Determination of Thermal Conductivities in Multilayer Materials

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Abstract: The objective of this work is the determination of the materials that make up a three-layer body, based on the simultaneous estimation of the thermal conductivity of the material of each layer. The body is exposed to a one-dimensional stationary, non-invasive, heat transfer process. It is assumed that the union of each pair of consecutive materials does not present thermal resistance. The parameters to be determined are estimated using three temperature measurements, one at each interface and another at the right edge of the body. The estimation is calculated analytically and a bound is given for the estimation error. In addition, an elasticity analysis is carried out to analyze the local dependence of each estimated parameter with respect to the data. A numerical example is included to illustrate and discuss the method proposed here.

Key–Words: Heat equation, solid-solid interface, inverse problems.

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1 Introduction

The determination of the thermal conductivity in heat transfer processes has several applications, for instance, in optimal control design of thermal processes. The thermal conductivity is a fundamental property that has a determining influence on the temperature distribution and heat flux density during thermal heating or cooling processes.

The estimation of thermal conductivity in heat transfer processes has been widely addressed, during the last decades. It was mainly studied using numerical techniques of inverse problems, see for example [3],[7],[14],[16]. In [4] the estimation was carried out under particular conditions using the conjugate gradient method. In [7] a finite difference method was used while in [18] an inverse linear model is proposed to estimate the temperature dependence of thermal conductivity. On the other hand, in [19] different iterative methods are used. Particular problems of estimation of the thermal conductivity coefficient that take into account multidimensional, inhomogeneous and/or composite materials in [2], [5], [6], [8], [9], [16], [17]. Other interesting estimation strategies applied to phase change materials can be seen in [10]-[13].

This work deals with the simultaneous determination of the thermal conductivity coefficients, namely κ_A, κ_B, κ_C $[W/m°C]$ of three materials, A, B and C, that compose a three-layer body. The simultaneous estimation of the parameters is performed based on three noisy temperature data; one at each interface and one at the right edge of the body.

2 Mathematical Framework

The problem to be analyzed can be considered as a stationary, one-dimensional transport process of thermal energy. For this reason, the multilayer material is modeled as a bar built with three consecutive sections of homogeneous and isotropic materials, A, B and C, so that the thermal diffusivity coefficients α_A^2, α_B^2, α_C^2 $[m^2/s]$ are assumed to be constant. The left part of the body (material A) has a length l_1 $[m]$; the middle section (material B) has a length $l_2 - l_1$ $[m]$ and the last section (material C) has a length $L - l_2$ $[m]$.

On the other hand, it is assumed that the union of the materials is perfectly assembled, i.e. no cracks or roughness is present, so there is no thermal resistance at the interfaces. Hence, continuity conditions for temperature and heat flow are considered at the solid-solid interfaces.

It is also assumed that the temperature at the left edge of the body is kept constant, at temperature F $[^\circ C]$ and the right edge remains free, in contact with the fluid, giving rise to the phenomenon of convection.

The problem described above can be modeled with the following system:

$$\begin{cases} u''(x) = 0, & 0 < x < l_1, \\ u''(x) = 0, & l_1 < x < l_2, \\ u''(x) = 0, & l_2 < x < L, \\ u(x) = F, & x = 0, \\ u(x^-) = u(x^+), & x = l_1, \\ u(x^-) = u(x^+), & x = l_2, \\ \kappa_A\, u'(x^-) = \kappa_B\, u'(x^+), & x = l_1, \\ \kappa_B\, u'(x^-) = \kappa_C\, u'(x^+), & x = l_2, \\ \kappa_C\, u'(x) = -h\,(u(x) - T_a), & x = L, \end{cases} \tag{1}$$

where u $[^\circ C]$ represents the stationary temperature, h $[W/(m^{2\circ}C)]$ the coefficient of heat transfer by convection, T_a $[^\circ C]$ the temperature outside the body and

$$\begin{cases} u(l_i^-) = \lim\limits_{x \to l_i^-} u(x), \\ u(l_i^+) = \lim\limits_{x \to l_i^+} u(x). \end{cases} \tag{2}$$

