Coefficient Identification Problem for the System of Heat and Wave Equations Associated with a Non-Characteristic Type Change Line

DURDIMUROD DURDIEV
Bukhara Branch of Romanovskii Institute of Mathematics,
Uzbekistan Academy of Sciences,
UZBEKISTAN

also with
Differential Equations, Bukhara State University,
11 M. Ikbal St. Bukhara 200100, Bukhara,
UZBEKISTAN

Abstract: The solvability of the inverse problem associated with the search for an unknown coefficient at the lowest term of a mixed parabolic-hyperbolic type equation with a non characteristic line of type change is studied. In the direct problem, we consider an analog of the Tricomi problem for this equation with a nonlocal condition on the characteristics in the hyperbolic part and initial-boundary conditions in the parabolic part of the domain. To determine unknown coefficient, with respect to the solution, defined in the parabolic part of the domain, the integral overdetermination condition is specified. The unique solvability of the inverse problem in the sense of the classical solution is proved.

Key-Words: - Parabolic-hyperbolic equation, characteristic, Green’s function, direct problem, inverse problem, contraction principle mapping.

Received: May 9, 2023. Revised: October 11, 2023. Accepted: December 2, 2023. Published: December 31, 2023.

1 Formulation of the Problem

Let $\Omega_T \subset \mathbb{R}^2$ be a finite open domain, bounded for $y > 0$ by segments AB, BC, CD, where $A(0,0)$, $B(0,1)$, $C(T,1)$, $D(T,0)$, T is fixed positive number, and for $y < 0$ - by the characteristics AE: $x + y = 0$ and DE: $x - y = T$ of the following equations:

$$Lu = \begin{cases} u_x - u_{yy} - q(x)u = 0, & y > 0, \\ u_{xx} - u_{yy} = 0, & y < 0. \end{cases} \quad (1)$$

Equation (1) is of mixed parabolic-hyperbolic type, and its type change line $y = 0$ is not a characteristic (parabolic degeneration of the first kind, [1]). In this case the parabolic boundary of equation (1) at $y > 0$ is $DA \cup AB \cup BC$.

Direct problem. Find in the domain Ω_{lT} the solution of the equation (1) satisfying the following boundary conditions:

$$u(0, y) = \varphi(y),\ y \in [0,1],\ u(x, 1) = 0,\ x \in [0, T], \quad (2)$$

$$u\left(\frac{x}{2}, -\frac{x}{2}\right) + u\left(\frac{x+T}{2}, \frac{x-T}{2}\right) = \psi(x), \quad x \in [0, T]. \quad (3)$$

where $\varphi_1(y)$, $\varphi_2(y)$, $\psi(x)$ are given functions.

By a *solution* (classical) of the direct problem (1)-(3) we mean the function $u(x, y)$ from the class $C(\overline{\Omega}_T) \cap C^1(\Omega_T) \cap C^{1,2}_{x,y}(\Omega_{1T}) \cap C^2(\Omega_{2T})$, which satisfies equation (1) and conditions (2), (3).

Let us formulate the *inverse problem* as the problem of finding a pair of functions $\big(u(x, y),\ q(x)\big) \in C(\overline{\Omega}_{lT}) \cap C^1(\Omega_{lT}) \cap C^{1,2}_{x,y}(\Omega_{1lT}) \cap C^2(\Omega_{2l}) \cup C[0, l]$, that satisfies the equation (1), boundary conditions (2), (3) and the following overdetermination condition:

$$\int_0^1 h(y)u(x, y)dy = f(x), \quad x \in [0, T], \quad (4)$$

where in (4) $h(y)$, $f(x)$ are given sufficiently smooth functions.

Direct and inverse problems for mixed type equations are not as well studied as similar problems for classical equations. Nevertheless, such problems are relevant from the point of view of applications. The importance of considering equations of mixed type, where the equation is of parabolic type in one part of the domain and hyperbolic in the other, was first pointed out in the work, [2]. Another example is the following phenomenon in electrodynamics: a mathematical study of the tension of an electromagnetic field in an inhomogeneous medium consisting of a dielectric and a conducting medium leads to a system consisting of a wave equation and a heat equation, [3]. There are many such examples.

The first results on the study of an analogue of the Tricomi problem for a hyperbolic-parabolic equation were obtained in [4]. Further, such problems with different classical and non-local boundary conditions for parabolic-hyperbolic equations with both characteristic and non-characteristic type change lines are formulated and studied in [5], [6], [7], [8].

