On the Structure of the Spectrum of Internal Vibrations for Stratified Rotating Compressible Flows in General Domains, in Rectangular Domains, in General Cylinders and in Spherical Volumes

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Abstract:—For exponentially stratified rotating compressible fluid, we investigate the localization and the structure of the spectrum of inner waves caused by the gravitational force and the Coriolis force. We find the essential spectrum for the first boundary value problem in general domains. Our main result is the explicit examples of the eigenvalues and the corresponding orthogonal eigenfunctions for parallelepipeds, for general cylinders and for spherical volumes.

Keywords:—Compressible flows, computational fluid dynamics, essential spectrum, internal waves, turbulence and multiphase flows.

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1. Introduction

LET us consider a bounded domain $\Omega \subset R^3$ with the boundary $\partial\Omega$ of the class C^∞ piecewise, and the following system of fluid dynamics

$$\begin{cases} \frac{\partial u_1}{\partial t} - \omega u_2 + \frac{\partial p}{\partial x_1} = 0 \\ \frac{\partial u_2}{\partial t} + \omega u_1 + \frac{\partial p}{\partial x_2} = 0 \\ \frac{\partial u_3}{\partial t} + \rho + \frac{\partial p}{\partial x_3} = 0 \\ \frac{\partial \rho}{\partial t} - N^2 u_3 = 0 \\ \alpha^2 \frac{\partial p}{\partial t} + \operatorname{div} \vec{u} = 0 \qquad x \in \Omega,\ t \geq 0. \end{cases} \tag{1}$$

In this system $x = (x_1, x_2, x_3)$ is the space variable,

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$\vec{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity field, $p(x,t)$ is the scalar field of the dynamic pressure and $\rho(x,t)$ is the dynamic density. We suppose that the stationary distribution of density is described by the function e^{-Nx_3}, where N is a positive constant. For the compressibility coefficient α, we assume $\alpha > 0$. In the model (1) the stratified fluid is rotating over the vertical axis with the constant angular velocity $\vec{\omega} = [0, 0, \omega]$.

For non-rotational case, the equations (1) are deduced, for example, in [1], [2].

Despite an extensive study of stratified flows from the physical point of view (see, for example, [3-6]) we would like to observe that there have been relatively few works considering the mathematical aspect of the problem. We associate the system (1) to the first boundary value (Dirichlet) condition

$$p\big|_{\partial\Omega} = 0.$$

The following separation of variables allows us to consider the problem of normal vibrations

$$\begin{aligned} \vec{u}(x,t) &= \vec{v}(x) e^{-\lambda t} \\ \rho(x,t) &= N v_4(x) e^{-\lambda t} \\ p(x,t) &= \tfrac{1}{\alpha} v_5(x) e^{-\lambda t},\ \lambda \in C. \end{aligned} \tag{2}$$

We denote $\widehat{v} = (\vec{v}, v_4, v_5)$ and write the system (1) in the matrix form

$$L\widehat{v} = 0 \tag{3}$$

where

$$L = M - \lambda I$$

and

$$M = \begin{pmatrix} 0 & -\omega & 0 & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_1} \\ \omega & 0 & 0 & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_2} \\ 0 & 0 & 0 & N & \frac{1}{\alpha}\frac{\partial}{\partial x_3} \\ 0 & 0 & -N & 0 & 0 \\ \frac{1}{\alpha}\frac{\partial}{\partial x_1} & \frac{1}{\alpha}\frac{\partial}{\partial x_2} & \frac{1}{\alpha}\frac{\partial}{\partial x_3} & 0 & 0 \end{pmatrix}.$$

We define the domain of the operator M as follows.

$$D(M) = \left\{ \begin{array}{l} \vec{v} \in (L_2(\Omega))^3 \,|\, \exists f \in L_2(\Omega) : \\ (\vec{v}, \nabla\phi) = (f, \phi) \ \forall \phi \in \overset{0}{W}{}_2^1(\Omega), \\ v_4 \in W_2^1(\Omega),\ v_5 \in \overset{0}{W}{}_2^1(\Omega) \end{array} \right\},$$

where $\overset{0}{W}{}_2^1(\Omega)$ is a closure of the functional space $C_0^\infty(\Omega)$ in the norm

$$\|f\| = \left(\int_\Omega \left[|\nabla f|^2 + f^2 \right] dx \right)^{\frac{1}{2}}.$$

In this paper, we investigate the structure and the localization of the spectrum of the operator M for general three-dimensional domains, and also we construct the explicit examples of the eigenvalues and eigenfunctions for rectangular, cylindrical and spherical domains.

From the point of view of applications, the separation of variables (2) may serve as a tool to represent every non-stationary motion described by (1) as a linear sum of the stationary modes. The knowledge of the spectrum of normal oscillations may be very useful for studying the stability of the flows. Besides, the spectrum of operator M plays an important role in the investigation of weakly non-linear flows, since the bifurcation points where the small non-linear solutions arise, belong to the spectrum of linear normal vibrations, i.e., to the spectrum of operator M.

It can be easily seen that the operator M is a closed operator, and its domain is dense in $(L_2(\Omega))^5$.

Let us denote by $\sigma_{ess}(M)$ the essential spectrum of operator M. We recall that the essential spectrum

$$\sigma_{ess}(M) = \{\lambda \in C : (M - \lambda I) \text{ is not of Fredholm type}\},$$

is composed of the points belonging to the continuous spectrum, limit points of the point spectrum and the eigenvalues of infinite multiplicity ([7-9]).

