On a system without critical points arising in heat conductivity theory

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Abstract: A two-point boundary value problem for the second order nonlinear ordinary differential equation, arising in the heat conductivity theory, is considered. Multiplicity and existence results are established for this problem, where the equation contains two parameters.

Keywords: heat conductivity, nullclines, phase portrait, Cauchy problems, bifurcation curves

Received: August 12, 2021. Revised: April 16, 2022. Accepted: May 17, 2022. Published: July 1, 2022.

1 Introduction

Mathematical modeling of biomass thermal conversion in reactors of regular shape can help in solving the problem of reduction greenhouse emissions and have a positive effect on climate change. The respective mathematical models of heat transfer in the presence of nonlinear heat sources can lead to nonlinear boundary value problems (BVP) for ordinary differential equations. The problem is to detect the number of positive solutions to BVP and trace the change of this number under the influence of built-in parameters. For more details the interested reader can consult the papers [11], [5], [4]. In this paper we consider one of these problems.

Consider the equation

$$T'' + aT' + Fe^T = 0, \quad ' = \frac{d}{dt}, \tag{1}$$

together with the boundary conditions

$$T(-1) = 0, \quad T(1) = 0. \tag{2}$$

The parameters a and F are real numbers, a can be of arbitrary sign, F is positive. The problem is to study the existence and the number of positive solutions. Each positive solution of the above problem corresponds to a temperature regime in a domain of parallelepipedal form. A solution $T(t)$ (T is the temperature, t is treated here as a spatial variable) is supposed to be C^2-smooth function. For general reading about problems, arising in heat conductivity theory, one may consult [1], [2], [3], [5], [16].

2 Reduction to a system

Replace T by x, x' by y and rewrite the equation in the form

$$\begin{cases} x' = y, \\ y' = -ay - Fe^x. \end{cases} \tag{3}$$

We are interested in positive solutions in the interval $I = (-1, 1)$, vanishing at the end-points of the interval I. Therefore, we have three equivalent boundary value problems

(BVP), respectively, the problem (1), (2), the problem

$$x'' + ax' + Fe^x = 0, \quad x = x(t), \tag{4}$$

$$x(-1) = 0,\ x(1) = 0,\ x(t) > 0,\ \forall t \in (-1, 1), \tag{5}$$

and the problem (3), (5).

We will refer to one of these problems when convenient.

2.1 Tools

In this section we consider several techniques (adapted for our case), used for investigation of autonomous ordinary differential equations and systems in the plane.

2.1.1 Nullclines and the phase plane

In order to analyze the phase plane of system (3) consider first the nullclines $N_1 = \{(x, y) : y = 0\}$ and $N_2 = \{(x, y) : -ay - Fe^x = 0\}$. The nullclines have not crosspoints, so the system has no critical points.

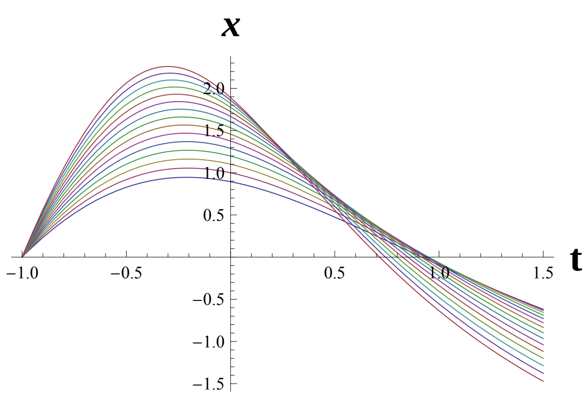

Figure 1: Solutions of (4) for $x(0) = -1$, $2.5 \leq x'(0) \leq 6$, $a = 1$, $F = 1$.

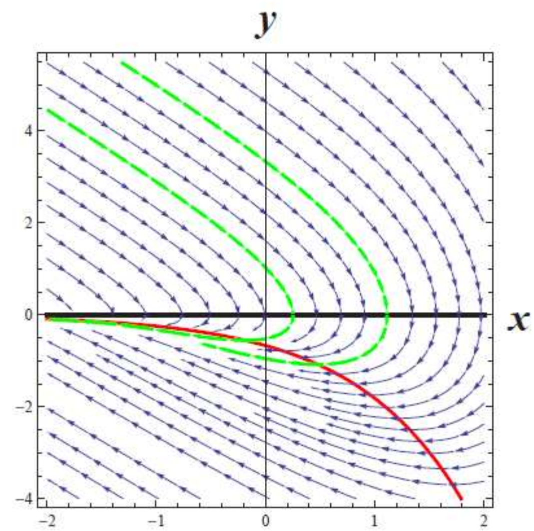

Figure 2: Trajectories of (3) for $x(0) = -1$, $2.5 \leq y(0) = x'(0) \leq 6$, $a = 1.5$, $F = 1$. Red - nullcline N_2, green - two trajectories passing through the points $(-1, 2.9)$ and $(-1, 5.0)$ respectively, correspond to solutions of the BVP (3), (5)