Methods for solving direct problems of finding the solution of an initial-boundary value problem for equations of the parabolic-hyperbolic type and inverse source problems for these equations in a rectangular domain were proposed in the monograph, [9].

Note that with various inverse problems for classical differential equations of hyperbolic and parabolic types of the second order, the reader can get acquainted in works, [10], [11], [12], [13], [14].

This article continues the study of the author [15], in which the local unique solvability of the inverse problem of determining the variable coefficient at the lowest term of a hyperbolic equation for a mixed hyperbolic-parabolic equation with a noncharacteristic line of type change is investigated.

Throughout this paper, with respect to the given ones, we will assume that the following conditions are satisfied:

(B1) $\varphi(y) \in C^3[0,1],\ \psi(x) \in C^2[0,T] \cap C^2(0,T)$;
(B2) $\varphi(1) = 0,\ \varphi_1(0) - \varphi_2(0) = \psi(0) - \psi(l)$;
(B3) $h(y) \in C^2[0,1]$, $h(0) = h(1) = h'(0) = h'(1) = 0$, $f(x) \in C^1[0,T]$, $\int_0^1 h(y)\varphi(y)dy = f(0)$, $|f(x)| \neq 0$ for all $x \in [0,T]$.

The study of inverse problems requires studying the differential properties of solutions of direct problems. This is most clearly seen in coefficient inverse problems (nonlinear problems), where, to obtain solvability theorems, one must carefully analyze the exact dependence of the differential properties of solutions of the direct problem on the smoothness of the coefficients and other data of the problem. That is why, let us study the direct problem first.

2 Investigation of the Direct Problem

Assume that the function $q(x)$ is known.

Theorem 1. *Let conditions (B1), (B2), $q(x) \in C[0,T]$ be satisfied.*

Then, in the domain Ω_T there exists an unique solution to the direct problem (1)-(3).

Let there be a solution $u(x,y)$ of the direct problem (1)-(3). Let us introduce the notation: $\tau(x) := u(x,0)$, $\nu(x) = \frac{\partial}{\partial y}u(x,0)$. Then, due to the unique solvability of the Cauchy problem for the wave equation, the solution to the equation (1) in the domain Ω_{2l} can be written using the d'Alembert formula

$$u(x,y) = \\ = \frac{1}{2}[\tau(x+y) + \tau(x-y)] - \frac{1}{2}\int_{x+y}^{x-y} \nu(s)ds. \quad (5)$$

Taking into account condition (3) and equalities $\tau(0) = \varphi(0)$ (a consequence of the definition of the classical solution), this implies the equality:

$$2\tau(x) + \varphi(0) + \tau(T) - \int_0^T \nu(s)ds = 2\psi(x), \quad x \in [0,T]. \quad (6)$$

Further it follows from (3) at $x = 0$: $u\left(\frac{T}{2}, -\frac{T}{2}\right) = \psi(0) - \varphi(0)$. Then, comparing this with (5) at $x = \frac{T}{2}$, $y = -\frac{T}{2}$, we have $\int_0^T \nu(s)ds = 3\varphi(0) - 2\psi(0) + \tau(T)$. Using this equality we eliminate $\int_0^T \nu(s)ds$ in (6) and we find

$$\tau(x) = \varphi(0) - \psi(0) + \psi(x). \quad (7)$$

Thus the function $\tau(x)$ becomes known.

Introduce the notations

$$G_k(x-\xi, y, \eta) = \\ = \frac{1}{2\sqrt{\pi(x-\xi)}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(y-\eta+2n)^2}{4(x-\xi)}\right) + (-1)^k \exp\left(-\frac{(y+\eta+2n)^2}{4(x-\xi)}\right)\right],\ k = 1,2.$$

Using the Green's function $G_1(x-\xi, y, \eta)$ of the first initial-boundary value problem for the heat equation in the domain Ω_{1T}, the solution of equation (1) with the conditions (2), and $u|_{AD} = \tau(x)$ represent in the form of integral equation:

$$u(x,y) = \int_0^1 G_1(x,y,\eta)\varphi(\eta)d\eta + \\ + \int_0^x G_{1\eta}(x-\xi, y, 0)\tau(\xi)d\xi +$$

$$+\int_0^x q(\xi)\int_0^1 G_1(x-\xi,y,\eta)u(\xi,\eta)d\eta d\xi. \quad (8)$$

Equations (8) represents a linear integral equation Volterra type of the second kind to determine the unknowns functions $u(x,y)$ ($\tau(x)$ is known). It is known from the general theory of integral equations that under the conditions of Theorem 1 this equation is solvable in the class of continuous in Ω_{1T} functions and determines the function $u(x,y)\in C^{1,2}_{x,y}(\Omega_{1T})$, that is, the solution of problem (1), (2) in the domain Ω_{1T}.

