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Neutral inhomogeneities in a two-dimensional steady-state heat conduction problem

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Abstract: A steady-state heat conduction problem is considered in a two-dimensional solid body which is filled up composite circular inclusions. The composite circular inclusions consist of a core and a coating both of which are cylindrically orthotropic. In this paper the neutral inhomogeneity is defined as a foreign body (inclusion) which can be introduced into the host body (matrix) without distributing the temperature field in the originally homogeneous body. The perfect thermal contacts are assumed to be between the different components of non-homogeneous bodies.

Key-Words: heat conduction, steady-state, neutral inhomogeneity, two-dimensional

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1 Introduction

The existence of the neutral inhomogeneities in a two-dimensional rectangular body in the case of steady-state heat conductance is proven. It is assumed that the original body which has no inclusions is subjected to constant temperature gradient. The considered two-dimensional body is shown in Fig. 1. The temperature field of the rectangle in the Cartesian coordinate x, y is prescribed as

$$T_0(y) = -\frac{t_1}{L} y + t_1 \frac{L_2}{L}, \tag{1}$$

where t_1 is a given temperature. It is evident that $T_0 = T_0(y)$ satisfies the boundary conditions

$$T_0(-L_1) = t_1, \quad T_0(L_2) = 0 \tag{2}$$

and the heat flux vector $\mathbf{q}_0$ is as follows

$$\mathbf{q}_0 = k_0 \frac{t_1}{L} \mathbf{e}_y, \tag{3}$$

where k_0 is the thermal conductance of the homogeneous isotropic rectangle. The unit vectors of the Cartesian coordinate system Oxy are $\mathbf{e}_x$ and $\mathbf{e}_y$ (Fig. 1).

The composite cylindrical inhomogeneity is introduced to the homogeneous isotropic rectangular body as shown in Fig. 2. The inclusion is placed to the origin of the Cartesian coordinate system Oxy and it contains two different material components. The first component is a coating occupying the hollow circular domain A_1 and the second component is the core occupying the solid circular domain A_2. In the polar coordinate system $Or\varphi$ the domains A_1 and A_2 are defined as

$$A_1 = \{(r,\varphi)\,|R_2 \le r \le R_1, 0 \le \varphi \le 2\pi\}, \tag{4}$$

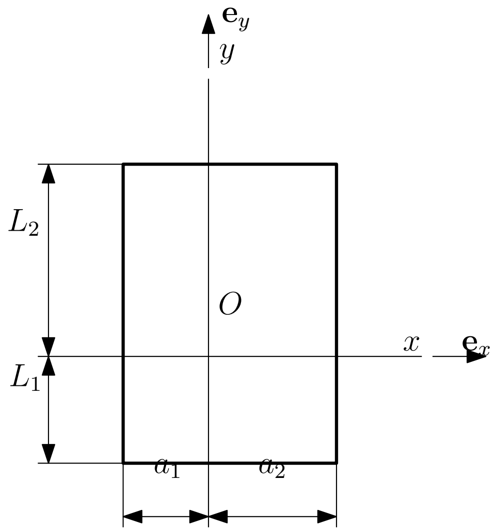

Figure 1: Rectangular body subjected to constant heat flux vector.

$$A_2 = \{(r,\varphi)\,|0 \le r \le R_2, 0 \le \varphi \le 2\pi\}. \tag{5}$$

The materials of the coating and core are homogeneous and cylindrically orthotropic with the thermal conductivities k_{1r}, $k_{1\varphi}$ and k_{2r}, $k_{2\varphi}$. In the polar coordinate system $Or\varphi$

$$T_0(r,\varphi) = -t_1 \frac{r}{L} \sin\varphi + t_1 \frac{L_2}{L}, \tag{6}$$

$$\mathbf{q}_0(r) = k \frac{t_1}{L} \left(\mathbf{e}_r \sin\varphi + \mathbf{e}_\varphi \cos\varphi\right). \tag{7}$$

The unit vectors of the polar coordinate system $Or\varphi$ are $\mathbf{e}_r(\varphi)$ and $\mathbf{e}_\varphi(\varphi)$ (Fig. 2).