The analytical solution of the problem described by equations (1)-(2) is given in the following result:

Theorem 1. *Given* $\kappa_A, \kappa_B, \kappa_C, T_a, F, L, l_1, l_2, h \in \mathbb{R}^+$ *such that* $F > T_a$, $L > l_2 > l_1$ *and* $u(\cdot) \in C^2\left((0, l_1) \cup (l_1, l_2) \cup (l_2, L)\right)$, *the elliptic problem* (1)-(2) *has a unique solution given by:*

$$u(x) = \begin{cases} F + \zeta x, \\ F + \zeta \left[d_1\, l_1 + \dfrac{\kappa_A}{\kappa_B} x \right], \\ F + \zeta \left[d_1\, l_1 + d_2\, l_2 + \dfrac{\kappa_A}{\kappa_C} x \right], \end{cases} \tag{3}$$

where the domains of each line are: $0 \leq x \leq l_1$, $l_1 \leq x \leq l_2$, $l_2 \leq x \leq L$, *respectively, and*

$$d_1 = 1 - \frac{\kappa_A}{\kappa_B}, \tag{4}$$

$$d_2 = \frac{\kappa_A}{\kappa_B} - \frac{\kappa_A}{\kappa_C}, \tag{5}$$

$$\zeta = -\frac{F - T_a}{L\, \zeta_0}, \tag{6}$$

with

$$\zeta_0 = \frac{\kappa_A}{hL} + d_1 \frac{l_1}{L} + d_2 \frac{l_2}{L} + \frac{\kappa_A}{\kappa_C}. \tag{7}$$

Proof. The theorem can be easily proved following the idea developed in [17]. Observe that ζ_0 is strictly positive, which assures the existence and uniqueness of the solution. □

3 Estimation of thermal conductivities

The solution to the forward problem, given by (3)-(7), allows us to approach the estimation problem.

3.1 Determination of the parameters

In this subsection, an analytical expression for the solution of the estimation problem is obtained from three temperature measurements, one at each interface (T_1 and T_2) and another at the right edge of the body (T_3).

Theorem 2. *Given* κ_A, κ_B, κ_C, T_a, F, l_1, l_2, L, h,T_1,T_2,$T_3 \in \mathbb{R}^+$ *such that,* $0 < l_1 < l_2 < L$ *and*

$$T_a < T_3 < T_2 < T_1 < F. \tag{8}$$

and the temperature function u *that satisfies* $u(\cdot) \in C^2\left((0, l_1) \cup (l_1, l_2) \cup (l_2, L)\right)$, *the solution to the problem of determining the thermal conductivities* κ_A, κ_B *and* κ_C *in the system* (1) *subjected to the overconditions*

$$\begin{cases} T_1 = u(x), & x = l_1, \\ T_2 = u(x), & x = l_2, \\ T_3 = u(x), & x = L, \end{cases} \tag{9}$$

is

$$\begin{cases} \kappa_A = h\, l_1 \dfrac{T_3 - T_a}{F - T_1}, \\ \kappa_B = h\,(l_2 - l_1) \dfrac{T_3 - T_a}{T_1 - T_2}, \\ \kappa_C = h\,(L - l_2) \dfrac{T_3 - T_a}{T_2 - T_3}. \end{cases} \tag{10}$$

Proof. Consider the system (1). Theorem 1 provides an explicit analytical relationship between the temperature function u and the physical parameters of the model, given by the expressions (3)-(7). Applying the conditions given in (9), it follows that

$$T_1 = F + \zeta\, l_1, \tag{11}$$

$$T_2 = F + \zeta \left[d_1\, l_1 + \frac{\kappa_A}{\kappa_B}\, l_2 \right], \tag{12}$$

$$T_3 = F + \zeta \left[d_1\, l_1 + d_2\, l_2 + \frac{\kappa_A}{\kappa_C}\, L \right]. \tag{13}$$