Note that the functions $G_k(x-\xi,y,\eta)$, $k=1,2$ have equivalent representations:

$$G_1(x-\xi,y,\eta)=$$
$$=2\sum_{n=1}^{\infty}\exp\left[-(n\pi)^2(x-\xi)\right]\sin n\pi y \sin n\pi\eta,$$
$$G_2(x-\xi,y,\eta)=$$
$$2\sum_{n=1}^{\infty}\exp\left[-(n\pi)^2(x-\xi)\right]\cos n\pi y\cos n\pi\eta \quad (9)$$

and are infinitely differentiable in Ω_{1T} [3].

Since the functions $\tau(x)$ and $u(x,y)$ are known in Ω_{1T}, let us now begin to find $\nu(x)$. For this, we calculate the derivatives of the first two terms on the right side of (8) using the obvious relations

$$G_{1y}(x-\xi,y,\eta)=-G_{2\eta}(x-\xi,y,\eta),$$
$$G_{1\eta}(x-\xi,y,\eta)=-G_{2y}(x-\xi,y,\eta),$$
$$G_{2\xi}(x-\xi,y,\eta)=-G_{2yy}(x-\xi,y,\eta). \quad (10)$$

Integrating by parts, we get:

$$\int_0^1 G_{1y}(x,y,\eta)\varphi(\eta)d\eta=$$
$$=-\int_0^1 G_{2\eta}(x,y,\eta)\varphi(\eta)d\eta$$
$$=G_2(x,y,0)\varphi(0)-G_2(x,y,1)\varphi(1)+$$
$$+\int_0^1 G_2(x,y,\eta)\varphi'(\eta)d\eta.$$

Using (10) and integrating by parts, we calculate the derivative with respect to y of the following term in formula (8):

$$\frac{\partial}{\partial y}\int_0^x G_{1\eta}(x-\xi,y,0)\tau(\xi)d\xi=$$
$$=-\frac{\partial}{\partial y}\int_0^x G_{2y}(x-\xi,y,0)\tau(\xi)d\xi$$
$$=\int_0^x G_{2\xi}(x-\xi,y,0)\tau(\xi)d\xi=$$
$$=-G_2(x,y,0)\tau(0)-$$
$$-\int_0^x G_2(x-\xi,y,0)\tau'(\xi)d\xi.$$

Taking into account the form of the function $\tau(x)$ according to the formula (7), we finally have:

$$\frac{\partial}{\partial y}\int_0^x G_{1\eta}(x-\xi,y,0)\tau(\xi)d\xi=$$
$$=-G_2(x,y,0)\varphi(0)-$$
$$-\int_0^x G_2(x-\xi,y,0)\psi'(\xi)d\xi.$$

Using the above equalities, we now differentiate (8) with respect to y and set $y=0$. Since $(\partial/\partial y)u(x,0)=\nu(x)$,, taking into account the matching conditions (B2), we obtain:

$$\nu(x)=\int_0^1 G_2(x,0,\eta)\varphi'(\eta)d\eta-$$
$$-\int_0^x G_2(x-\xi,0,0)\psi'(\xi)d\xi+$$
$$+\int_0^x q(\xi)\int_0^1 G_{1y}(x-\xi,0,\eta)u(\xi,\eta)d\eta d\xi,$$
$$x\in[0,T]. \quad (11)$$

Eliminating the function $\tau(x)$ in (8) using equality (7), we obtain integral equation for the function $u(x,y)$

$$u(x,y)=\int_0^1 G_1(x,y,\eta)\varphi(\eta)d\eta$$
$$+2\int_0^x G_{1\eta}(x-\xi,y,0)\psi\left(\frac{\xi}{2}\right)d\xi$$
$$+\int_0^x G_{1\eta}(x-\xi,y,0)\int_0^{\xi}\nu(s)ds d\xi$$
$$+\int_0^x q(\xi)\int_0^1 G_1(x-\xi,y,\eta)u(\xi,\eta)d\eta d\xi. \quad (12)$$