Consider the Cauchy problem

$$x'' + ax' + Fe^x = 0, \quad a, F > 0,$$

$$x(-1) = 0, \quad x'(-1) = p > 0. \tag{6}$$

Proposition 2.1. *Any solution of the Cauchy problem (6) has the first zero t_1 and a unique maximum in the interval $(-1, t_1)$.*

$\triangleright$ Consider the vector field $\{y, -ay - Fe^x\}$, directed as shown in Fig. 2 for any positive a and F. The vector field is directed vertically downward only on the $x' = y = 0$ axis. Since there are no critical points, any trajectory, starting at $x = 0$ with $y > 0$, can be continued to $x = 0$ with $y < 0$. $\triangleleft$

Figures (1), (2) and (3) provide the first impressions on the behavior of solutions of the Cauchy problems (6). These figures shows that $t_1(p)$ (the first zero) monotonically increases together with p, and, after reaching some critical value p_*, monotonically decreases. In Fig. 1 the value of $t_1(p_*)$ is less than 1. This is indication that there

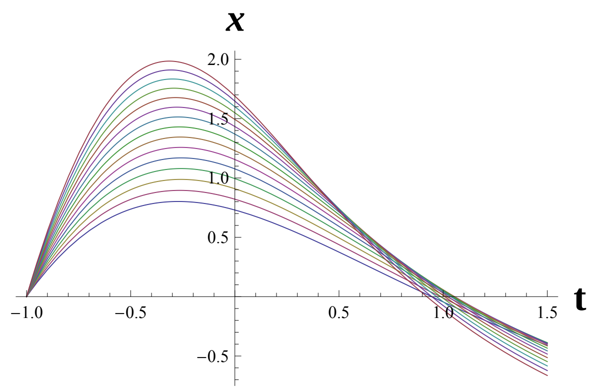

Figure 3: Trajectories of (3) for $x(0) = -1$, $2.5 \leq y(0) = x'(0) \leq 6$, $a = 1.5$, $F = 1$. Two solutions of the BVP correspond to $x'(0) = 1.9$, $x'(0) = 5.0$.

are no positive solutions of the BVP (4,) (5) for $a = 1$, $F = 1$. On the other hand, Fig. 3 shows that $t_1(p_*)$ is greater than 1. Therefore $t_1(p)$ passes the value $t = 1$ twice, giving rise to two solutions of the BVP (4,) (5) for $a = 1.5$, $F = 1$. The two trajectories, corresponding to solutions of the BVP, are depicted in Fig. 2 (in green).

2.1.2 Energy dissipation

Conservative (Newtonian) systems are known to save energy along trajectories. This is not the case for system (3).

Proposition 2.2. *Any solution of the Cauchy problem (6) has the first zero t_1 and the initial value p is greater than $|x'(t_1(p))|$ by $2a\int_{-1}^{t_1} x'^2(s)\,ds$.*

$\triangleright$ One has for equation (4)

$$
\begin{aligned}
&x'' + ax' + Fe^x = 0,\\
&2x'x'' + 2ax'^2 + 2Fe^x x' = 0,\\
&d(x'^2 + 2Fe^x) = -2ax'^2 dt,\\
&x'^2(t) + 2Fe^{x(t)} = x'^2(-1) + 2Fe^{x(-1)}\\
&-2a\int_{-1}^{t} x'^2(s)\,ds,\\
&x'^2(t_1) + 2Fe^{x(t_1)} = x'^2(-1) + 2Fe^{x(-1)}\\
&-2a\int_{-1}^{t_1} x'^2(s)\,ds,\\
&x'^2(t_1) + 2F = x'^2(-1) + 2F\\
&-2a\int_{-1}^{t_1} x'^2(s)\,ds,\\
&x'^2(t_1) - x'^2(-1) = -2a\int_{-1}^{t_1} x'^2(s)\,ds < 0,\\
&x'^2(t_1) - p^2 = -2a\int_{-1}^{t_1} x'^2(s)\,ds < 0.
\end{aligned}
$$