From expressions (11)-(13) it results

$$\vartheta_1 = \frac{\kappa_A}{\kappa_B} = \frac{T_2 - T_1}{T_1 - F}\frac{l_1}{l_2 - l_1}, \tag{14}$$

$$\vartheta_2 = \frac{\kappa_A}{\kappa_C} = \frac{T_3 - T_2}{T_1 - F}\frac{l_1}{L - l_2}. \tag{15}$$

Replacing the expressions (14) and (15) in (6)-(7), it follows

$$\zeta = -\frac{F - T_a}{\frac{\kappa_A}{h} + l_1(1 - \vartheta_1) + l_2(\vartheta_1 - \vartheta_2) + \vartheta_2\, L}, \tag{16}$$

Finally, (10) is derived by substituting (16) in equations (11)-(13). □

3.2 Error estimate

An analytical expression is obtained for a bound of the estimation error of the thermal conductivities κ_A, κ_B and κ_C, when using three noisy temperature data T_1^ϵ, T_2^ϵ and T_3^ϵ, assuming

$$\begin{cases} |T_1 - T_1^\epsilon| \le \epsilon\,(F - T_a), \\ |T_2 - T_2^\epsilon| \le \epsilon\,(F - T_a), \\ |T_3 - T_3^\epsilon| \le \epsilon\,(F - T_a), \end{cases} \tag{17}$$

where $\epsilon > 0$ (small enough) denotes the noise level.

Theorem 3. *The inverse problem of simultaneous determination of the thermal conductivities κ_A, κ_B and κ_C from* (1)*,* (8) *and* (9) *is considered. Let $\widehat{\kappa_A}$, $\widehat{\kappa_B}$ and $\widehat{\kappa_C}$ be the approximated solutions that depend on noisy temperature measurements T_1^ϵ at $x = l_1$, T_2^ϵ at $x = l_2$, and T_3^ϵ at $x = L$, that satisfy the condition* (17)*.*

There exist dimensionless constants $M_1, M_2, M_3 \in (0,1)$ such that

$$M_1 \le \frac{F - T_1}{F - T_a}, \quad M_2 \le \frac{T_1 - T_2}{F - T_a}, \quad M_3 \le \frac{T_2 - T_3}{F - T_a}, \tag{18}$$

that satisfy

$$|\kappa_A - \widehat{\kappa_A}| \le \frac{2\,h\,l_1}{M_1(M_1 - \epsilon)}\,\epsilon, \tag{19}$$

$$|\kappa_B - \widehat{\kappa_B}| \le \frac{3\,h\,(l_2 - l_1)}{M_2(M_2 - 2\epsilon)}\,\epsilon, \tag{20}$$

and

$$|\kappa_C - \widehat{\kappa_C}| \le \frac{3\,h\,(L - l_2)}{M_3(M_3 - 2\epsilon)}\,\epsilon, \tag{21}$$

for

$$0 < \epsilon < \min\left\{ M_1, \frac{M_2}{2}, \frac{M_3}{2} \right\}. \tag{22}$$

Proof. By using the noisy temperature data T_1^ϵ, T_2^ϵ and T_3^ϵ in (10) it results

$$\begin{cases} \widehat{\kappa_A} = h\,l_1 \dfrac{T_3^\epsilon - T_a}{F - T_1^\epsilon}, \\ \widehat{\kappa_B} = h\,(l_2 - l_1)\dfrac{T_3^\epsilon - T_a}{T_1^\epsilon - T_2^\epsilon}, \\ \widehat{\kappa_C} = h\,(L - l_2)\dfrac{T_3^\epsilon - T_a}{T_2^\epsilon - T_3^\epsilon}. \end{cases} \tag{23}$$