Note that the following holds for the function $G_2(x-\xi,0,0)$ equality:

$$G_2(x-\xi,0,0)=$$
$$=\frac{1}{\sqrt{\pi(x-\xi)}}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{n^2}{x-\xi}\right)=$$

$$= \frac{1}{\sqrt{\pi(x-\xi)}} + \frac{2}{\sqrt{\pi(x-\xi)}} \sum_{n=1}^{\infty} \exp\left(-\frac{n^2}{x-\xi}\right). \tag{13}$$

Equations (11) and (12) represent a system of linear integral equations Volterra type of the second kind to determine the unknowns functions $u(x,y)$ and $\nu(x)$. By virtue of formula (13), integral equation (11) has a weak polar singularity. It is known from the general theory of integral equations that the system of equations (11) and (12) is solvable in the class of continuous in $\overline{\Omega}_{1T}$ functions. This solution can be found, for example, by the method of successive approximations and $\nu(0) = 0$, due to $\lim_{x\to 0} G_2(x,0,\eta) = 0$ for $\eta \in (0,1)$.

Considering equality:

$$\int_0^1 G_{2x}(x,0,\eta)\varphi'(\eta)d\eta = \int_0^1 G_{2\eta\eta}(x,0,\eta)\varphi'(\eta)d\eta,$$

using integration by parts, based on conditions (B1), (B2), we find

$$\int_0^1 G_{2\eta\eta}(x,0,\eta)\varphi'(\eta)d\eta = \int_0^1 G_2(x,0,\eta)\varphi'''(\eta)d\eta. \tag{14}$$

Assuming now the existence of a derivative of the solution $\nu(x)$, taking into account conditions (B1), (B2) and (14), we obtain for $\nu'(x)$ the equation

$$\nu'(x) = \int_0^1 G_2(x,0,\eta)\varphi'''(\eta)d\eta - -\frac{1}{2}\int_0^x G_2(x-\xi,0,0)\psi''\left(\frac{\xi}{2}\right)d\xi - -\int_0^x G_2(x-\xi,0,0)\nu'(\xi)d\xi + +\int_0^x q(\xi)\int_0^1 G_{1yx}(x-\xi,0,\eta)u(\xi,\eta)d\eta d\xi,$$
$$x \in (0,T],$$

which is also solvable in the class of continuous functions instead of with equation (12). Thus, $\nu(x) \in C[0,T] \cap C^1(0,T]$. From the found function $\nu(x)$, the function $\tau(x)$ is found from formulas (8). Due to conditions (B2) and $\nu(x) \in C^1(0,T]$, we have $\tau(x) \in C^1[0,T] \cap C^2(0,T]$. And the function $u(x,y)$, constructed as solution of equation (1) with conditions (2), and $u|_{AD} = \tau(x)$ when conditions (B1), (B2) are met and inclusion $q(x) \in C[0,T]$ belongs to the class $C^{1,2}_{x,y}(\Omega_{1T})$.

Thus, found in Ω_{1T} solution $u(x,y)$ and function (6) in Ω_{2T} together determine the classical solution to the direct problem (1)-(3) in the domain Ω_T. Theorem 1 is proved.

3 Study of the Inverse Problem

Assume that conditions (B3) are satisfied. Multiplying the equation (1) in the domain Ω_{1T} by the function $h(y)$ and integrating over the segment [0,1], in view of (4), we find

$$q(x) = \frac{f'(x)}{f(x)} - \frac{1}{f(x)}\int_0^1 h''(y)\,u(x,y)dy,$$
$$x \in [0,T]. \tag{15}$$

Now, substituting the right side of (15) instead of $q(x)$ in (8), we write the resulting equation in the operator form:

$$u(x,y) = U[u](x,y), \quad (x,y) \in \overline{\Omega}_{1lT}, \tag{16}$$

where the operator U is defined by the equality:

$$Uu(x,y) = u_0(x,y) + +\int_0^x\int_0^1 G_1(x-\xi,y,\eta)\left[\frac{f'(\xi)}{f(\xi)} - \frac{1}{f(\xi)}\int_0^1 h''(s)u(\xi,s)ds\right]u(\xi,\eta)d\eta d\xi, \tag{17}$$

and in (17) u_0 denotes the sum of terms of integral equation (8) which are free from unknown function:

$$u_0(x,y) := \int_0^1 G_1(x,y,\eta)\varphi(\eta)d\eta + \int_0^x G_{1\eta}(x-\xi,y,0)\tau(\xi)d\xi.$$

Recall that the function $\tau(x)$ is defined by the formula (7).