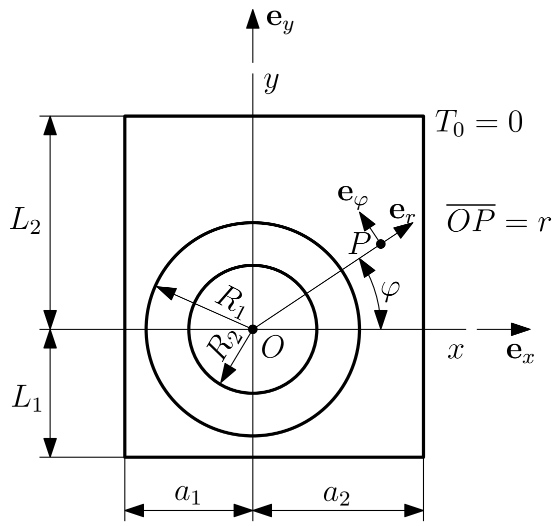

Figure 2: A coated circular inhomogeneity in homogeneous rectangular body.

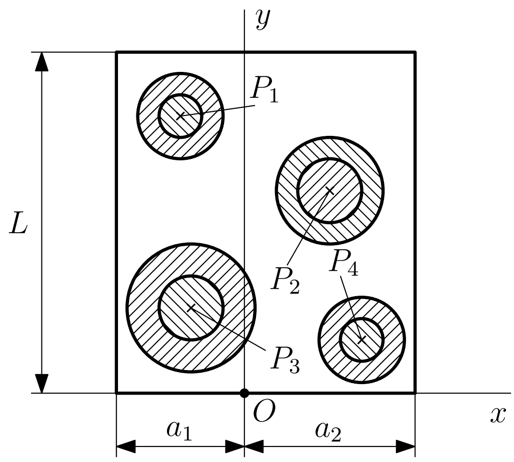

Figure 3: Several circular inhomogeneities in rectangular homogeneous two-dimensional body.

The existence of neutral inhomogeneity for imperfect thermal contact was analysed by Benveniste and Miloh [1].

For the three-dimensional steady-state heat conduction problem the neutral inhomogeneity with spherical orthotropic inclusions was studied in paper [2], where the volume fraction of the core in all inhomogeneities is the same. The functionally graded material properties of inclusions are also discussed in paper by Ecsedi and Baksa [2].

The existence of the neutral inhomogeneities in different boundary-value problems of elasticity was studied by Ru [3], Ru et al. [4], Benveniste and Chen [5], Ecsedi and Baksa [6].

The steady-state temperature field in a two-dimensional cylindrically anisotropic homogeneous body is described by the following partial differential equation in cylindrical coordinate system $Or\varphi$ [7]

$$k_{ir}\frac{\partial^2 T_i}{\partial r^2} + \frac{k_{ir}}{r}\frac{\partial T_i}{\partial r} + \frac{k_{i\varphi}}{r^2}\frac{\partial^2 T_i}{\partial \varphi^2} = 0, \quad (r,\varphi) \in A_i, \quad (i = 1, 2). \tag{8}$$

The expression of the heat flux in radial direction is

$$q_{ir}(r,\varphi) = -k_{ir}\frac{\partial T_i}{\partial r}, \ (r,\varphi) \in A_i, \ (i = 1, 2). \tag{9}$$

2 Governing equations

According to equation (6) and

$$q_{0r} = \mathbf{q}_0 \cdot \mathbf{e}_r = k_0 \frac{t_1}{L}\sin\varphi, \tag{10}$$

where the dot between two vectors denotes their scalar product we assume that

$$T_1(r,\varphi) = F_1(r)\sin\varphi + t_1\frac{L_2}{L}, \quad (r,\varphi) \in A_1, \tag{11}$$

$$T_2(r,\varphi) = F_2(r)\sin\varphi + t_1\frac{L_2}{L}, \quad (r,\varphi) \in A_2, \tag{12}$$