From (10) and (23), the estimation errors are obtained as follows

$$|\kappa_A - \widehat{\kappa_A}| = h\,l_1 \left| \frac{T_3 - T_a}{F - T_1} - \frac{T_3^\epsilon - T_a}{F - T_1^\epsilon} \right|. \tag{24}$$

$$|\kappa_B - \widehat{\kappa_B}| = h\,(l_2 - l_1) \left| \frac{T_3 - T_a}{T_1 - T_2} - \frac{T_3^\epsilon - T_a}{T_1^\epsilon - T_2^\epsilon} \right|, \tag{25}$$

$$|\kappa_C - \widehat{\kappa_C}| = h\,(L - l_2) \left| \frac{T_3 - T_a}{T_2 - T_3} - \frac{T_3^\epsilon - T_a}{T_2^\epsilon - T_3^\epsilon} \right|. \tag{26}$$

Adding and subtracting $T_1^\epsilon\,T_3^\epsilon$ in the numerator and rewriting the expression in terms of the errors in the data, (24) can be expressed by

$$\frac{|\kappa_A - \widehat{\kappa_A}|}{h\,l_1} = \left| \frac{(F - T_1^\epsilon)(T_3^\epsilon - T_3) + (T_1 - T_1^\epsilon)(T_3^\epsilon - T_a)}{(F - T_1)(F - T_1^\epsilon)} \right|$$

and by the triangular inequality it follows that

$$\frac{|\kappa_A - \widehat{\kappa_A}|}{h\,l_1} \leq \frac{|T_3^\epsilon - T_3|\,|F - T_1^\epsilon| + |T_1 - T_1^\epsilon|\,|T_3^\epsilon - T_a|}{(F - T_1)\,|F - T_1^\epsilon|},$$

and assumptions (17) lead to

$$\frac{|\kappa_A - \widehat{\kappa_A}|}{h\,l_1} \leq \frac{F - T_a}{F - T_1}\left(1 + \frac{|T_3^\epsilon - T_a|}{|F - T_1^\epsilon|}\right)\epsilon. \quad (27)$$

Note that

$$T_1^\epsilon \leq T_1 + \epsilon(F - T_a) \Longrightarrow F - T_1^\epsilon \geq F - T_1 - \epsilon(F - T_a),$$

and $F - T_1 - \epsilon(F - T_a) > 0$ for ϵ sufficiently small. Hence, from expresions (8), (17) and (27) it follows that

$$\begin{aligned}\frac{|\kappa_A - \widehat{\kappa_A}|}{h\,l_1} &\leq \frac{F - T_a}{F - T_1}\left(1 + \frac{|T_3^\epsilon - T_3| + |T_3 - T_a|}{F - T_1 - \epsilon(F - T_a)}\right)\epsilon,\\ &\leq \frac{F - T_a}{F - T_1}\left(1 + \frac{(1+\epsilon)\,(F - T_a)}{F - T_1 - \epsilon(F - T_a)}\right)\epsilon,\\ &\leq \frac{F - T_a}{F - T_1}\left(\frac{F - T_1 + F - T_a}{F - T_1 - \epsilon(F - T_a)}\right)\epsilon,\\ &\leq 2\,\frac{F - T_a}{F - T_1}\left(\frac{F - T_a}{F - T_1 - \epsilon(F - T_a)}\right)\epsilon.\end{aligned}$$

Since $0 < F - T_1 < F - T_a$ then $0 < \dfrac{F - T_1}{F - T_a} < 1$ and so, there exists a constant $M_1 \in (0, 1)$ that satisfies

$$M_1 \leq \frac{F - T_1}{F - T_a},$$

yielding (19) for $\epsilon < M_1$. Analogously, there exist constants $M_2, M_3 \in (0, 1)$ satisfying (18). Finally, from (25)-(26) we obtain (20) and (21) for $\epsilon < \dfrac{1}{2}\min\{M_2, M_3\}$. □

Remark 4. *Expressions* (19)-(21) *indicate that if* $\epsilon \longrightarrow 0$ *then the estimation errors satisfy* $|\kappa_A - \widehat{\kappa_A}| \longrightarrow 0$, $|\kappa_B - \widehat{\kappa_B}| \longrightarrow 0$ *and* $|\kappa_C - \widehat{\kappa_C}| \longrightarrow 0$.

