$ can be referred to as the worst-case interval (Hanss [42]).

Notation: *All the fuzzy parameters $[\tilde{X}_1]_\alpha,[\tilde{X}_2]_\alpha,\ldots\ldots,[\tilde{X}_n]_\alpha$ can be seen as the coordinates of points on the n-dimensional hypersurfaces $[\tilde{X}_1]_\alpha\times[\tilde{X}_2]_\alpha,\ldots\ldots,\times[\tilde{X}_n]_\alpha$, nested according to their level of membership. In the case of the Reduced Transformation Form, only the 2^n vertex points of the n-dimensional cuboids are considered for the evaluation of the problem. Fig. 4 illustrates the case n=3, which is a cube with 2^3 vertices and α=0, 1/3, 2/3, 1. The cuboid for the membership level μ=1, is degenerated to one single point.*

2.4.2 Transformation of the intervals

For the case of the Reduced Transformation Form, the intervals $X_i^{(j)}, i=1,2,\ldots.,n$ are transformed into arrays of the following form:

$$\hat{X}_i^{(j)}=\overbrace{((a_i^{(j)},b_i^{(j)}),(a_i^{(j)},b_i^{(j)}),\ldots\ldots,(a_i^{(j)},b_i^{(j)})}^{2^{i-1}set} \tag{21}$$

$$a_i^{(j)}=\underbrace{(\alpha_i^{(j)},\ldots\ldots.\alpha_i^{(j)})}_{2^{n-i}elements}, b_i^{(j)}=\underbrace{(\beta_i^{(j)},\ldots\ldots.\beta_i^{(j)})}_{2^{n-i}elements} \tag{22}$$

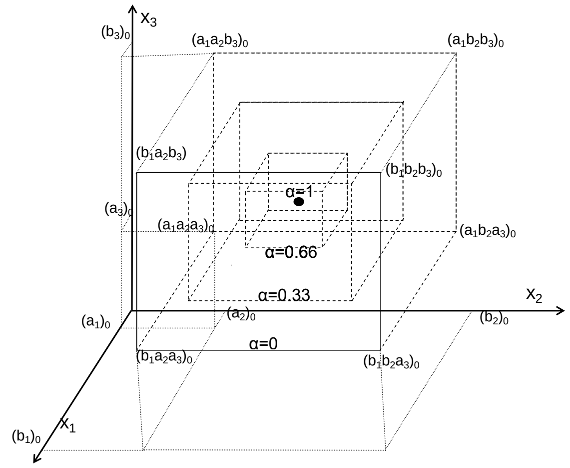


Figure 4. Geometric interpretation of the transformation scheme for n=3.

In the present case for $\tilde{Y}$, (n=3 variables) it is:

$$\begin{bmatrix}\hat{X}_1^j\\ \hat{X}_2^j\\ \hat{X}_3^j\end{bmatrix}=\begin{bmatrix}\alpha_1^{(j)},\alpha_1^{(j)},\alpha_1^{(j)},\alpha_1^{(j)},\beta_1^{(j)},\beta_1^{(j)},\beta_1^{(j)},\beta_1^{(j)}\\ \alpha_2^{(j)},\alpha_2^{(j)},\beta_2^{(j)},\beta_2^{(j)},\alpha_2^{(j)},\alpha_2^{(j)},\beta_2^{(j)},\beta_2^{(j)}\\ \alpha_3^{(j)},\beta_3^{(j)},\alpha_3^{(j)},\beta_3^{(j)},\alpha_3^{(j)},\beta_3^{(j)},\alpha_3^{(j)},\beta_3^{(j)}\end{bmatrix}=[3\times 8] \tag{23}$$

where $j=0,1,\ldots\ldots,$ is the index of the α-level.
In case for $\tilde{d}$ (n=2 variables) it is:

$$\begin{bmatrix}\tilde{Y}^j\\ \hat{X}_4^j\end{bmatrix}=\begin{bmatrix}\alpha_1^{(j)},\alpha_1^{(j)},\beta_1^{(j)},\beta_1^{(j)}\\ \alpha_2^{(j)},\beta_2^{(j)},\alpha_2^{(j)},\beta_2^{(j)}\end{bmatrix}=[2\times 4] \tag{24}$$

Notation*: Between the matrices of the Reduced Transformation Form (RTF) and the VERTEX method there is the following relation*:

$$[Matrix\ of\ Transformation\ Method]=[Matrix\ of\ VERTEX\ method]^T,$$

i.e. the columns of the Reduced Transformation Form are the rows of the VERTEX method. Besides, the columns are the coordinates of n- dimensional hyperface vertices.