From the partial differential equation (8) it follows that

$$\frac{\mathrm{d}^2 F_i}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}F_i}{\mathrm{d}r} - \frac{k_i^2}{r^2}F_i = 0, \quad k_i = \sqrt{\frac{k_{i\varphi}}{k_{ir}}}, \quad \text{for } i = 1,\ R_2 \le r \le R_1, \quad \text{for } i = 2,\ 0 \le r \le R_2. \tag{13}$$

The solution of the ordinary differential equation (13) for $F_1 = F_1(r)$ and $F_2 = F_2(r)$ are

$$F_1(r) = C_1 r^{k_1} + C_2 r^{-k_1}, \quad R_2 \le r \le R_1, \tag{14}$$

$$F_2(r) = C_3 r^{k_2} + C_4 r^{-k_2}, \quad 0 \le r \le R_2. \tag{15}$$

The function $F_2 = F_2(r)$ is bounded at $r = 0$, from this fact it follows that

$$C_4 = 0. \tag{16}$$

The temperature fields in inhomogeneity are

$$T_1(r,\varphi) = \left(C_1 r^{k_1} + C_2 r^{-k_1}\right)\sin\varphi + t_1\frac{L_2}{L}, \quad (r,\varphi) \in A_1, \tag{17}$$

$$T_2(r,\varphi) = C_3 r^{k_2}\sin\varphi + t_1\frac{L_2}{L}, \quad (r,\varphi) \in A_2. \tag{18}$$

The whole temperature field of the rectangular body with cylindrically anisotropic inclusion is as follows

$$\begin{aligned} T(r,\varphi) &= (H(r) - H(r-R_2))\,T_2(r,\varphi) + \\ &+ (H(r-R_2) - H(r-R_1))\,T_1(r,\varphi) + \\ &+ H(r-R_1)T_0(r,\varphi), \quad (r,\varphi) \in A = \\ &= \{(x,y) | -a_1 \le x \le a_2, -L_1 \le y \le L_2\}\,. \end{aligned} \tag{19}$$

Here, $H = H(r)$ is the Heaviside function [8]. In the case of perfect thermal contact the following equations are valid

$$T_1(R_1,\varphi) = T_0(R_1,\varphi), \quad 0 \le \varphi \le 2\pi, \tag{20}$$

$$q_{1r}(R_1,\varphi) = q_{0r}(R_1,\varphi), \quad 0 \le \varphi \le 2\pi, \tag{21}$$

$$T_1(R_2,\varphi) = T_2(R_2,\varphi), \quad 0 \le \varphi \le 2\pi, \tag{22}$$

$$q_{1r}(R_2,\varphi) = q_{2r}(R_2,\varphi), \quad 0 \le \varphi \le 2\pi. \tag{23}$$

The system of equations (20–23) contains only three unknown C_1, C_2 and C_3 which has unique solution if the geometrical and material properties satisfy certain conditions. Section 3 of this paper deals with the answering the question what kind of connection must exist between R_1, R_2 and k_0, k_{1r}, $k_{1\varphi}$, k_{2r}, $k_{2\varphi}$ to compute the constants C_1, C_2 and C_3 from the system of equations (20–23).