$\triangleleft$

2.1.3 Continuity of t_1

Consider the Cauchy problem (6). Then the first zero $t_1(p)$ exists. The value t_1 depends on three parameters, respectively a, F and p. So we may write $t_1(a, F, p.)$ This function is continuous on an open set of parameters, say, for $a > 0$, $F > 0$, $p > 0$. One has that $t_1(1, 1, 3) < 1$ (by calculation). On the other hand, $t_1(0.1, 0.1, 4) > 1$ (by calculation). By continuity of t_1, there exists at least one point (a_*, F_*, p_*) on any continues 3D-curve, lying in $\{(a, F, p) : a > 0, F > 0, p > 0\}$ and connecting the points $(0.1, 0.1, 4)$ and $(1, 1, 3)$.

Generally, solutions $x(t; a, F, p)$ of the Cauchy problem (6) continuously depend on the parameters (a, F, p). Consider intersection of a ball $B_\varepsilon = \{a^2 + F^2 + p^2 = \varepsilon^2\}$ with $R^3_+ := \{(a, F, p) : a > 0, F > 0, p > 0\}$. For ε^2 tending to zero, solutions $x(t; a, F, p)$ tend to a solution of $T'' = 0, \quad T(-1) = 0,\ T'(-1) = 0$, that is, to the trivial solution. Similarly, solutions $x(t; a_n, F_n, 1)$ tend to a solution of the problem $T'' = 0, \quad T(-1) = 0,\ T'(-1) = 1$, as a_n and F_n tend to zero.

The following therefore is true.

Proposition 2.3. *Solutions $x(t; a, F, 1)$ of the Cauchy problems*

$$T'' + aT' + Fe^T = 0,$$

$$T(-1) = 0, \quad T'(-1) = 1, \tag{7}$$

where $a^2 + F^2 = \varepsilon^2$, have not zeros in the interval $(-1, 1]$ for ε^2 sufficiently small.

2.1.4 Considering partial cases: $F = 1$

Numerical experiments show that: a) let a be fixed and less than approximately $a_0 = 1.3$ (this constant can be made more precise). Let $t_1(p)$ be the first zero of the Cauchy problem (6). The values $t_1(p)$ are monotonically increasing together with p

until $p = p_0(a)$ (can be calculated). The values $t_1(p)$ are monotonically decreasing for $p \in (p_0, +\infty)$. The value $p_0(a)$ reaches $t = 1$ for a_0. The following assertion is

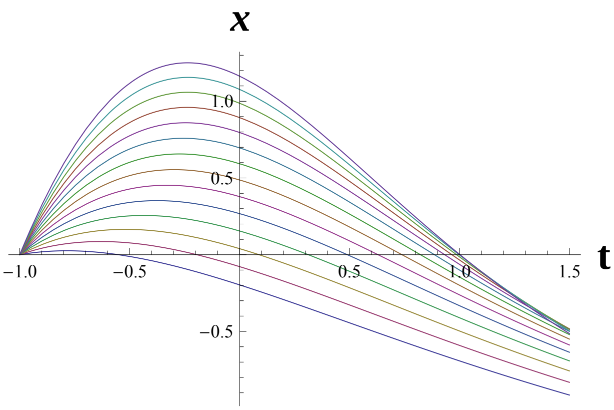

Figure 4: Solutions of (6) for $x(0) = -1$, $0.25 \le x'(0) \le 3.5$, $a = 1.28$, $F = 1$. A unique solution of the BVP exists.

confirmed by calculations using Wolfram Mathematica.

Consider the BVP (1), (2). Let $F = 1$. Consider the Cauchy problem (6). Let $t_1(a,p)$ be the first zero function.

Proposition 2.4. *The function $t_1(a,p)$ for fixed $a < a_0$ is monotonically increasing for $p \in (0, p_0)$ and monotonically decreasing for $p \in (p_0, +\infty)$ with a maximum at p_0. For $a < a_0$ one has $t_1(a, p_0) < 1$, for $a > a_0$ $t_1(a, p_0) > 1$.*

Therefore no positive solutions of BVP (1), (2) for $a < a_0$ (recall $F = 1$).

There exists exactly one positive solution for $a = a_0$.