In the present case, the dimensions of the matrices are 3×8 and for each column corresponding to a vertex , are created 2^3=8 values of the function $({F_1}_\ell{}^{(j)}, \ell = 1,2,\ldots..8)$ and 2^2=4 for function $d\ {}_\ell^{(j)}, \ell = 1,2,\ldots..4$. It is examined now the following relation:

$$k_{\hat{z}}(\ell)^j = F(\hat{X}_1(\ell), \hat{X}_2(\ell), \hat{X}_3(\ell))^j,$$

(ℓ =the number of the matrix (23) column)

$$k_{\hat{z}}(\ell)^j = \left\{\frac{4X_1^\ell\left(X_3^\ell\right)^2 + 8X_1^\ell X_4^\ell X_3^\ell}{X_2^\ell}\right\}^j, \ell = 1,2,\ldots,8\ j = 0,1,\ldots,m$$

or:

$$[\tilde{F}]_\ell^j = [min(k_{\hat{z}_j}(\ell)), max(k_{\hat{z}_j}(\ell)],$$
$$j = 0,1,2,\ldots,m, \ell = 1,2,\ldots..,8 \qquad (25)$$

3 Results

In the present section a soil sample was used with the following parameters: A hydraulic conductivity K with mean value $\bar{x} = 1.6m/d$ and standard deviation $\sigma = 0.4m/d$, a recharge rate R per unit surface area with mean value $\bar{x} = 0.002m/d$ and standard deviation $\sigma = 0.0005m/d$, a distance D above the impervious floor to the drains level with mean value $\bar{x} = 5m/d$ and standard deviation $\sigma = 0.5m/d$. All these parameters follow a normal distribution law. The above (K, R, D) random sample is of size N=40 observations. The drainpipes with a radius of r=0.1 m, are placed at a depth of 1.8 m below the soil surface and the height h above the drain level, midway between the two drains is 0.6 m.

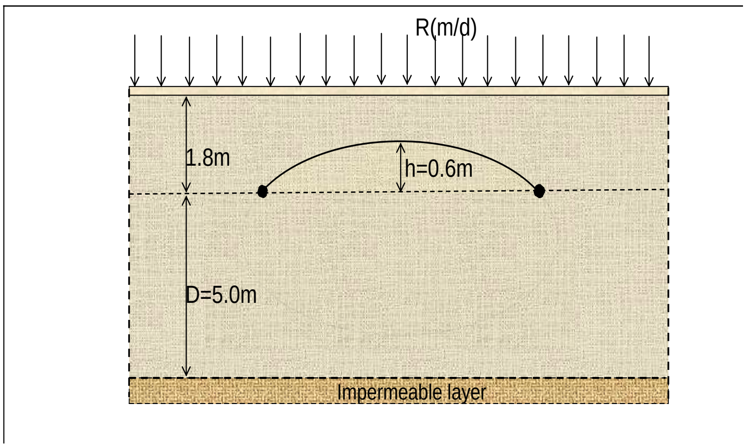


Figure 5. Definition sketch.

3.1 Fuzzy estimators

We apply the theory of non-asymptotic fuzzy estimators (NAFE) (Sfiris and Papadopoulos, [59]). A non-asymptotic fuzzy estimator is a complete triangular form fuzzy number, whose a-cuts are as follows:

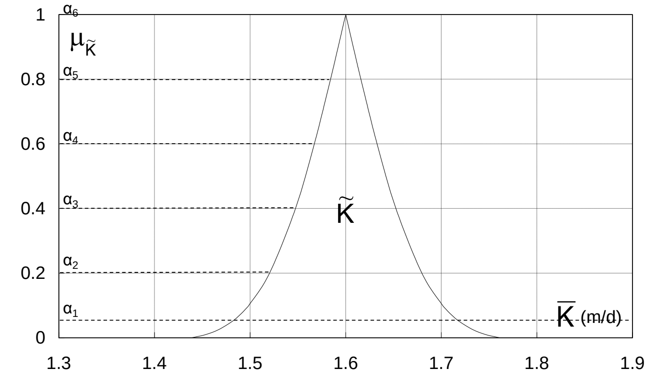
$$\alpha\mu_\gamma = [\bar{x} - z_{\square(\alpha)}\frac{\sigma}{\sqrt{N}}, \bar{x} + z_{\square(\alpha)}\frac{\sigma}{\sqrt{N}}],$$