3 Formulation of the conditions of neutral inhomogeneity

The detailed form of system of equations (20–23) is

$$C_1 R_1^{k_1} + C_2 R_1^{-k_1} + t_1 \frac{R_1}{L} = 0, \tag{24}$$

$$\kappa_1 C_1 R_1^{k_1} - \kappa_1 C_2 R_1^{-k_1} + k_0 t_1 \frac{R_1}{L} = 0, \tag{25}$$

$$C_1 R_2^{k_1} + C_2 R_2^{-k_1} - C_3 R_2^{k_2} = 0, \tag{26}$$

$$\kappa_1 C_1 R_2^{k_1} - \kappa_1 C_2 R_2^{-k_1} - \kappa_2 C_3 R_2^{k_2} = 0, \tag{27}$$

where

$$\kappa_1 = \sqrt{k_{1r} k_{1\varphi}}, \quad \kappa_2 = \sqrt{k_{2r} k_{2\varphi}}. \tag{28}$$

System of equations (24–27) generates a homogeneous system of linear equations for the C_1, C_2, C_3 and

$$X = t_1 \frac{R_1}{L}. \tag{29}$$

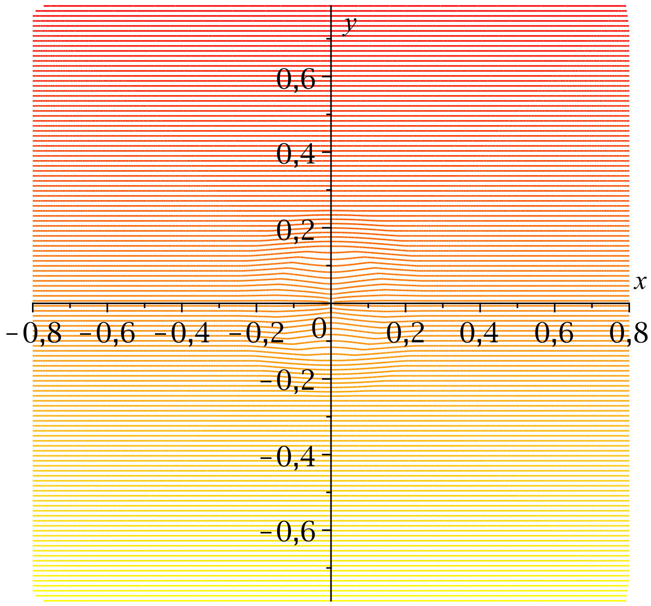

Figure 4: The contour lines of the temperature function.

This system of equations has only non-trivial solutions for C_1, C_2, C_3 and X if its determinant D_0 vanishes, that is

$$D_0 = \begin{vmatrix} R_1^{k_1} & R_1^{-k_1} & 0 & 1 \\ \kappa_1 R_1^{k_1} & -\kappa_1 R_1^{-k_1} & 0 & k_0 \\ R_2^{k_1} & R_2^{-k_1} & -R_2^{k_2} & 0 \\ \kappa_1 R_2^{k_1} & -\kappa_1 R_2^{-k_1} & -\kappa_2 R_2^{k_2} & 0 \end{vmatrix} = 0. \tag{30}$$

After some manipulations the determinant D_0 can be written in the form

$$D_0 = \begin{vmatrix} R_1^{k_1} & R_1^{-k_1} & 0 & 1 \\ d_1 R_1^{k_1} & -d_2 R_1^{-k_1} & 0 & 0 \\ R_2^{k_1} & R_2^{-k_1} & -R_2^{k_2} & 0 \\ d_3 R_2^{k_1} & -d_4 R_2^{-k_1} & 0 & 0 \end{vmatrix} = 0, \tag{31}$$

where

$$\begin{aligned} d_1 &= \left(1 - \frac{k_0}{\kappa_1}\right), \quad d_2 = \left(1 + \frac{k_0}{\kappa_1}\right), \\ d_3 &= \left(1 - \frac{\kappa_2}{\kappa_1}\right), \quad d_4 = \left(1 + \frac{\kappa_2}{\kappa_1}\right). \end{aligned} \tag{32}$$

The condition of the neutral inhomogeneity which assures that there is a non-trivial solution of homogeneous system of linear equations for C_1, C_2, C_3 and X can be formulated as

$$\begin{aligned} &\left(1 + \frac{k_0}{\kappa_1}\right)\left(1 - \frac{\kappa_2}{\kappa_1}\right) R_1^{-k_1} R_2^{k_1} - \\ &- \left(1 - \frac{k_0}{\kappa_1}\right)\left(1 + \frac{\kappa_2}{\kappa_1}\right) R_1^{k_1} R_2^{-k_1} = 0. \end{aligned} \tag{33}$$