Remark 5. *Note that the condition* (22) *means that precise measurements are required in order to obtain the above results.*

4 Elasticity analysis

The local relationships between the estimated parameters $(\widehat{\kappa_A}, \widehat{\kappa_B}, \widehat{\kappa_C})$ and the data $(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon)$ used for the estimation, are studied. For this purpose, the elasticity function is used, which provides the percentage estimation error when an error of 1% is made in the measurement of the data.

Since three parameters are estimated from three different data, nine elasticity functions arise for this problem, they are the elasticity of each estimation $\widehat{\kappa_A}$, $\widehat{\kappa_B}$, $\widehat{\kappa_C}$ with respect to each datum T_1^ϵ, T_2^ϵ, T_3^ϵ, given by

$$E_{\widehat{\kappa_i}}^{T_j^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = \frac{T_j^\epsilon}{\widehat{\kappa_i}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon)}\,\frac{\partial \widehat{\kappa_i}}{\partial T_j^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon), \quad (28)$$

where $i = A, B, C$ and $j = 1, 2, 3$.

The system (23) lead to the following analytical expressions for the elasticity functions defined in (28)

$$\begin{cases} E_{\widehat{\kappa_A}}^{T_1^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = \dfrac{T_1^\epsilon}{F - T_1^\epsilon},\\ E_{\widehat{\kappa_A}}^{T_2^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = 0,\\ E_{\widehat{\kappa_A}}^{T_3^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = \dfrac{T_3^\epsilon}{T_3^\epsilon - T_a}, \end{cases} \quad (29)$$

$$\begin{cases} E_{\widehat{\kappa_B}}^{T_1^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = \dfrac{T_1^\epsilon}{T_2^\epsilon - T_1^\epsilon},\\ E_{\widehat{\kappa_B}}^{T_2^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = \dfrac{T_1^\epsilon}{T_1^\epsilon - T_2^\epsilon},\\ E_{\widehat{\kappa_B}}^{T_3^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = \dfrac{T_3^\epsilon}{T_3^\epsilon - T_a} \end{cases} \quad (30)$$

and

$$\begin{cases} E_{\widehat{\kappa_C}}^{T_1^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = 0,\\ E_{\widehat{\kappa_C}}^{T_2^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = \dfrac{T_2^\epsilon}{T_3^\epsilon - T_2^\epsilon},\\ E_{\widehat{\kappa_C}}^{T_3^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = \dfrac{T_3^\epsilon}{T_3^\epsilon - T_a}\,\dfrac{T_2^\epsilon - T_a}{T_2^\epsilon - T_3^\epsilon}. \end{cases} \quad (31)$$

Remark 6. *Note that* $E_{\widehat{\kappa_A}}^{T_2^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = 0$ *because the estimated value* $\widehat{\kappa_A}$ *is independent of* T_2^ϵ. *Analogously,* $E_{\widehat{\kappa_C}}^{T_1^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = 0$ *since* $\widehat{\kappa_C}$ *is independent of* T_1^ϵ. *Furthermore,* $E_{\widehat{\kappa_A}}^{T_3^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = E_{\widehat{\kappa_B}}^{T_3^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon)$, *while* $E_{\widehat{\kappa_C}}^{T_3^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = E_{\widehat{\kappa_A}}^{T_3^\epsilon}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon)\dfrac{T_2^\epsilon - T_a}{T_2^\epsilon - T_3^\epsilon}$.

5 Numerical example

Example 1. *A Nickel-Lead-Iron material is assumed where* $L = 10\,m$; $F = 100\,°C$ *and* $Ta = 25\,°C$. *The convective fluid is assumed to be air and convective heat transfer coefficient* (h) *are determined as explained in [15]. It is also assumed that* $l_1 = 4\,m$ *and* $l_2 = 7\,m$.