The main result of this section is the following assertion:

Theorem 2. *Let conditions (B1)-(B3) be satisfied. Then, there are positive numbers T^* such that equation* (16) *has a unique continuous solution in the domain Ω_{1T} for $T \in (0,T^*)$.*

Proof. It is clear from (17) that under the conditions of the theorem the operator U translates the functions $u(x,y) \in C(\overline{\Omega}_{1lT})$ into functions also belonging to the space $C(\bar{\Omega}_{1T})$. The u norm in $C(\bar{\Omega}_{1T})$ we define as follows:

$$\|u\|_T = \max_{(x,y)\in\bar{\Omega}_{1T}} |u(x,y)|.$$

For brevity, we also introduce notations

$$f_0 := \min_{x\in[0,T]} |f(x)|,\ \ f_1 := \max_{x\in[0,T]} \left|f'^{(x)}\right|,$$
$$h_0 := \max_{x\in[0,1]} |h''(y)|.$$

Let us now show that, for sufficiently small T, the operator U performs a contraction mapping of the ball:

$$S(u_0,r) := \{u \in C(\bar{\Omega}_{1T}) : \|u-u_0\|_T \le r\} \subset C(\bar{\Omega}_{1T})$$

with radius r (r is a known number) and centered at the point $u_0(x,y)$ of the functional space $C(\bar{\Omega}_{1T})$ into itself. Thus, we will prove that equation (16) has in the domain $\bar{\Omega}_{1T}$ an unique continuous solution satisfying the inequality $\| u-u_0 \|_T \le r$.
It is obvious that for the element $u \in S(u_0,r)$ there holds an estimate:

$$\|u\|_T \le \|u_0\|_T + r =: R,$$

where R denotes a known positive number.

Let us estimate $\|u_0\|_T$. To do this, we need estimates for integrals involving the functions G_1, $G_{1\eta}$ in the definitions of the function $u_0(x,y)$. In this case, we use the equality:

$$\int_0^1 G\,(x,\xi,y)d\xi = 1,$$

which follows from the definition of the function G. Taking this into account, the first term of $u_0(x,y)$ can be easily estimated in modulo:

$$\left|\int_0^1 G\,(x,\xi,y)\varphi(\xi)d\xi\right| \le \|\varphi\|_{C[0,T]}. \quad (18)$$

Based on (9), we have the equalities:

$$G_\eta(x-\xi,y,0) = = \frac{2}{l}\sum_{n=1}^{\infty} \exp\left[-\left(\frac{n\pi}{l}\right)^2 (x-\xi)\right]\frac{n\pi}{l}\sin n\pi y$$
$$= \int_0^1 G_{1\xi}\,(x-\xi,y,\eta)(1-\eta)d\eta,$$

which are checked directly. Using these relations, we transform the following integral:

$$\int_0^x G_{1\eta}\,(x-\xi,y,0)\tau(\xi)d\xi =$$
$$= \int_0^1 (l-\eta)\int_0^x G_{1\xi}\,(x-\xi,y,\eta)\tau(\xi)d\xi d\eta =$$
$$= \int_0^1 (l-\eta)\left\{[G_1(x-\xi,y,\eta)\tau(\xi)]_0^x - \int_0^x G_1\,(x-\xi,y,\eta)\tau'(x)d\xi\right\}d\eta$$
$$= (1-y)\tau(\xi) - - \int_0^1 (1-\eta)\left[G_1(x-\xi,y,\eta)\tau(0) + \int_0^x G_1\,(x-\xi,y,\eta)\tau'(\xi)d\xi\right]d\eta.$$

Here, in intermediate calculations, we used the relation $\lim_{\xi\to x} G_1(x-\xi,y,\eta) = \delta(y-\eta)$, $\delta(\cdot)$ is the Dirac's delta function and the main property of the function $\delta(\cdot)$: $\int_0^1 a\,(\xi)\delta(x-\xi)d\xi = a(x)$ which is valid for any continuous function $a(x)$ on the interval (0,1).