To find the essential spectrum of the operator M, we will use the following property ([10]):

$$\sigma_{ess}(M) = Q \cup S,$$

where

$$Q = \left\{ \begin{array}{l} \lambda \in C : (M - \lambda I) \text{ is not elliptic in sense of} \\ \text{Douglis-Nirenberg} \end{array} \right\}$$

and

$$S = \left\{ \begin{array}{l} \lambda \in C \setminus Q : \text{the boundary conditions for the operator} \\ (M - \lambda I) \quad \text{do not satisfy Lopatinski conditions} \end{array} \right\}.$$

We recall the following two definitions.

Definition 1. Let us consider a differential matrix operator

$$L = \begin{pmatrix} l_{11} & \dots & l_{1N} \\ \dots & \dots & \dots \\ l_{N1} & \dots & l_{NN} \end{pmatrix},\ l_{ij} = \sum_{|\alpha| \le n_{ij}} a_{ij}^{(\alpha)} D^\alpha, \alpha = (\alpha_1, \dots, \alpha_n),$$

$$|\alpha| = \alpha_1 + \dots + \alpha_n,\ D_j = \frac{\partial}{\partial x_j},\ D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

Let $\{s_i\}_{i=1}^N$, $\{t_j\}_{j=1}^N$ be two sets of integer numbers such that, if $l_{ij} \ne 0$, then $n_{ij} = \deg l_{ij} \le s_i + t_j$. In case $l_{ij} = 0$, we do not require any condition for the sum $s_i + t_j$. Now, we construct the main symbol of $L(D)$ as follows.

$$\hat{L}(D) = \begin{pmatrix} \hat{l}_{11}(D) & \dots & \hat{l}_{1N}(D) \\ \dots & \dots & \dots \\ \hat{l}_{N1}(D) & \dots & \hat{l}_{NN}(D) \end{pmatrix},$$

$$\hat{l}_{ij} = \begin{cases} 0 \ \text{ if } l_{ij}(D) = 0 \text{ or } \deg l_{ij}(D) < s_i + t_j \\ \sum\limits_{|\alpha| = s_i + t_j} a_{ij}^{(\alpha)} D^\alpha \ \text{ if } \deg l_{ij}(D) = s_i + t_j \end{cases}.$$

If there exist the sets s and t which satisfy the above conditions and, additionally, if the following condition holds,

$$\det \hat{L}(\xi) \ne 0 \ \text{ for all } \xi \in R^n \setminus \{0\},$$

then the operator $L(D)$ is called *elliptic in sense of Douglis-Nirenberg* (see[11]).

Definition 2. Let us consider $\xi = (\xi_1, \xi_2, \xi_3)$, $\hat{\xi} = (\xi_1, \xi_2)$, $\hat{L}(\xi)-$ the matrix of the algebraic complements of the main symbol matrix $\hat{L}(\xi)$, $G(\xi)$ is the main symbol of the matrix $G(D)$ which defines the boundary conditions, $M^+(\hat{\xi}, \tau) = \prod(\tau - \tau_j(\hat{\xi}))$, $\tau_j(\hat{\xi})$ are the roots of the equation $\det \hat{L}(\hat{\xi}, \tau) = 0$ with positive imaginary part. If the rows of the matrix $G(\hat{\xi}, \tau)\hat{L}(\hat{\xi}, \tau)$ are linearly independent with respect to the module $M^+(\hat{\xi}, \tau)$ for $|\hat{\xi}| \ne 0$, then we will say that *the conditions of Lopatinski are satisfied* (see [10]).

Now we establish the following two theorems.

Remark 1. We note that, for the boundary condition $\vec{u} \cdot \vec{n}|_{\partial\Omega} = 0$, the results analogous to the Theorems 1 and 2, were proved in [12]. The operator M with boundary condition $p|_{\partial\Omega} = 0$ has not been considered previously.

Theorem 1. The operator M is skew-selfadjoint.

Proof. We represent M as

$$M = M_0 + B_\omega + B_N,$$

where

$$B_\omega = \begin{pmatrix} 0 & -\omega & 0 & 0 & 0 \\ \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N & 0 \\ 0 & 0 & -N & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

are anti-symmetric bounded operators. From [7] we have that it is sufficient to prove the skew-selfadjointness for the operator M_0 with the domain $D(M_0) = D(M)$. We note first that, integrating by parts, for $\hat{u}, \hat{v} \in D(M_0)$ we obtain

$$\left(M_0 \hat{u}, \hat{v}\right)_{L_2} = -\left(\hat{u}, M_0 \hat{v}\right)_{L_2}.$$

Now, let $\hat{v} \in D(M_0^*)$. It means that $\hat{v} \in L_2(\Omega)$ and there exists $\hat{f} \in L_2(\Omega)$ such that

$$\left(M_0 \hat{u}, \hat{v}\right)_{L_2} = \left(\hat{u}, \hat{f}\right)_{L_2} \text{ for all } \hat{u} \in D(M_0).$$

In particular, for $\hat{u} = (0,0,0,0,u_5)$ $\forall u_5 \in \overset{0}{W}{}_2^1(\Omega)$ we have

$$\frac{1}{\alpha}(\nabla u_5, \vec{v}) = (u_5, f_5).$$

Now, for $\hat{u} = (\vec{u}, 0, 0)$ we obtain

$$\frac{1}{\alpha}(\operatorname{div} \vec{u}, v_5) = (\vec{u}, \vec{f}).$$

Keeping in mind that $\nabla v_5 = -\alpha \vec{f}$, we integrate by parts the last identity and thus have that for any $\vec{u} \in L_2(\Omega) \big| \operatorname{div} \vec{u} \in L_2(\Omega)$, the relation holds:

$$\frac{1}{\alpha} \int_{\partial\Omega} (\vec{u} \cdot \vec{n}) v_5 ds - \frac{1}{\alpha}(\vec{u}, \nabla v_5) = (\vec{u}, \vec{f}).$$

Therefore,

$$\int_{\partial\Omega} (\vec{u} \cdot \vec{n}) v_5 ds = 0 \quad \forall \vec{u} \in L_2(\Omega) \big| \operatorname{div} \vec{u} \in L_2(\Omega),$$

from which it follows that

$$v_5\big|_{\partial\Omega} = 0,$$

which implies

$$v_5 \in \overset{0}{W}{}_2^1(\Omega).$$

Since M_0 is not acting on the fourth component of the vector $\hat{u}$, we may consider $u_4 = v_4 = f_4 = 0$.