The analytical (exact) data for this example are $T^* = (T_1, T_2, T_3) = (87.71, 64.01, 52.65)$ $[°C]$, obtained from (11)-(13) where the values of the coefficient of thermal conductivity are $\kappa_A = 90\ W/m°C$, $\kappa_B = 35\ W/m°C$ and $\kappa_C = 73\ W/m°C$ (see [1]).

Firstly, the estimation errors for different values of T_1^ϵ, T_2^ϵ and T_3^ϵ close to T_1, T_2 and T_3, are analyzed. Let us define

$$Err_{\kappa_A}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = |\kappa_A - \widehat{\kappa_A}|,$$
$$Err_{\kappa_B}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = |\kappa_B - \widehat{\kappa_B}|,$$
$$Err_{\kappa_C}(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon) = |\kappa_C - \widehat{\kappa_C}|.$$

Table 1: Relative estimate errors $\frac{Err_{\kappa_i}}{\kappa_i}$, $i = A, B, C$ for Example 1.

T_1^ϵ	T_2^ϵ	T_3^ϵ	$\frac{Err_{\kappa_A}}{\kappa_A}$	$\frac{Err_{\kappa_B}}{\kappa_B}$	$\frac{Err_{\kappa_C}}{\kappa_C}$
87.2	63.5	52.1	0.060	0.020	0.023
87.3	63.6	52.2	0.048	0.016	0.019
87.4	63.7	52.3	0.037	0.012	0.015
87.5	63.8	52.4	0.025	0.009	0.012
87.6	63.9	52.5	0.014	0.005	0.008
87.7	64.0	52.6	0.002	0.001	0.005
87.8	64.1	52.7	0.009	0.002	0.001
87.9	64.2	52.8	0.021	0.005	0.002
88.0	64.3	52.9	0.033	0.009	0.005
88.1	64.4	53.0	0.045	0.012	0.009
88.2	64.5	53.1	0.058	0.016	0.012

Table 1 shows the relative estimation errors for some values T_1^ϵ, T_2^ϵ and T_3^ϵ close to T_1, T_2 and T_3. It can be seen that good relative estimation errors are obtained for the conductivity values. Furthermore, it is observed that the estimate worsens as the error in data increases. In this range of temperature values, a maximum error of 6% is obtained for the estimate of κ_A, of 2% for the estimate of κ_B and of 2.3% for that of κ_C.

It is interesting to determine the directions of maximum increase in estimation errors. The theory of calculus in several variables proves that the direction of maximum growth of a function at a point is the direction of the gradient at that point. In this case, they are the directions of $\nabla Err_{\kappa_A}(T^*)$,$\nabla Err_{\kappa_B}(T^*)$ and $\nabla Err_{\kappa_C}(T^*)$, respectively, shown in Figure 1.

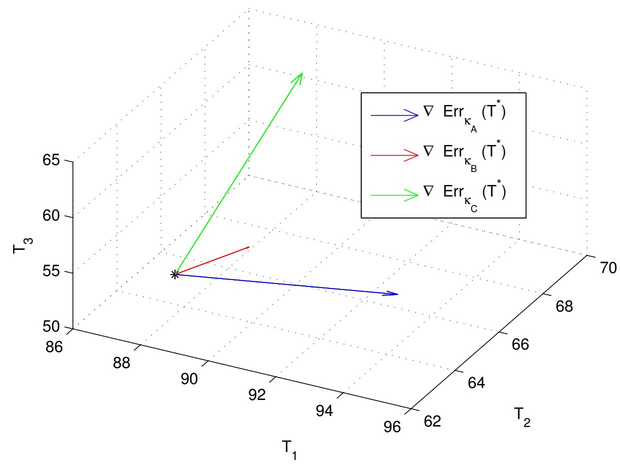

Figure 1: Gradients of the estimation errors at (87.71, 64.01, 52.65) (Example 1).