From these relations for $(x,y) \in C(\bar{\Omega}_{1T})$ easily it follows the estimate

$$\left|\int_0^x G_{1\eta}\,(x-\xi,y,0)\tau(\xi)d\xi\right| \le$$
$$\le \|\varphi\|_{C[0,T]} + (2+T)\|\psi\|_{C^1[0,T]}. \quad (19)$$

Then, inequalities (18), (19) imply the estimate:

$$\|u_0\|_T \le 3\|\varphi\|_{C^1[0,T]} + (2+T)\|\psi\|_{C^1[0,1]}.$$

We now turn to obtaining conditions for T under which it is possible to apply the fixed point theorem to the operator U. Let $u \in S(u_0,r)$, then, for all $(x,y) \in \bar{\Omega}_{1T}$, we obtain the inequalities:

$$|Uu-u_0| \le$$
$$\le \int_0^x\int_0^1 G\,(x,\xi,y-\eta)\left[\frac{|f'(\xi)|}{|f(\xi)|} + \frac{1}{|f(\xi)|}\int_0^T \left|h''^{(s)}\right| |u(\xi,s)|ds\right]|u(\xi,\eta)|d\xi d\eta \le$$
$$\le (f_1+h_0R)\frac{R}{f_0}T =: m_1(T).$$

Condition $\| u-u_0 \|_T \le r$ (that is $Uu \in S(u_0,r)$) will be valid if T is chosen from the condition $m_1(T) < r$. This condition is satisfied by all $T \in (0,T_1)$, where $T_1 := rf_0/R(f_1+h_0R)$.

Let us now to show that the operator U contracts the distance between elements of the ball $S(u_0, r)$. To prove this, we take any two elements $(u^1, u^2) \in S(u_0, r)$ and estimate the norm of the difference between their images Uu^1, Uu^2. For this purpose, using (17) we have the inequality

$$|Uu^1 - Uu^2| \leq$$
$$\leq \int_0^x \int_0^1 G(x, \xi, y-\eta) \times$$
$$\times \left[\frac{|f'(\xi)|}{|f(\xi)|} |u^1(\xi, s) - u^2(\xi, s)| + \right.$$
$$+ \frac{1}{|f(\xi)|} \int_0^1 |h''(s)| \, |u^1(\xi, s) u^1(\xi, \eta)$$
$$\left. - u^2(\xi, s) u^2(\xi, \eta)| ds \right] d\xi d\eta. \quad (20)$$

Here to estimate the expression $|u^1(\xi, s) u^1(\xi, \eta) - u^2(\xi, s) u^2(\xi, \eta)|$, we use inequality

$$|u^1(\xi, s) u^1(\xi, \eta) - u^2(\xi, s) u^2(\xi, \eta)| \leq$$
$$\leq |u^1(\xi, s)| |u^1(\xi, \eta) - u^2(\xi, \eta)| +$$
$$+ |u^2(\xi, \eta)| |u^1(\xi, s) - u^2(\xi, s)| \leq$$
$$\leq 2R \|u^1 - u^2\|_{lT}, (s, \xi, \eta) \in [0,1] \times [0,1] \times [0, x],$$

which holds for arbitrary $(u^1, u^2) \in S(u_0, r)$.

Continuing the estimate (20), we get

$$\|Uu^1 - Uu^2\|_T \leq (f_1 + 2h_0 R) \frac{T}{f_0} \|u^1 - u^2\|_T$$
$$=: m_2(T) \|u^1 - u^2\|_T.$$

We choose T so that the inequality $m_2(T) < 1$ holds, then the operator U contracts the distance between elements of the ball $S(u_0; r)$. This condition is satisfied by $T \in (0, T_2)$, where $T_2 := f_0/(f_1 + 2h_0 R)$. Let $T^* = \min(T_1, T_2)$. Since $r/R < 1$, then, it is easy to see that $T^* = r f_0 / R(f_1 + 2h_0 R)$. Hence, the operator U for $T \in (0, T^*)$ performs a contraction mapping of the ball $S(u_0, r)$ to itself. Hence, according to the contraction mapping principle, equation (16) defines a unique solution $u(x, y) \in S(u_0, r)$. Theorem 2 is proved.
After finding the function $u(x, y)$ the functions $q(x)$ is determined by the formula (15).
Thus the following assertion is valid:
Theorem 3. *Let conditions* (B1)-(B3) *be satisfied and* $T \in (0, T^*)$. *Then, the formula* (15) *defines* $q(x)$ *on any fixed segment* $[0, l]$.