Summing up the obtained results, we have verified that

$$D(M_0^*) \subset D(M_0).$$

The reciprocal inclusion can be proved analogously and thus the theorem is proved.

Remark 2. Since M is skew-selfadjoint, then its spectrum belongs to the imaginary axis. Indeed, $\forall \hat{v} \in D(M)$

$$\left(M\hat{v}, \hat{v}\right) + \overline{\left(M\hat{v}, \hat{v}\right)} = \left(M\hat{v}, \hat{v}\right) + \left(\hat{v}, M\hat{v}\right) = 0,$$

from which it follows that $\left(M\hat{v}, \hat{v}\right)$ is imaginary. If λ is an eigenvalue of M with the corresponding eigenfunction $\hat{v}$, then λ also is imaginary since $\lambda = \frac{(M\hat{v}, \hat{v})}{\|\hat{v}\|^2}$. If we remove from the spectrum of M all the isolated points which are eigenvalues of finite multiplicity, the remaining set will form the essential spectrum of the operator M.

2. Stratified Compressible Rotating Fluid in General Domains

Theorem 2. Let $a = \min\{\omega, N\}$, $A = \max\{\omega, N\}$. Then, the essential spectrum of M is the following symmetrical set of the imaginary axis:

$$\sigma_{ess}(M) = \{0\} \cup [-iA, -ia] \cup [ia, iA] \qquad (5)$$

Moreover, the points $\{0\}, \pm\{ia\}, \pm\{iA\}$ are eigenvalues of infinite multiplicity.

Proof. We observe that, according to [9], [11], we can choose the numbers $s_i = t_j = 0$ for $i, j = 1,2,3,4$ and $s_5 = t_5 = 1$. In this way, the main symbol of the operator $L = M - \lambda I$ is:

$$\hat{L}(\xi) = \begin{pmatrix} -\lambda & -\omega & 0 & 0 & \frac{1}{\alpha}\xi_1 \\ -\omega & -\lambda & 0 & 0 & \frac{1}{\alpha}\xi_2 \\ 0 & 0 & -\lambda & N & \frac{1}{\alpha}\xi_3 \\ 0 & 0 & -N & -\lambda & 0 \\ \frac{1}{\alpha}\xi_1 & \frac{1}{\alpha}\xi_2 & \frac{1}{\alpha}\xi_3 & 0 & 0 \end{pmatrix}.$$

We calculate the determinant of the last matrix

$$\det\left(\hat{L}\right)(\xi) = \frac{\lambda}{\alpha^2}\left[\left(\lambda^2 + N^2\right)\left(\xi_1^2 + \xi_2^2\right) + \left(\lambda^2 + \omega^2\right)\xi_3^2\right]. \qquad (6)$$

From (6) we can see that, if λ does not belong to the set (5), then L is elliptic in sense of Douglis-Nirenberg. It is easy to prove that the boundary condition $p\big|_{\partial\Omega} = 0$ satisfies Lopatinski conditions. Indeed, if we write it as $G\hat{u}\big|_{\partial\Omega} = 0$, we obtain immedeately that $G = (0,0,0,0,1)$ which is a vector row. Since $\hat{L}(\hat{\xi}, \tau)$ is a 5x5-matrix, then $G\hat{L}$ is a non-zero row with fife components and the Lopatinski condition is satisfied.

Now, let us consider the system (3) for $\lambda = 0$:

$$\begin{cases} -\omega v_2 + \frac{1}{\alpha}\frac{\partial v_5}{\partial x_1} = 0 \\ \omega v_1 + \frac{1}{\alpha}\frac{\partial v_5}{\partial x_2} = 0 \\ N v_4 + \frac{1}{\alpha}\frac{\partial v_5}{\partial x_3} = 0 \\ -N v_3 = 0 \\ \frac{1}{\alpha}\operatorname{div}\vec{v} = 0 \end{cases}.$$

Evidently, every vector-function of the form

$$\hat{v} = \left(\frac{-1}{\alpha\omega}\frac{\partial\phi}{\partial x_2}, \frac{1}{\alpha\omega}\frac{\partial\phi}{\partial x_1}, 0, \frac{-1}{\alpha N}\frac{\partial\phi}{\partial x_3}, \phi\right), \ \phi \in C_0^\infty(\Omega),$$

satisfies the last system and thus $\lambda = 0$ is an eigenvalue of infinite multiplicity. The cases $\lambda = \pm\{ia\}, \pm\{iA\}$, are analogous, for example, for $\lambda = i\omega$ the system (3) has an infinite set of solutions

$$\hat{v} = (\phi, -i\phi, 0, 0, 0),$$

where $\phi \in C_0^\infty(\Omega)$ and has the form $\phi(x) = \psi(x_1 - ix_2, x_3)$.
In this way, the theorem is proved.