where $\quad z_{h(\alpha)} = \Phi^{-1}(1 - h(\alpha))$

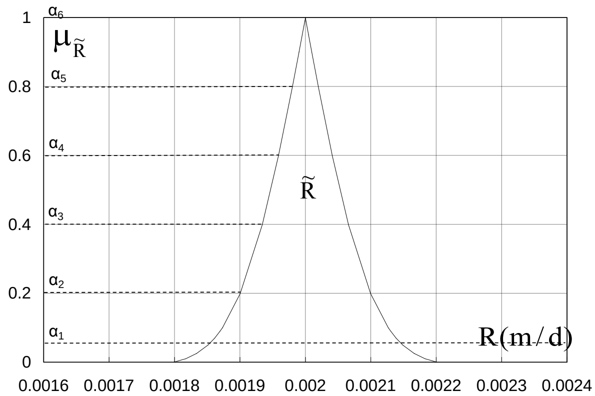
and $\quad h(\alpha) = (\frac{1}{2} - \frac{\gamma}{2})\alpha + \frac{\gamma}{2}.$

Φ denotes the cumulative distribution function of the standard normal distribution. In our case γ=0.01 and

$$h(\alpha) = (0.495)\alpha + 0.005.$$



(a)



(b)

Figure 5. Fuzzy estimators: (a) Hydraulic Conductivity K, (b) Rainfall Intensity R.

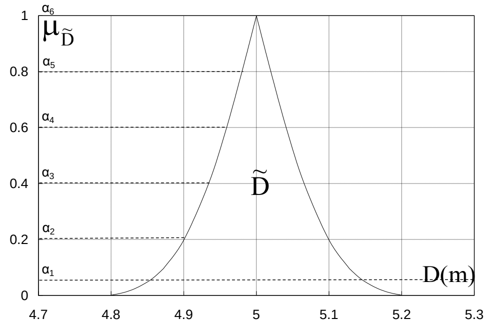


Figure 6. Fuzzy estimator of D.

3.2 Fuzzy Estimation of equivalent depth and drain spacing.

To find the drain spacing an iterative process has started in two steps:

1rst step. With the aid of equation 10(a) a first value of drain spacing estimator$\tilde{L}_1$ is calculated. This value was given by the transformation method and six α-cuts (α_1=0.05, α_2=0.2, α_3=0.4, α_4=0.6, $\alpha_{5=0.8}$ and α_6=1) were used for each variable $\tilde{X}_i, i = 1,2,3$.

2nd step. This value $\tilde{L}_1$ was used to estimate the variable estimator $\tilde{d}_1$ (first iteration) using the equation 10(b).

The iterative process was stopped when the absolute difference of two successive values was negligible. Figures 6 and 7 illustrate the estimators of $\tilde{L}_i$ and $\tilde{d}_i$ and the successive forms in every iteration.

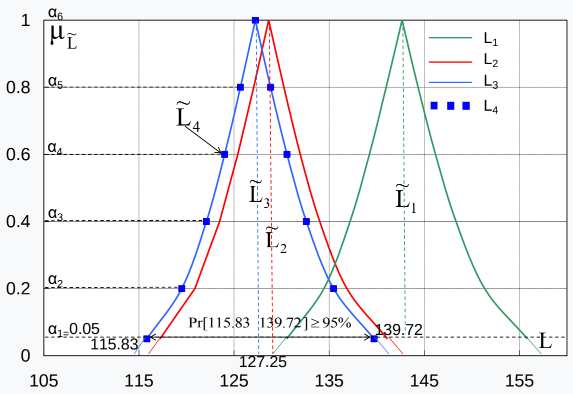


Figure 6. Successive forms of drain spacing estimators.