There exist exactly two positive solutions for $a > a_0$.

2.1.5 Time intervals evaluation

Consider the equation

$$x'' + ax' + Fe^x = 0 \quad (8)$$

together with the initial conditions

$$x(-1) = 0, \quad x'(-1) = p. \quad (9)$$

A solution of this problem has the first zero $t_1(p)$. It has a unique point of maximum $t_{max}(p)$. A solution is monotonically increasing in the interval $[-1, t_{max}]$ and monotonically decreasing in $[t_{max}, t_1]$. In the intervals of monotonicity the function $x(t)$ has an inverse function $t(x)$. Using the standard formulas for inverse functions

$$t'_x(x) = 1/x_t(t), \quad t''_{x^2} = -x''_{t^2}/(x'_t)^3 \quad (10)$$

the equation (8) can be rewritten as

$$-t'' + at'^2 + Ft'^3 e^x = 0, \quad t = t(x), \quad ' = \frac{d}{dx}. \quad (11)$$

Replacing $t' = u$,

$$-u' + au^2 + Fu^3 e^x = 0, \quad u = u(x). \quad (12)$$

This is the first order (non-autonomous) equation and it is easier to treat numerically.

Let us make comparison of two solutions, $x_1(t)$ and $x_2(t)$, of the equation (8), satisfying the initial conditions

$$x_1(-1) = 0, \quad x_1'(-1) = p_1 \quad (13)$$

$$x_2(-1) = 0, \quad x_2'(-1) = p_2. \quad (14)$$

Let $p_1 = 1$, $p_2 = 3$. Both solutions are depicted in Fig. 5.

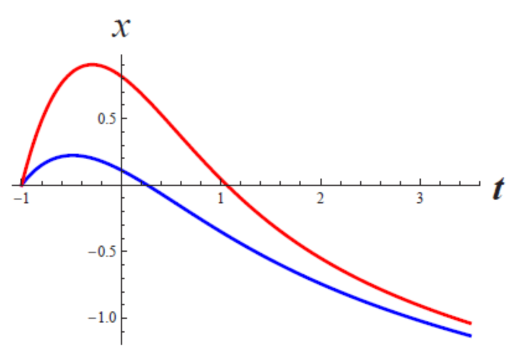

Figure 5: Solutions of (13) and (14), $a = 1.8$, $F = 1$.

Let t_{1max} and t_{2max} be points of maximum for $x_1(t)$ and $x_2(t)$ respectively. For our choice of p_1 and p_2 we have $t_{1max} < t_{2max}$.

Let us consider the inverse functions $t_1(x)$ and $t_2(x)$, defined respectively in the intervals $[0, x(t_{1max})$ and $[0, x(t_{2max})$. The functions $t_1'(x)$ and $t_2'(x)$ are solutions of the Cauchy problems

$$u' = au^2 + Fu^3e^x,$$
$$u = u(x), \quad ' = \frac{d}{dx}, \quad u(0) = 1/p_1 = 1 \tag{15}$$

and

$$v' = av^2 + Fv^3e^x,$$
$$v = v(x), \quad ' = \frac{d}{dx}, \quad v(0) = 1/p_2 = 1/3. \tag{16}$$

By comparison theorem for the first order equations $u(x) = t_1'(x) > v(x) = t_2'(x)$ on the interval $(0, x_{1max})$ and, due to $t_1'(0) > t_2'(0)$, $t_1(x) > t_2(x)$ for $x \in (0, x_{1max})$. The graphs of both functions $t_1(x)$ and $t_2(x)$ are depicted in Fig. 6.

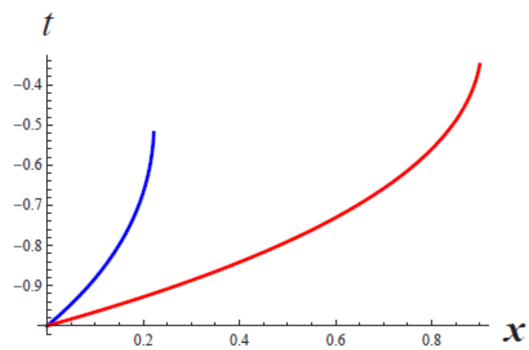

Figure 6: Solutions $t_1(x)$(blue) and $t_2(x)$ (red) of (11), $a = 1.8$, $F = 1$, $t_1(0) = t_2(0) = -1$, $t_1'(0) = 1$, $t_2'(0) = 1/3$, $t_1' = +\infty$ at $x \approx 0.222$, $t_2' = +\infty$ at $x \approx 0.904$.