Remark 3. There exists an alternative criterion of the essential spectrum which is attributed to Weyl [8]: a necessary and sufficient condition that a real finite value λ be a point of the essential spectrum of a selfadjoint operator M is that there exists a sequence of elements $v_n \in D(M)$ such that $\|v_n\| = 1$, v_n tends weakly to zero, and $\|(M - \lambda I)v_n\| \to 0$.
In the proof of the analogous result for the operator M with boundary condition $\vec{u} \cdot \vec{n}\big|_{\partial\Omega} = 0$ in [12], there was constructed the following explicit Weyl sequence for the essential spectrum of the operator M:

$$\begin{cases} v_j^k(x) = \eta_j e^{ik^3\langle x,\xi\rangle}\left(\psi_k + \dfrac{i}{k^3\xi_j}\dfrac{\partial\psi_k}{\partial x_j}\right), \; j = 1,2,3 \\ v_4^k(x) = \eta_4 \psi_k e^{ik^3\langle x,\xi\rangle} \\ v_5^k(x) = -\dfrac{i}{k^3}\psi_k e^{ik^3\langle x,\xi\rangle} \\ \langle x,\xi\rangle = x_1\xi_1 + x_2\xi_2 + x_3\xi_3, \qquad k = 1,2,... \end{cases}$$

$$\psi_k(x) = k^{\frac{3}{2}}\psi_0(k(x - x_0)), k = 1,2,...$$

$$x_0 \in \Omega, \;\; \psi_0 \in C_0^\infty(\Omega), \; \int\limits_{\|x\|\le 1} \psi_0^2(x)dx = 1,$$

where the components η_k are defined as follows

$$\begin{cases} \eta_1 = \dfrac{\lambda_0\xi_1 - \omega\xi_2}{\alpha(\lambda_0^2 + \omega^2)}, \; \eta_2 = \dfrac{\lambda_0\xi_2 + \omega\xi_1}{\alpha(\lambda_0^2 + \omega^2)}, \\ \eta_3 = \dfrac{\lambda_0\xi_3}{\lambda_0^2 + N^2}, \; \eta_4 = \dfrac{-N\xi_3}{\alpha(\lambda_0^2 + N^2)}, \; \eta_5 = 1. \end{cases}$$

$$\lambda_0 \in \pm(ia, iA) \setminus \{0\}.$$

We observe that the above Weyl sequence is also valid for the boundary condition $p\big|_{\partial\Omega} = 0$, which was proved in [13].

Remark 4. The statement of Theorem 2 corresponds clearly to the previously studied particular cases of $\omega = 0$ [9], [14], where it was proved that $\sigma_{ess}(M) = [-iN, iN]$, as well as the particular case of $N = 0$ [9], where it was proved that $\sigma_{ess}(M) = [-i\omega, i\omega]$.

Remark 5. The case of essential spectrum of stratified (non-rotational) viscous fluid was considered in [15].

3. Spectrum of Internal Waves in Rectangular Domains

We consider the boundary value problem (3) in its component representation, where, without loss of generality, we put $\alpha = 1$:

$$\begin{cases} -\lambda v_1 - \omega v_2 + \frac{\partial v_5}{\partial x_1} = 0 \\ \omega v_1 - \lambda v_2 + \frac{\partial v_5}{\partial x_2} = 0 \\ -\lambda v_3 + N v_4 + \frac{\partial v_5}{\partial x_3} = 0 \\ -N v_3 - \lambda v_4 = 0 \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \\ v_5\big|_{\partial\Omega} = 0 \end{cases}. \qquad (7)$$

Theorem 3. Let Ω be a rectangular parallelepiped in R^3 : $\Omega = [0,a] \times [0,b] \times [0,c]$. Then, the eigenfunctions v_5 of the problem (7) have the form

$$v_{5\,k,j,n}(x) = \frac{2\sqrt{2}}{\sqrt{abc}} \sin\left(\frac{\pi k x_1}{a}\right) \sin\left(\frac{\pi j x_2}{b}\right) \sin\left(\frac{\pi n x_3}{c}\right), \qquad (8)$$

$k, j, n = 1,2,3,...$

and the corresponding eigenvalues are $\{\pm\lambda_{kjn}^\pm\}$, where

$$\lambda_{kjn}^\pm = \frac{i}{\sqrt{2}}\left[\begin{array}{c} N^2 + \omega^2 + \pi^2\left(\frac{k^2}{a^2} + \frac{j^2}{b^2} + \frac{n^2}{c^2}\right) \pm \\ \pm\sqrt{\begin{array}{l}\left(N^2 + \omega^2 + \pi^2\left(\frac{k^2}{a^2} + \frac{j^2}{b^2} + \frac{n^2}{c^2}\right)\right)^2 - \\ -4\left(\omega^2\frac{\pi^2 n^2}{c^2} + N^2\left[\pi^2\left(\frac{k^2}{a^2} + \frac{j^2}{b^2}\right) + \omega^2\right]\right)\end{array}} \end{array}\right]^{\frac{1}{2}}. \qquad (9)$$

The set $\{\pm\lambda_{kjn}^+\}$ forms a discrete spectrum outside the set (5), while the set $\{\pm\lambda_{kjn}^-\}$ is dense in (5) $\setminus\{0\}$. Every point of the set (5) $\setminus\{0\}$ is a limit point of the eigenvalues (9).

Remark 6. After substituing $v_{5\,k,j,n}$ in (7), the rest of the coordinates of the eigenfunctions (v_1, v_2, v_3, v_4) can be easily found from the resulting algebraic system.