Moreover, the maximum values of the derivatives in these directions are

$$\|\nabla Err_{\kappa_A}(T^*)\| = 8.01,$$
$$\|\nabla Err_{\kappa_B}(T^*)\| = 2.44,$$
$$\|\nabla Err_{\kappa_C}(T^*)\| = 11.11.$$

Hence, taken into account the values for κ_A, κ_B and κ_C, it follows that the maximum growth at T^* for the relative errors of the estimates of κ_A, κ_B, κ_C are, respectively, 0.089, 0.069 and 0.152, i.e., about 9%, 7% and 15%.

Finally, the local relationships between the estimated parameters $(\widehat{\kappa_A}, \widehat{\kappa_B}, \widehat{\kappa_C})$ and the data $(T_1^\epsilon, T_2^\epsilon, T_3^\epsilon)$ used for the estimation, are analyzed. In Figures 2-4 the elasticity functions defined in (29)-(31) for the Nickel-Lead-Iron material described above, are plotted.

Figure 2 shows that the estimate error of κ_A increases with T_1^ϵ. It can also be observed that $E_{\widehat{\kappa_A}}^{T_1^\epsilon}(T^*) \approx 7$ while $E_{\widehat{\kappa_B}}^{T_1^\epsilon}(T^*) \approx 3.7$ and $E_{\widehat{\kappa_C}}^{T_1^\epsilon}(T^*) = 0$. In other words, an error of 1% in the measurement of T_1 leads to an error of about 7% in the estimate of κ_A and 3.7 % in the estimate of κ_B, which means that $\widehat{\kappa_A}$ is more sensitive to the error in T_1^ϵ than the $\widehat{\kappa_B}$. As it was mentioned before κ_C does not depend on T_1^ϵ.

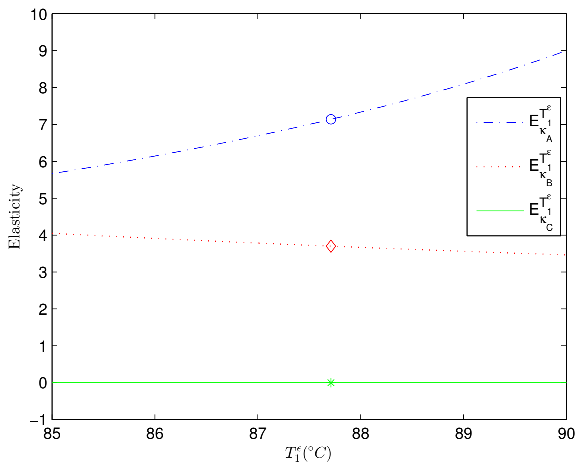

Figure 2: Elasticity of the conductivities with respect to T_1 for a Nickel-Lead-Iron material (Example 1).

In Figure 3 it is observed that the estimate of κ_C is more sensitive to the measured value T_2^ϵ than the other estimates since $E^{T_2^\epsilon}_{\widehat{\kappa_C}}(T^*) \approx 6$ (about 6% of estimate error when there is a 1% in the measurement error), while $E^{T_2^\epsilon}_{\widehat{\kappa_B}}(T^*) \approx 2.8$ (about 2.8% of error in the estimate when there is a 1% in the measurement error), and $E^{T_2^\epsilon}_{\widehat{\kappa_A}}(T^*) = 0$ (κ_A does not depend on T_2^ϵ).