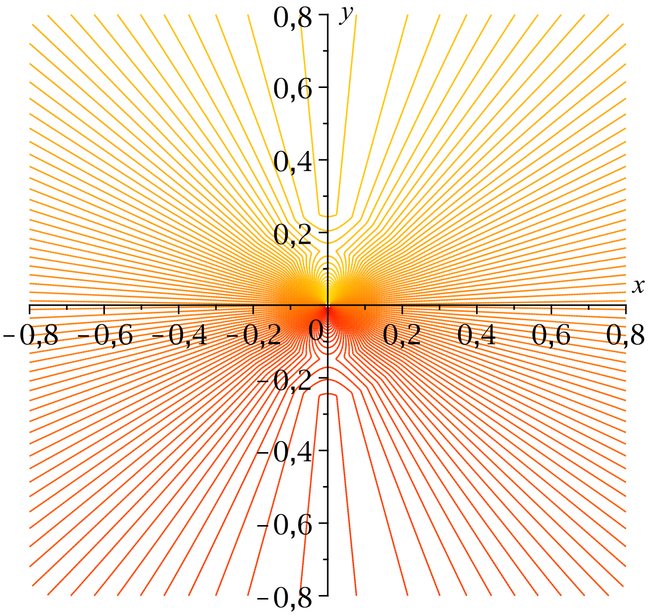

Figure 5: The contour lines of the radial component of heat flux vector.

Independently of the size and the position of neutral inhomogeneity equation (33) satisfies if

$$k_0 = \kappa_1 = \kappa_2. \tag{34}$$

It must be noted that there is no restriction to the position of the origin of the coordinate system Oxy.

From equations (24–27) the following formulae can be derived for C_1, C_2 and C_3

$$C_1 = -t_1\frac{R_1^{1-k_1}}{L}, \quad C_2 = 0, \qquad C_3 = -t_1\frac{R_1^{1-k_1}}{L}R_2^{k_1}R_2^{-k_2}. \tag{35}$$

The expression of radial component of the heat flux vector in the domain $A_1 \cup A_2$ is

$$q_{2r}(r,\varphi) = t_1 k_{2r}\frac{R_1^{1-k_1}}{L}R_2^{k_1}R_2^{-k_2}r^{k_2-1}\sin\varphi, \qquad 0 \le r \le R_1, \quad 0 \le \varphi \le 2\pi, \tag{36}$$

$$q_{1r}(r,\varphi) = t_1 k_{1r}\frac{R_1^{1-k_1}}{L}r^{k_1-1}\sin\varphi, \qquad R_2 \le r \le R_1, \quad 0 \le \varphi \le 2\pi. \tag{37}$$

On the whole rectangular domain the radial component of the heat flux vector is as follows

$$q_r(r,\varphi) = (H(r) - H(r-R_1))\, q_{1r}(r,\varphi) + {} + (H(r-R_1) - H(r-R_2))\, q_{2r}(r,\varphi) + {} + H(r-R_2) q_{0r}(r,\varphi). \tag{38}$$

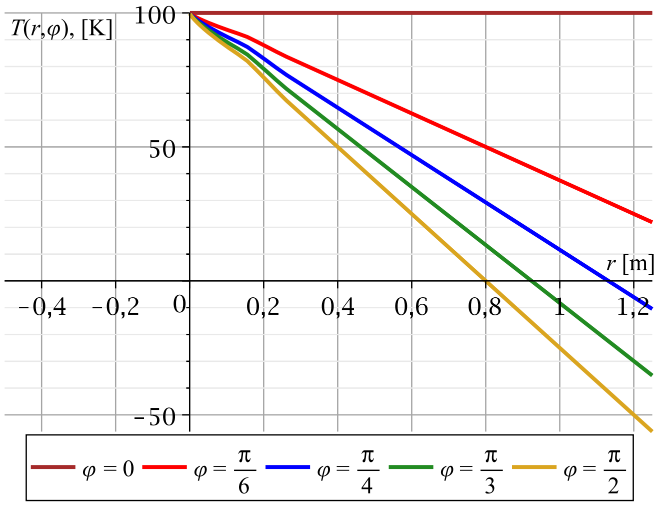

Figure 6: The plots of temperature function as a function of r.