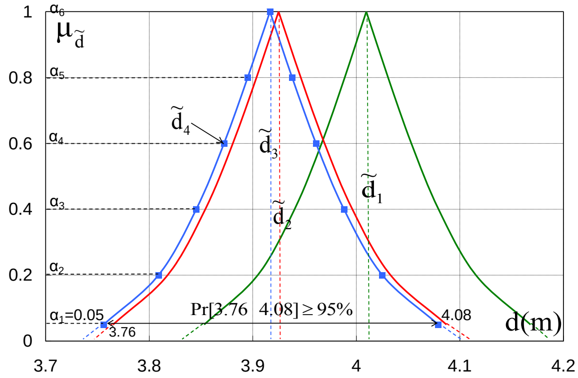


Figure 7. Successive forms of equivalent depth estimators.

After a cycle of four iterations, the iterative process stopped because the absolute difference ε between the two last iterations was negligible.

$$\varepsilon = \left|\frac{Iter4-Iter3}{Iter4}\right|.$$

Tables 1 and 2 present the absolute difference ε between iterations 3 and 4 for every α_i (i=1,2,…6) and the mean value of ε is 9.96 E-04 for $\tilde{L}_i$and 2.16 E-04 for $\tilde{d}_i$.

Table 1. Reduced absolute difference between two iterations for $\tilde{L}$.

	α^-		ε	α^+		ε
	Iteration 3	Iteration 4		Iteration 3	Iteration4	
α_1	115.96	115.83	1.12E-03	139.84	139.72	8.84E-04
α_2	119.63	119.50	1.07E-03	135.58	135.46	9.19E-04
α_3	122.22	122.09	1.05E-03	132.73	132.61	9.44E-04
α_4	124.13	124.01	1.03E-03	130.69	130.57	9.63E-04
α_5	125.80	125.67	1.01E-03	128.97	128.84	9.79E-04
α_6	127.37	127.25	9.94E-04	127.37	127.25	9.94E-04

Table 2. Reduced absolute difference between two iterations for $\tilde{d}$.

	α-		ε	α+		ε
	Iteration 3	Iteration 4		Iteration 3	Iteration4	
α_1	3.7560	3.7550	2.52E-04	4.0793	4.0785	1.84E-04
α_2	3.8089	3.8080	2.39E-04	4.0251	4.0244	1.94E-04
α_3	3.8455	3.8446	2.31E-04	3.9881	3.9873	2.01E-04
α_4	3.8722	3.8713	2.25E-04	3.9611	3.9603	2.06E-04
α_5	3.8951	3.8942	2.20E-04	3.9381	3.9373	2.11E-04
α_6	3.9166	3.9157	2.16E-04	3.9166	3.9157	2.16E-04

4 Discussion

In classical logic in mathematical models, there are imprecisions and fuzziness, which are rejected as being instability factors. However, the higher the precision achieved, the higher the fuzziness becomes. This problem is covered by the Fuzzy Logic and Possibility theories, which introduce the notion of fuzzy numbers, providing fuzzy numerical operations on the base of the Fuzzy Sets Theory.

The present paper, presents the solution of the Hooghout equation based on Fuzzy Logic and Possibility theories, using the Reduced Transformation Method for the related numerical calculations. This results in a fuzzy estimator for the drain spacing, whose α-cuts, provide, according to Possibility Theory, the confidence intervals of the drain spacing with probability greater than the α-level. Results are presented in Figures 6 and 7 after an iterative process, witch after four iterations attained a value of estimator $\tilde{L}$ (drains spacing) and $\tilde{d}$ (equivalent depth) very close to real values. According to Figure 6 presented above, the drain spacing on the base of classical logic, is 127.25m, while based on Fuzzy Logic and Possibility theories the drain spacing is the interval [115.53,139.73] with a probability greater than 95%.

5 Conclusion

From the above, it can be concluded that with the application of the Fuzzy Logic and Possibility theories to the problem of the design of drainage networks, the designer has obtained confidence intervals for the spacing of the drains for any probability level. Consequently, the Fuzzy Logic and Possibility theories application, enables the engineers and designers of irrigation, drainage, and water resources projects to gain knowledge of hydraulic properties and take the right decision for rational and productive engineering studies and design the networks based on these values.

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Christos Tzimopoulos: conceptualization, methodology, software, validation, formal analysis, resources, data curation, writing - original draft preparation.
George Papaevangelou: investigation, visualization, supervision, project administration, writing – review and editing.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.