Similar treatment is possible for the equation (8) on the intervals $[t_{max}, t_1]$, where both t_{max} and t_1 are dependent on $x'(-1) = p$.

2.1.6 Polar coordinates

Sometimes passage to polar coordinates can be useful. Introduce polar coordinates, using the formulas

$$x = \rho \sin\varphi, \quad y = \rho\cos\varphi. \tag{17}$$

The system

$$\begin{cases} x' = y, \\ y' = -ay - Fe^x \end{cases} \tag{18}$$

takes the form

$$\begin{cases} \rho' = \rho\sin\varphi\cos\varphi - a\rho\cos^2\varphi - F\cos\varphi e^{\rho\sin\varphi}, \\ \varphi' = \cos^2\varphi + a\sin\varphi\cos\varphi + \dfrac{1}{\rho}\sin\varphi F e^{\rho\sin\varphi}. \end{cases} \tag{19}$$

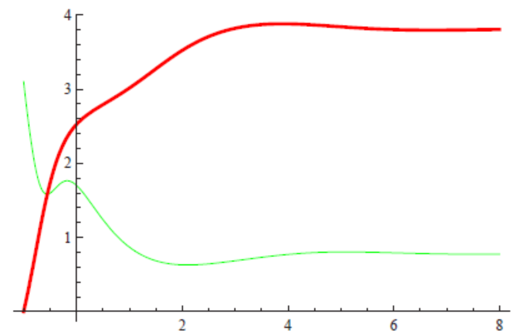

Figure 7: Solutions $\rho(t)$ (green) and $\varphi(t)$ (red), $a = 1.8$, $F = 1$.

Notice that polar function $\rho(t)$ need not to be monotone. That means that the value $x^2(t) + x'^2(t)$ can have maxima and minima.

3 Phase portrait analysis

The typical phase portrait for system (3) is depicted in Fig. 8 below.

Any trajectory starts at $(x, y) = (0, p)$. It rotates clock-wise following the vector field $(y, -ax - Fe^x)$. The end point of any trajectory corresponds to $(x(1), y(1))$. If the end point is on the axis $x = 0$, then this trajectory corresponds to a solution of the BVP (3), (5). Then equation (4) is

$$x'' + Fe^x = 0, \quad x = x(t). \tag{20}$$

This equation together with the boundary

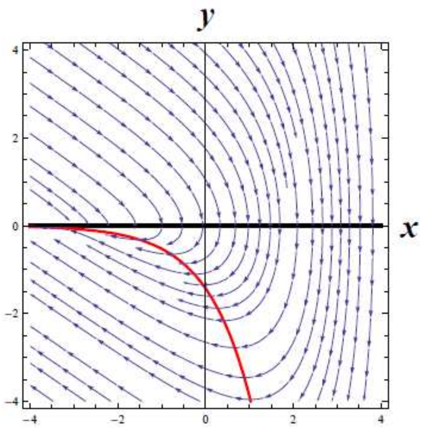

Figure 8: Phase plane with the nullclines N_1 (black) and N_2 (red)

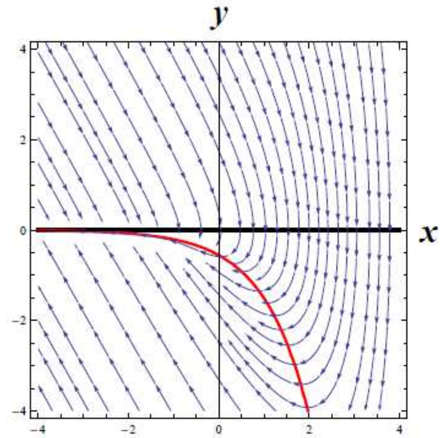

Figure 10: Phase plane with the nullclines N_1 (black) and N_2 (red)

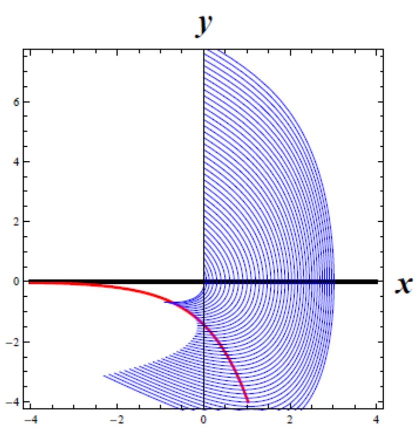

Figure 9: Segments of trajectories parameterized by $t \in [-1, 1]$, $a = 0.7$, $F = 1.0$.