Proof. By consecutive differentiation and substitution, we can exclude the unknown functions (v_1, v_2, v_3, v_4) from the system (7)and thus obtain the following scalar equation for v_5

$$(\lambda^2 + N^2)\Delta_2 v_5 + (\lambda^2 + \omega^2)\frac{\partial^2 v_5}{\partial x_3^2} - (\lambda^2 + \omega^2)(\lambda^2 + N^2)v_5 = 0 \quad (10)$$

with the boundary condition $v_5\big|_{\partial\Omega} = 0$, where $\Delta_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

We put $\lambda = i\eta$ and solve the problem (1)0 by using the separation of variables

$$v_5(x) = w(x')z(x_3), \qquad (11)$$

where $x' = (x_1, x_2)$.

Therefore, we obtain

$$\frac{(N^2 - \eta^2)}{(\eta^2 - \omega^2)}\frac{\Delta_2 w}{w} + (N^2 - \eta^2) = \frac{\frac{d^2 z}{dx_3^2}}{z} = -\mu. \qquad (12)$$

From (12), we solve first the problem for the function $z(x_3)$

$$\begin{cases} \frac{d^2 z}{dx_3^2} + \mu z = 0 \\ z(0) = z(c) = 0 \end{cases}. \qquad (13)$$

The solutions of the problem (13) are

$$\begin{cases} \mu_n = \left(\frac{\pi n}{c}\right)^2 \\ z_n(x_3) = \sqrt{\frac{2}{c}}\sin\left(\frac{\pi n x_3}{c}\right), \ n = 1,2,3,... \end{cases}.$$

For the function $w(x')$, we obtain the problem

$$\begin{cases} \Delta_2 w + \beta w = 0 \\ w\big|_{[0,a]\times[o,b]} = 0 \end{cases}, \tag{14}$$

where

$$\beta = \frac{(\eta^2-\omega^2)}{(N^2-\eta^2)}\left[\left(\frac{\pi n}{c}\right)^2 + (N^2-\eta^2)\right].$$

It is easy to see that the solutions of the problem (14) are

$$\begin{cases} \beta_{kj} = \pi^2\left(\frac{k^2}{a^2}+\frac{j^2}{b^2}\right) \\ w_{kj}(x_1,x_2) = \frac{2}{\sqrt{ab}}\sin\left(\frac{\pi k x_1}{a}\right)\sin\left(\frac{\pi j x_2}{b}\right), k,j = 1,2,3,... \end{cases}.$$

Thus we conclude that the eigenvalues of the problem (10) and (7) are found from the equation

$$(\eta^2-\omega^2)\left[\left(\frac{\pi n}{c}\right)^2 + (N^2-\eta^2)\right] = (N^2-\eta^2)\pi^2\left(\frac{k^2}{a^2}+\frac{j^2}{b^2}\right), \tag{15}$$

which can be written as

$$\eta^4 - \left\{\pi^2\left(\frac{k^2}{a^2}+\frac{j^2}{b^2}+\frac{n^2}{c^2}\right)+N^2+\omega^2\right\}\eta^2 + \\ +\omega^2\frac{\pi^2 n^2}{c^2} + N^2\left[\pi^2\left(\frac{k^2}{a^2}+\frac{j^2}{b^2}\right)+\omega^2\right] = 0.$$

The roots of the last equation are

$$2\eta^2 = N^2+\omega^2+\pi^2\left(\frac{k^2}{a^2}+\frac{j^2}{b^2}+\frac{n^2}{c^2}\right) \pm \\ \pm\sqrt{\left(N^2+\omega^2+\pi^2\left(\frac{k^2}{a^2}+\frac{j^2}{b^2}+\frac{n^2}{c^2}\right)\right)^2 - 4\left(\omega^2\frac{\pi^2 n^2}{c^2}+N^2\left[\pi^2\left(\frac{k^2}{a^2}+\frac{j^2}{b^2}\right)+\omega^2\right]\right)}. \tag{16}$$

Keeping in mind that $\lambda = i\eta$, we obtain the eigenvalues (9).

The eigenfunctions $v_{5\,k,j,n}(x)$, according to the separation of variables (11), are represented by

$$v_{5\,k,j,n}(x) = \frac{2\sqrt{2}}{\sqrt{abc}}\sin\left(\frac{\pi k x_1}{a}\right)\sin\left(\frac{\pi j x_2}{b}\right)\sin\left(\frac{\pi n x_3}{c}\right), \\ k,j,n = 1,2,3,...$$

We note that the sign " $\pm$ " before λ in $\{\pm\lambda^{\pm}_{kjn}\}$ means that the spectrum is symmetrical with respect to zero.

Evidently, the subset $\{\pm\lambda^{+}_{kjn}\}$ forms a discrete spectrum on the imaginary axis outside the set (5).

Let us show that the subset $\{\pm\lambda^{-}_{kjn}\}$ is dense in (5) $\setminus\{0\}$ and thus every point of (5) $\setminus\{0\}$ is a limit point of the eigenvalues (9). For that, we will consider a positive function

$$f_L(Q) = \frac{1}{2}\left(N^2+\omega^2+L+Q - \sqrt{(N^2+\omega^2+L+Q)^2 - 4\left[\omega^2 L + N^2(Q+\omega^2)\right]}\right). \tag{17}$$

We observe that for $L = \frac{\pi^2 n^2}{c^2}$ and $Q = \pi^2\left(\frac{k^2}{a^2}+\frac{j^2}{b^2}\right)$ we have that $f_L(Q) = \eta^2_{kjn}$, where η^2_{kjn} are defined in (16).