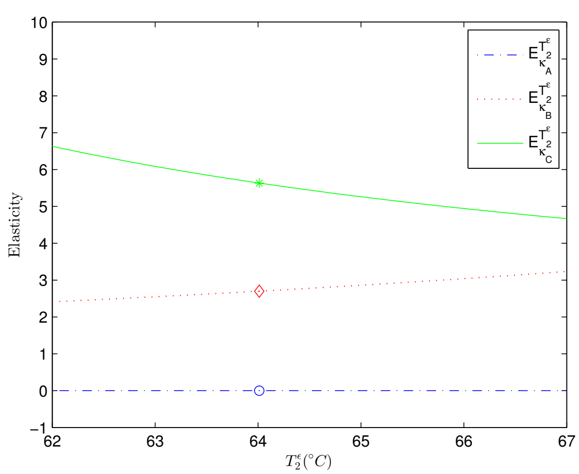

Figure 3: Elasticity of the conductivities with respect to T_2 for a Nickel-Lead-Iron material (Example 1).

Figure 4 indicates that the estimate of κ_C is also more sensitive to the measurement of T_3 than the other estimates since $E^{T_3^\epsilon}_{\widehat{\kappa_C}}(T^*) \approx 6.5$ (about 6.5% of estimate error when there is a 1% in the measurement error), while $E^{T_3^\epsilon}_{\widehat{\kappa_A}}(T^*) = E^{T_3^\epsilon}_{\widehat{\kappa_B}}(T^*) \approx 2$ (about 2% of error in the estimates when there is a 1% in the measurement error).

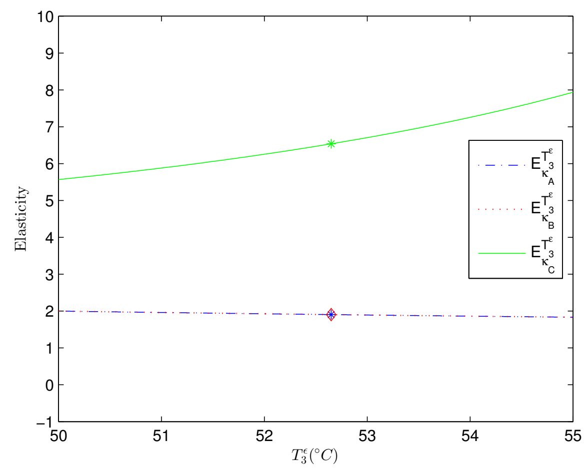

Figure 4: Elasticity of the conductivities with respect to T_3 for a Nickel-Lead-Iron material (Example 1).

The graphs show that the estimates of κ_A and κ_B are more sensitive to measurement error in T_1 while the estimate of κ_C is more sensitive to errors in the measurements of T_2 and T_3.

Denoting $E_{\widehat{\kappa_i}} = (E^{T_1^\epsilon}_{\widehat{\kappa_i}}, E^{T_2^\epsilon}_{\widehat{\kappa_i}}, E^{T_3^\epsilon}_{\widehat{\kappa_i}})$, $i = A, B, C$ from the numerical experiment for this example results in $\|E_{\widehat{\kappa_A}}(T^*)\| = 7.39$, $\|E_{\widehat{\kappa_B}}(T^*)\| = 4.96$, $\|E_{\widehat{\kappa_C}}(T^*)\| = 8.63$, so larger errors in the temperature measurements lead to larger errors in the estimation of κ_A and κ_C than in κ_B. These results agree with the conclusion reached previously when discussing the maximum growth values of the relative errors for the estimation.

6 Conclusion

This article analyzes the simultaneous estimation of the thermal conductivity coefficients for a stationary heat transfer problem with two solid-solid interfaces. A technique is proposed for the estimation of these physical properties based on three noisy temperature over-conditions, one at each interface and another at the right end of the material. The necessary and sufficient conditions for the existence and uniqueness of the solution to the estimation problem are provided, and analytical bounds for the determination errors are derived.

The local influence of the data on the estimated parameters is studied by means of elasticity analysis. For a numerical example, the directions and the values of maximum growth of the relative errors are also studied, noting that the results are consistent with the elasticity analysis.

The results obtained suggest that the approach presented here is useful to determine the three thermal conductivities for each body material. However, it is important to measure the temperatures as accurately as possible, since the estimated values are sensitive to measurement errors.

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