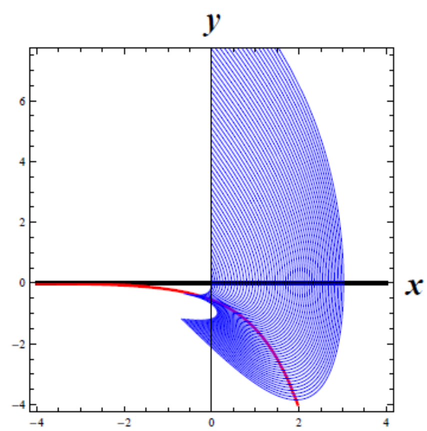

Figure 11: Segments of trajectories parameterized by $t \in [-1, 1]$, $a = 1.8$, $F = 1.0$.

conditions (5) was previously studied[1]. The following result is true due to findings of the above mentioned article.

Theorem 3.1. *There exists* $F_0 \approx 0.878458$ *with the properties:*
1) if $0 < F < F_0$*, then the problem (20),(5) has exactly two positive solutions;*
(2) if $F = F_0$*, then the problem (20),(5) has exactly one positive symmetrical solution* x_{F_0} *with* $x_{max} = x(0) = 1.18684$*;*
(3) if $F > F_0$*, then the problem (20),(5) has no positive solutions.*

[1]S.-Y. Huang, S.-H. Wang, Proof of a Conjecture for the One-Dimensional Perturbed Gelfand Problem from Combustion Theory, Arch. Ration. Mech. Anal., Vol. 222, No. 2, 2016, pp. 769825. https://link. springer.com/article/10.1007/s00205-016-1011-1

The exact bifurcation curve F against $x(0)$ is available for positive solutions of (20),(5).

3.1 $a < 0$

The independent variable change t to $-t$ turns equation $x'' + ax' + Fe^x = 0$ to the equation $X'' - aX' + Fe^X = 0$, where the coefficient at X' is positive. The boundary conditions remain unchanged. Therefore this case is reduced to the previously studied one, where $a > 0$.

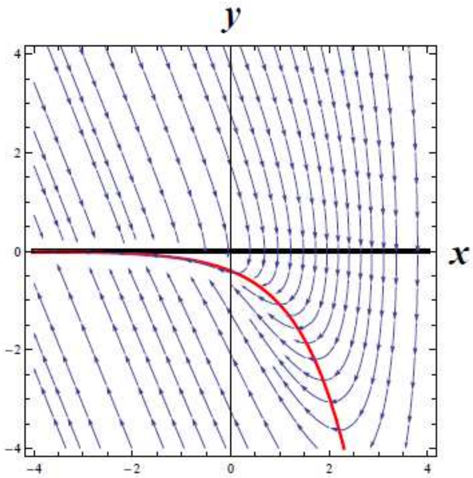

Figure 12: Phase plane with the nullclines N_1 (black) and N_2 (red)

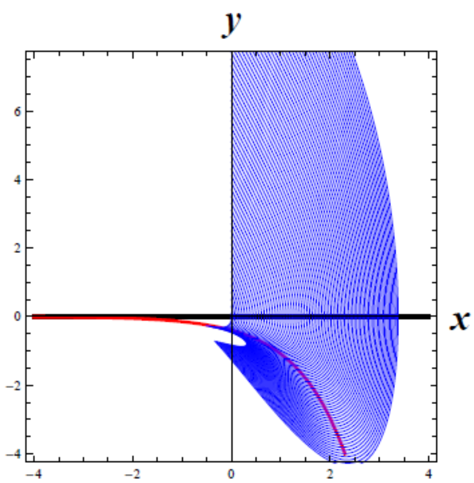

Figure 13: Segments of trajectories parameterized by $t \in [-1, 1]$, $a = 2.5$, $F = 1.0$.