We also represent the function $f_L(Q)$ as

$$f_L(Q) = \frac{2\left[\omega^2 L + N^2(Q+\omega^2)\right]}{N^2+\omega^2+L+Q+\sqrt{(N^2+\omega^2+L+Q)^2 - 4\left[\omega^2 L + N^2(Q+\omega^2)\right]}}. \tag{18}$$

Using (18), for L and Q sufficiently large, we can estimate $f_L(Q)$ as

$$F_L(Q) \le f_L(Q) \le G_L(Q), \tag{19}$$

where

$$F_L(Q) = \frac{\omega^2 L + N^2 Q + N^2\omega^2}{N^2+\omega^2+L+Q},$$

$$G_L(Q) = \frac{2\left[\omega^2 L + N^2 Q + N^2\omega^2\right]}{N^2+\omega^2+L+Q+\left|N^2+L-\omega^2-Q\right|}.$$

For fixed L, we have

$$\lim_{Q\to\infty} f_L(Q) = N^2. \tag{20}$$

On the other hand, for fixed Q, we have

$$\lim_{L\to\infty} f_L(Q) = \omega^2. \tag{21}$$

Evidently, the properties (20), (21) are also valid for the functions $F_L(Q)$ and $G_L(Q)$.

Now, if we denote $a = \min\{\omega, N\}$, $A = \max\{\omega, N\}$, then we can easily see that, for sufficiently large L and Q, the values of the functions $F_L(Q)$ and $G_L(Q)$ (and thus the values of the function $f_L(Q)$) will belong to the interval $[a^2, A^2]$. Additionally, it can be easily seen that every point of the interval $[a^2, A^2]$ can be represented as a limit point of the functions $F_L(Q)$ and $G_L(Q)$ (and thus as a limit point of the function $f_L(Q)$) for appropriate election of $L,Q\to\infty$.

Indeed, for example, let $\omega < N$, $p \in [\omega^2, N^2]$, $p = \omega^2 + \delta(N^2-\omega^2)$, $0<\delta<1$. We will show, for example, that for suitable election of $L,Q\to\infty$ for arbitrary small $\varepsilon > 0$, the estimate will hold:

$$0 < p - F_L(Q) < \varepsilon. \tag{22}$$

Indeed,

$$p - F_L(Q) = \omega^2(1-\delta) + N^2\delta - \frac{\omega^2 L + N^2 Q + N^2\omega^2}{N^2+\omega^2+L+Q} =$$

$$=\frac{\left[N^2-\omega^2\right]\left[\delta L-(1-\delta)Q\right]+\delta N^4+(1-\delta)\omega^2}{N^2+\omega^2+L+Q}. \quad (23)$$

In (23) we choose $L=\left(\frac{1-\delta}{\delta}\right)Q$ and thus we have

$$p-F_L(Q)=\frac{\delta N^4+(1-\delta)\omega^2}{N^2+\omega^2+\frac{Q}{\delta}}. \quad (24)$$

From (24) we obtain that for every arbitrarily small $\varepsilon>0$ there exists sufficiently large $Q>0$ such that (22) will hold, where $L=\left(\frac{1-\delta}{\delta}\right)Q$. Therefore, every point of the interval $\left[\omega^2,N^2\right]$ is a limit point of the function $F_L(Q)$ for $L,Q\to\infty$. The same property for the function $G_L(Q)$ can be verified analogously and thus the Theorem is proved.

4. Spectrum of Internal Waves in Circular Cylinders and in General Cylinders

Theorem 4. Let Ω be a circular cylinder in R^3 :

$$\Omega=\left\{(x_1,x_2,x_3)\,:\,x_1^2+x_2^2\le R^2,\,0\le x_3\le c\right\}.$$

Then, the eigenfunctions v_5 of the problem (7) have the form

$$v_{5\,k,j,n}(x)=\frac{\sqrt{2}}{R\left|J_k'\left(\gamma_j^k\right)\right|\sqrt{\pi c}}J_k\left(\frac{\gamma_j^k r}{R}\right)\sin\left(\frac{\pi n x_3}{c}\right)\exp(ik\phi)\ ,$$

$$k,j,n=1,2,3,...\quad r=\sqrt{x_1^2+x_2^2}\ ,\ \phi=\arctan\left(\frac{x_2}{x_1}\right), \quad (25)$$

γ_j^k are positive roots of the Bessel function $J_k(\gamma)$

and the corresponding eigenvalues are $\left\{\pm\lambda_{kjn}^{\pm}\right\}$, where

$$\lambda_{kjn}^{\pm}=\frac{i}{\sqrt{2}}\left[\begin{array}{c}N^2+\omega^2+\frac{\pi^2n^2}{c^2}+\frac{\left(\gamma_j^k\right)^2}{R^2}\pm\\ \pm\sqrt{\left(N^2+\omega^2+\frac{\pi^2n^2}{c^2}+\frac{\left(\gamma_j^k\right)^2}{R^2}\right)^2-4\left(\omega^2\frac{\pi^2n^2}{c^2}+N^2\left[\frac{\left(\gamma_j^k\right)^2}{R^2}+\omega^2\right]\right)}\end{array}\right]^{\frac{1}{2}}. \quad (26)$$

The set $\left\{\pm\lambda_{kjn}^{+}\right\}$ forms a discrete spectrum outside the set (5), while the set $\left\{\pm\lambda_{kjn}^{-}\right\}$ is dense in (5) $\backslash\{0\}$. Every point of the set (5) $\backslash\{0\}$ is a limit point of the eigenvalues (26).