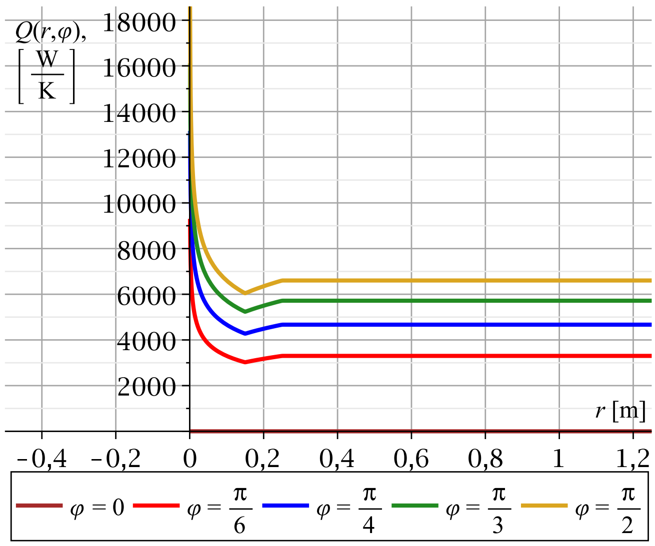

Figure 7: The graphs of the radial component of heat flux vector as a function of r.

It is evident that in the case of several circular cylindrically anisotropic inclusions (Fig. 3) when the temperature field of the host body is given by equation (1), if

$$k_0 = \sqrt{k_{ir} k_{i\varphi}}, \quad (i = 1, 2 \ldots, N) \tag{39}$$

then the temperature field does not disturb outside of the inclusions.

4 Numerical example

The numerical example uses the following data: $t_1 = 200$ K, $L_1 = 0.8$ m, $L_2 = 0.8$ m, $a_1 = 0.8$ m, $a_2 = 0.8$ m, $R_1 = 0.25$ m, $R_2 = 0.15$ m, $k_{1r} = 45$ W/mK, $k_{1\varphi} = 62$ W/mK, $k_{2r} = 67.5$ W/mK, $k_{2\varphi} = 41.33333$ W/mK, $k_0 = 52.82045058$ W/mK.

The contour lines of the temperature function $T = T(r, \varphi)$ is shown in Fig. 4. The contour lines of radial component of the heat flux vector is given in Fig. 5. The plots of function $T = T(r, \varphi)$ for five different values of φ ($\varphi = 0$, $\varphi = \frac{\pi}{6}$, $\varphi = \frac{\pi}{4}$, $\varphi = \frac{\pi}{3}$, $\varphi = \frac{\pi}{2}$) as a function of r for $0 \leq r \leq 5R_1$ are shown in Fig. 6. The graphs of the radial component of heat flux vector are presented for five different values of φ ($\varphi = 0$, $\varphi = \frac{\pi}{6}$, $\varphi = \frac{\pi}{4}$, $\varphi = \frac{\pi}{3}$, $\varphi = \frac{\pi}{2}$) as a function of radial coordinate r for $0 \leq r \leq 5R_1$ in Fig. 7.

5 Conclusion

Paper gives the existence conditions of neutral inhomogeneities in a rectangular domain for a one dimensional steady-state heat flow problem. The composite inclusions consist of a core and coating which are cylindrically orthotropic. A numerical example illustrates the validity of the presented theory. The main result of the paper is a contribution to the existing exact benchmark solution for heat conduction in composite solid bodies.

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