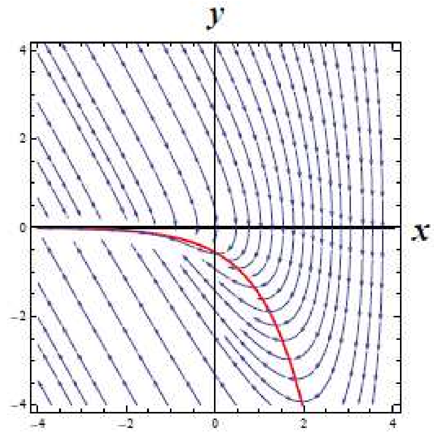

Figure 14: Phase plane: $a = 1.8$, $F = 1.0$.

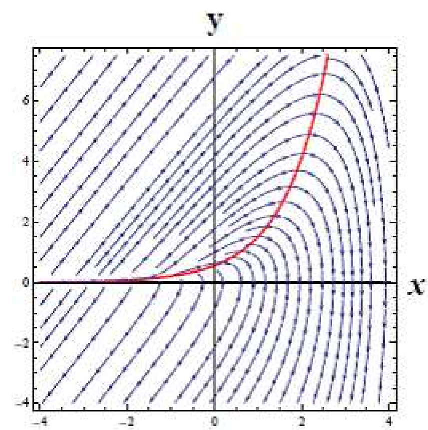

Figure 15: Phase plane: $a = -1.8$, $F = 1.0$.

4 Bifurcation curves

An alternative method can be used to study bifurcations in the problem (20),(5).

Consider the Cauchy problem

$$x'' + ax' + Fe^x = 0,$$

$$x(-1) = 0,\ x'(-1) = p > 0. \tag{21}$$

Assume that $a > 0$ is given. The first zero function $t_1(F, p)$ known also as the time-map function could be studied. The equation

$$t_1(F, p) = 1 \tag{22}$$

defines the bifurcation curve that bears information on the number of positive solutions and values of $x'(-1) = p$ which produce positive solutions of (20),(5).

This bifurcation curve can be constructed numerically for particular values of the coefficient a. Below the bifurcation curves, $x'(-1) = p$ against F are shown for $a = 2$ and $a = -2$.

Material in this section is based on the private communication by A. Gritsans.

The technique of bifurcation curves for detecting of solutions of boundary value problems was used in [6], [7], [8], [10], [11]. The number of solutions in related problems were studied in [9], [12], [13], 14], [15].

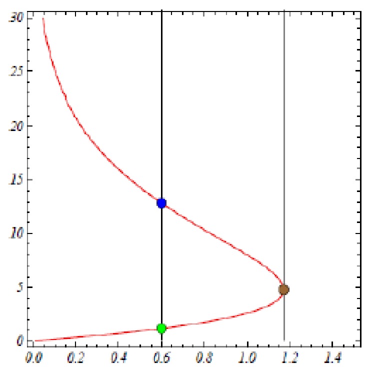

Figure 16: $a = 2$, two solutions for $F = 0.6$

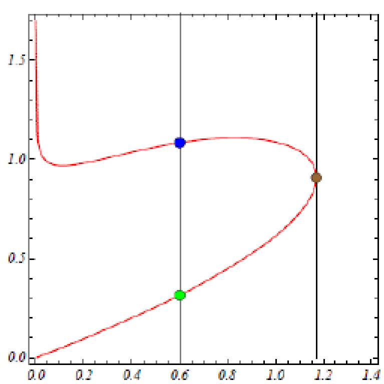

Figure 17: $a = -2$

5 Conclusions

Both cases $a > 0$ and $a < 0$ are symmetrical. It is enough to consider the case $a > 0$. The following observations were made for the problem (1), (2) with $a > 0$:

- There are at most two positive solutions of the problem;
- For some values of parameters a and F the existence of a single positive solution and no positive solutions are possible;
- For a given pair of parameters a and F a complete numerical analysis can be made;
- Bifurcation curves were constructed if one of the parameters a and F is given;
- The approximate initial values $T'(-1)$ for the positive solutions of the problem can be found by numerical inspection;
- Turning points on bifurcation curves, corresponding to transition from no solutions to two solutions can be found numerically;
- Generic properties of the time-map function $t_1(p, a, F)$ can be described.

The analysis (both theoretical and numerical) of similar tasks for ordinary differential equations can help to treat problems, arising in the theory of heat transfer and fluid mechanics with applications to clean energy production, biomass thermal conversion and utilization of waste.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Author Contributions:
All authors have contributed equally to creation of this article.

Sources of funding for research presented in a scientific article or scientific article itself

Fundamental and applied research project of the Latvian Council of Science. lzp-2020/1-0076

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