Proof. As in Theorem 3, we solve the problem (10) using the separation of variables (11). For the function $z(x_3)$ we have

$$z_n(x_3)=\sqrt{\frac{2}{c}}\sin\left(\frac{\pi n x_3}{c}\right),\ n=1,2,3,... \quad (27)$$

For $w(x')$ we obtain the boundary value problem

$$\begin{cases}\Delta_2 w+\beta w=0\\ w\big|_{|x'|=R}=0\end{cases}, \quad (28)$$

where

$$\beta=\frac{\left(\eta^2-\omega^2\right)}{\left(N^2-\eta^2\right)}\left[\left(\frac{\pi n}{c}\right)^2+\left(N^2-\eta^2\right)\right].$$

Solving the problem (28) in polar coordinates, we have that for

$$\beta_{kj}=\frac{\left(\gamma_j^k\right)^2}{R^2}\ ,\ k,j=1,2,3,...$$

the solutions have the form

$$w_{k,j}(r,\phi)=\frac{J_k\left(\frac{\gamma_j^k r}{R}\right)\exp(ik\phi)}{\sqrt{\pi}R\left|J_k'\left(\gamma_j^k\right)\right|}, \quad (29)$$

where γ_j^k are positive roots of the Bessel function $J_k(\gamma)$.

In this case, the eigenvalues of the problem (10) are found from the equation

$$\left(\eta^2-\omega^2\right)\left[\left(\frac{\pi n}{c}\right)^2+\left(N^2-\eta^2\right)\right]=\left(N^2-\eta^2\right)\frac{\left(\gamma_j^k\right)^2}{R^2}.$$

Therefore, the eigenvalues of the problem (10) (and also of the problem (7)) are:

$$\lambda_{kjn}^{\pm}=\frac{i}{\sqrt{2}}\left[\begin{array}{c}N^2+\omega^2+\frac{\pi^2n^2}{c^2}+\frac{\left(\gamma_j^k\right)^2}{R^2}\pm\\ \pm\sqrt{\left(N^2+\omega^2+\frac{\pi^2n^2}{c^2}+\frac{\left(\gamma_j^k\right)^2}{R^2}\right)^2-4\left(\omega^2\frac{\pi^2n^2}{c^2}+N^2\left[\frac{\left(\gamma_j^k\right)^2}{R^2}+\omega^2\right]\right)}\end{array}\right]^{\frac{1}{2}}.$$

From (11), (27) and (29) we have that the corresponding eigenfunctions have the form:

$$v_{5\,k,j,n}(x)=\frac{\sqrt{2}}{R\left|J_k'\left(\gamma_j^k\right)\right|\sqrt{\pi c}}J_k\left(\frac{\gamma_j^k r}{R}\right)\sin\left(\frac{\pi n x_3}{c}\right)\exp(ik\phi).$$

To prove the properties that the set $\left\{\pm\lambda_{kjn}^{-}\right\}$ is dense in (5) $\backslash\{0\}$ and that every point of the set (5) $\backslash\{0\}$ is a limit point of the eigenvalues (26), we can follow exactly the reasoning of the proof of Theorem 3, using the same functions $F_L(Q)$, $f_L(Q)$ and $G_L(Q)$ with $L=\frac{\pi^2n^2}{c^2}$ and $Q=\frac{\left(\gamma_j^k\right)^2}{R^2}$. We only need the fact that γ_j^k be infinite, countable, do not have finite limit points and posess the property

$$\forall k\lim_{j\to\infty}\gamma_j^k=\infty. \quad (30)$$

For the Bessel functions $J_k\left(\frac{\gamma_j^k r}{R}\right),$ these properties, including (30), as well as the properties of orthogonality and completeness in L_2 , are established, for example, in [16].

Additionally, not only for the circle but also for more general domains, it is well known that for the problem

$$\begin{cases}\Delta w+\beta w=0\\ w\big|_{\partial\Omega}=0\end{cases}$$

the same properties are valid (see, for example, [17]). In particular, the eigenvalues are positive, tend to infinity, have finite multiplicity and do not have finite limit points. The

eigenfunctions form a complete system in L_2 and can be chosen orthogonal.
In this way, the Theorem is proved.

Remark 7. The reasoning we used in the proofs of the Theorems 3 and 4 may be easily extended to the cases of more general domains of arbitrary cylinders

$$C=\left\{x\in R^3\ :\ \left(x_1,x_2\right)\in\Omega\subset R^2,\ 0\le x_3\le c\right\}.$$

Remark 8. For arbitrary domains $G\subset R^2$ with Lipshchitz boundary , the property of positiveness of the eigenvalues and orthogonality of the eigenfunctions for the problem

$$\begin{cases}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\lambda v=0\ ,\ \left(x,y\right)\in G\\ \left.v\right|_{\partial G}=0\end{cases},\tag{31}$$

can be deduced directly from the following simple calculations. Indeed, let v be a solution of (31). Integrating the identity

$$\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2=\frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(v\frac{\partial v}{\partial y}\right)-v\Delta v$$

in the domain G , we have

$$\int_G\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]dxdy=$$
$$=\int_{\partial G}v\frac{\partial v}{\partial\vec{n}}ds-\int_G v\Delta v dxdy=\lambda\int_G v^2dxdy,$$

from which we obtain the positiveness of λ . Now, if $\lambda_k\ne\lambda_m$ are eigenvalues of (31) with the corresponding eigenfunctions v_k and v_m , then, after integrating the equation

$$\frac{\partial}{\partial x}\left(v_k\frac{\partial v_m}{\partial x}-v_m\frac{\partial v_k}{\partial x}\right)+\frac{\partial}{\partial y}\left(v_k\frac{\partial v_m}{\partial y}-v_m\frac{\partial v_k}{\partial y}\right)=$$
$$=v_k\Delta v_m-v_m\Delta v_k$$

in the domain G , we obtain

$$\int_G\left[v_k\Delta v_m-v_m\Delta v_k\right]dxdy=\left(\lambda_k-\lambda_m\right)\int_G v_kv_mdxdy=0.$$