4 Conclusion

In this paper, the solvability of the inverse problem associated with the search for an unknown coefficient at the lowest term of a mixed parabolic-hyperbolic type equation with a non-characteristic line of type change is investigated. In the direct problem, an analog of the Tricomi problem for this equation with a nonlocal condition on the characteristics in the hyperbolic part and initial-boundary conditions in the parabolic part of the domain is considered. To determine the unknown coefficient, with respect to the solution of the direct problem, defined in the parabolic part of the domain, the integral overdetermination condition is specified. The unique solvability of the inverse problem in the sense of the classical solution is proved.
Note that the zero-coefficient of the parabolic equation is defined here. Many applied problems require consideration of more general equations than (1) and determination of other coefficients in both parabolic and hyperbolic equations. Similar problems and the numerical study of the inverse problem considered in this article are open problems.

Acknowledgement:
The author thanks the anonymous reviewers who, after reading the article, made very useful comments.

References:

[1] Babich V.M., Kapilevich M.B., Mikhlin S.G., Natanson G.I., Riz P.M., Slabodetskii L.N., Smirnov M.M., *Linear Equations of Mathematical Physics. Reference Mathematical Library*. Moscow: Science, 1964. [Babich V.M., Kapilevich M.B., Mikhlin S.G., Natanson G.I., Riz P.M., Slabodetskii L.N., Smirnov M.M., *Lineinye uravneniya matematicheskoi fiziki. Spravochnaya matematicheskaya biblioteka*, Moskva: Nauka, 1964.]

[2] Gel'fand I.M., Some questions of analysis and differential equations, *Usp. Mat. Nauk*, vol. 14, no. 3, 1959, pp. 3-19.

[3] Tikhonov A.N., Samarskii A.A. *Equations of Mathematical Physics*. Oxford: Pergamon Press, 1963.

[4] Zolina L.A., Boundary value problem for the model equation of the hyperbolic-parabolic type, *U.S.S.R. Comput. Math. Math. Phys.*, vol. 6, no. 6, 1966, pp. 63–78 [Zolina

L.A., O kraevoi zadache dlya modelnogo uravneniya giperbolo-parabolicheskogo tipa, *Zh. Vychisl. Mat. Mat. Fiz.*, 6:6 (1966), 991–1001].

[5] Rassias J.M., *Lecture Notes on Mixed Type Partial Differential Equations,* World Scientific, Singapore, 1990.

[6] Kapustin, N.Yu, The Tricomi problem for a parabolic-hyperbolic equation with a degenerate hyperbolic side. II, *Differ. Equ.*, 1988, vol. 24, no. 8, pp. 898-903.

[7] Kapustin, N.Yu, A problem for a parabolic-hyperbolic equation with a nonlocal gluing condition for the normal derivatives on the line of change of type, *Dokl. Math.*, 1989, vol. 39, no. 2, pp. 248-250.

[8] Wen G.C., The Tricomi problem for second order linear mixed equations with parabolic degeneracy, *Appl. Anal.,* 2005, Vol. 84, pp. 1267–1286.

[9] Romanov V.G. *Inverse problems of mathematical physics.* Utrecht: VNU Science Press, The Netherlands, 1987.

[10] Hasanoğlu A. Hasanov, Romanov V. G. *Introduction to Inverse Problems for Differential Equations*. Springer International Publishing, 2017.

[11] Sun H.S., Tricomi problem for nonlinear equation of mixed type, *Sci. China Ser. A,* 1992, vol. 35, pp. 14–20.

[12] Salawu S.O. Evaluation of Thermo-diffusion and Diffusion-thermo Phenomenon on the Reactive Micropolar Fluid Motion Over an Extending Device, *WSEAS: Engineering World,* 2023, vol. 5, pp. 75-87.

[13] Aslam A., Mahomed F.M., Qadir A., Safdar M. Invariants for a System of Two Linear Hyperbolic Equations by Complex Methods, *WSEAS: Engineering World*, 2020, vol. 2, pp. 86-95.

[14] Ecsedi I, Lengyel A.J. Analytical Solutions of Heat Conduction Problems for Anisotropic Solid Body, *WSEAS: Transactions on Heat and Mass Transfer*, 2021, vol. 16, pp. 127-136

[15] Durdiev D.K. Determining the coefficient of a mixed parabolic–hyperbolic equation with noncharacteristic type change line, *Differential Equations*, 2022, vol. 58, no. 12, pp. 1618–1629.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflicts of interest to declare.