5. Spectrum of Internal Waves in Spherical Volumes

Theorem 5. Let Ω be a spherical volume in R^3 :
$\Omega=\left\{\left(x_1,x_2,x_3\right)\ :\ x_1^2+x_2^2+x_3^2\le R^2\right\}.$

Then, the eigenfunctions v_5 of the problem (7) have the form

$$v_{5\,k,j,n}\left(x\right)=\frac{c_{njm}}{\sqrt{r}}J_{n+\frac{1}{2}}\left(\frac{\mu_j^{\left(n+\frac{1}{2}\right)}r}{R}\right)P_n\left(\cos\vartheta\right)$$

n=0,1,... j=1,2,... $m=0,\pm1,...,\pm n$,

$\mu_j^{\left(n+\frac{1}{2}\right)}$ are positive roots of the Bessel function $J_{n+\frac{1}{2}}\left(\mu\right)$

$P_n\left(\xi\right)$ are Legendre polynomials $P_n\left(\xi\right)=\dfrac{1}{2^nn!}\dfrac{d^n\left(\xi^2-1\right)^n}{d\xi^n}$, (32)

and the constants c_{njm} are chosen such that the normalization condition holds:

$$\frac{1}{c_{njm}}=\sqrt{\int_0^R\int_0^\pi\int_0^{2\pi}J^2_{n+\frac{1}{2}}\left(\mu_j^{\left(n+\frac{1}{2}\right)}\frac{r}{R}\right)P_n^2\left(\cos\vartheta\right)rdrd\vartheta d\phi}=$$
$$=R\left|J'_{n+\frac{1}{2}}\left(\mu_j^{\left(n+\frac{1}{2}\right)}\right)\right|\sqrt{\pi\left(\frac{1+\delta_{nm}}{2l+1}\right)\frac{\left(l+|m|\right)!}{\left(l-|m|\right)!}}$$

and the corresponding eigenvalues are $\left\{\pm\lambda_{nj}^{\pm}\right\}$, where

$$\lambda_{nj}^{\pm}=\frac{i}{\sqrt{2}}\left[2N^2+\frac{n(n+1)}{R^2}+\frac{\left(\mu_j^{\left(n+\frac{1}{2}\right)}\right)^2}{R^2}\pm\sqrt{\left(2N^2+\frac{n(n+1)}{R^2}+\frac{\left(\mu_j^{\left(n+\frac{1}{2}\right)}\right)^2}{R^2}\right)^2-4\left(\omega^2\frac{\left(\mu_j^{\left(n+\frac{1}{2}\right)}\right)^2}{R^2}+N^2\left[\frac{n(n+1)}{R^2}+\omega^2\right]\right)}\right]^{\frac{1}{2}}.\tag{33}$$

The set $\left\{\pm\lambda_{nj}^{+}\right\}$ forms a discrete spectrum outside the set (5), while the set $\left\{\pm\lambda_{nj}^{-}\right\}$ is dense in (5) $\backslash\{0\}$. Every point of the set (5) $\backslash\{0\}$ is a limit point of the eigenvalues (32).

Proof. We use the spherical coordinates

$$x_1=r\sin\vartheta\cos\varphi$$
$$x_2=r\sin\vartheta\sin\varphi$$
$$x_3=r\cos\vartheta.$$

Assuming that $v_5\left(x\right)=v_5\left(r,\vartheta\right)$ does not depend on φ, we solve the problem (10) using the separation of variables $v_5\left(r,\vartheta\right)=z\left(r\right)Y\left(\vartheta\right)$. In this way, for the function $Y\left(\vartheta\right)$ we have the problem

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dY}{d\vartheta}\right)+\mu Y=0\ .\tag{34}$$

After the substitution $\xi=\cos\vartheta$ $y=Y(\arccos\xi)$ we obtain that (34) has a bounded solution only if $\mu=\mu_n=n\left(n+1\right)$, and that the solutions of (34) are Legendre polynomials $P_n\left(\xi\right)=\dfrac{1}{2^nn!}\dfrac{d^n\left(\xi^2-1\right)^n}{d\xi^n}$.

For $z(r)$ we obtain the boundary value problem

$$\begin{gathered}\left(r^2z'\right)'+\left(A^2r^2-\mu_n\right)z=0\\ \mu_n=n\left(n+1\right),\quad\left|z(0)\right|<\infty,\quad z\left(R\right)=0\end{gathered},$$

which has the solutions expressed in terms of the Bessel functions ([17]): $z_n\left(R\right)=\dfrac{1}{\sqrt{r}}J_{n+\frac{1}{2}}\left(Ar\right)$. Proceeding analogously as in the proof of Theorem 4, we obtain that the eigenvalues are found from the equation

$$\left(\eta^2-\omega^2\right)\left[\frac{n(n+1)}{R^2}+\left(N^2-\eta^2\right)\right]=\left(N^2-\eta^2\right)\frac{\left[\mu_j^{\left(n+\frac{1}{2}\right)}\right]^2}{R^2}.$$

Keeping in mind that $\lambda=i\eta$, we obtain finally that the eigenvalues have the form (33). From the orthogonality and completeness of Bessel functions in $L_2\left((0,R)\right)$ and Legendre polynomials in $L_2\left(S_1\right)$, we obtain the useful property that the found set of the eigenfunctions (32) is complete and orthonormal in $L_2(\Omega)$. The rest of the proof is totally analogous to the proofs of the Theorems 3 and 4.

6. Conclusion

For the considered particular cases of parallelepipeds, cylinders and spheres, the explicitly calculated spectrum clearly corresponds to the essential spectrum for general domains.

The constructed systems of eigenfunctions (8), (25) and (32) are complete and orthonormal in $L_2(\Omega)$, which can be used for solving more general problems in various applications modelling rotating stratified comressible fluid.

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