A Modified Beavers and Joseph Condition for Gravity-Driven Flow over Porous Layers

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Abstract: - Gravity-driven flow through an inclined channel over a semi-infinite porous layer is considered in order to obtain a modification to the usual Beavers and Joseph slip condition that is suitable for this type of flow. Expressions for the velocity, shear stress, volumetric flow rates, and pressure distribution across the channel are obtained together with an expression for the interfacial velocity. In the absence of values for the slip parameter when the flow is over a Forchheimer porous layer, this work provides a relationship between the slip parameters of the Darcy and Forchheimer layers. Expressions for the interfacial velocities in both cases are obtained. This original work is intended to provide baseline analysis and a benchmark with which more sophisticated types of flow, over porous layers in an inclined domain can be compared.

Key-Words: - Beavers and Joseph slip hypothesis; inclined channel flow; gravity-driven flow.

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1. Introduction

The study of fluid flow over porous layers is of considerable importance in many practical applications that include lubrication theory, design of cooling systems, and oil recovery, [1], [2]. Conditions at the interface between a fluid layer in free space and a fluid layer in a porous medium influence heat and mass transfer across layers, [3]. While the no-slip velocity condition at the interface was implemented prior to 1967, [4], the experiments of Beavers and Joseph, [5], replaced the no-slip condition with a slip hypothesis that resulted in what is now known as the Beavers and Joseph condition, (BJ condition). Beavers and Joseph, [5], attempted to explain the increase in mass flux in the channel associated with the presence of a permeable boundary, as compared to the mass flux in the channel when the no-slip condition is applied at an impermeable boundary. Their analysis resulted in the empirical slip-flow condition of Beavers and Joseph, which ascertains that the shear stress at the interface between free space and a porous layer is proportional to the difference between the interfacial velocity and the velocity in the porous medium, [5]. Their slip hypothesis agreed well with their experiment of flow through a Navier-Stokes channel terminated by a semi-infinite Darcy porous layer. The constant of proportionality in the BJ condition includes what is termed a slip parameter, α, which is a semi-empirical, dimensionless slip coefficient that is independent of fluid viscosity and is dependent on the porous medium properties, Reynolds number and flow direction at the interface, (*c.f.* [6] to [10]).

For over half of a century, the BJ condition has received considerable attention in the literature and has been modified and adjusted to fit different types of porous media. Many excellent reviews are available and discuss various types of conditions that have been proposed to handle permeability discontinuity at the interface between a porous layer and free space (*c.f.* [11] to [18]). Ehrhardt, [16], provided an elegant account of the available interfacial conditions, including the jump condition of Ochoa-Tapia and Whitaker, [19].

Models of flow through porous media may be divided into two groups, one is compatible in differential order with the Navier-Stokes equations, such as the Brinkman equation, and a group that is of lower differential order than the Navier-Stokes equations, such as Darcy's equation and the Forchheimer equation. In the flow over a porous layer of finite thickness, Rudraiah, [2], suggested that flow in the porous layer should be governed by an equation compatible with the Navier-Stokes equations, namely Brinkman's equation. Neale and Nader, [12], proposed the use of velocity and shear stress continuity at the interface with the porous layer when using Brinkman's equation. They obtained a solution of flow over a thick porous that is the same as the solution obtained using Darcy's equation with BJ condition when choosing the slip parameter $\alpha = \sqrt{\mu_{eff} / \mu}$, where μ is the base fluid viscosity and μ_{eff} is the effective viscosity. It is worth noting here that μ_{eff} is a semi-empirical quantity, not unlike α. The case of flow through a channel bounded by a Brinkman layer of constant permeability results in permeability discontinuity at the interface. This has been remedied by Nield and Kuznetsov, [15] with the introduction of a transition layer between a constant permeability layer and the free-space channel. In the flow through a channel over a porous layer of infinite depth, the flow through which is governed by an equation of lower differential order than Navier-Stokes equations, Rudraiah, [2], suggested the use of the BJ condition. This adds to the already established understanding that in the flow through a channel over a Forchheimer porous layer the use of the Beavers and Joseph condition is appropriate and justified, [11], [19].

However, this raises some important questions with regard to the appropriate value(s) of the coefficient α when the Forchheimer layer is used. For flow over a Darcy layer, Nield, [11], discussed that α ranges from 0.01 to 5. In the experiments of Beavers and Joseph, values of α used were 0.78, 1.45, and 4.0 for Foametal with average pore sizes of 0.016, 0.034, and 0.045 inches, respectively, and 0.1 for Aloxite with average pore size of 0.013 or 0.027 inches.

The semi-empirical nature of the BJ slip parameter and its flow-direction dependence, combined with the absence of parametric its values for different types of porous media and their governing equations, raise the following questions which motivate this current work:

(i) If the flow is through a Forchheimer porous layer what values does the slip parameter take, and how are its values determined?

(ii) Some important flows include flow through composite porous layers, where one layer is underlain by another. If the flow is through a porous layer underlain by either a Darcy layer or a Forchheimer layer, and the slip hypothesis is assumed to be valid, how are the values of the slip parameter determined?

(iii) A third question of equal important stems from the fact that the slip parameter is flow direction-dependent, and in the Beavers and Joseph experiment, they considered flow through a horizontal channel underlain by a porous layer under a common driving pressure gradient. What is the form of the slip hypothesis, and what are the values of slip parameter when the flow is down an incline and driven by gravity?

The first two of the above questions have received some attention, and partial answers were provided in [20] and [21]. Abu Zaytoon and Hamdan, [20], considered flow through a Navier-Stokes channel underlain by a Forchheimer porous layer and derived a relationship between the slip parameter in the Beavers and Joseph experiment and the slip parameter associated with the slip hypothesis for flow over a Forchheimer porous layer.

Silva-Zea *et.al.*, [21], considered flow through a Brinkman porous layer underlain by a Darcy porous layer and a Forchheimer porous layer, and derived expressions for values of their associated slip parameters in terms of the Beavers and Joseph slip parameter.

Providing a partial answer to the third question, above, is the scope of the current work in which we consider flow through a free-space channel inclined at an angle to the horizontal, over an inclined porous layer of semi-infinite extent. The porous layers considered are a Darcy layer and a Forchheimer layer, and the flow in each configuration is driven only by the action of gravity. The governing equations in each case are solved and expressions for the velocity, shear stress, volumetric flow rates, pressure distributions in the channel and porous layers, and velocities at the interfaces are obtained. Furthermore, a relationship between slip parameters associated with the Darcy and the Forchheimer layers is derived. Numerical values for the slip velocities and slip parameters are obtained and analyzed.

In addition to attempting to shed some light on the effects of flow direction on the slip parameter, this work is intended to provide a baseline analysis and some bench mark with which more real-life flow problems can be compared. As an example, in recent developments of flow through porous media, considerable attention has been devoted to the flow of fluids with pressure-dependent viscosities, due to the importance of this type of fluid flow in enhanced oil recovery and carbon sequestration, [22], [23]. An important flow configuration that is in use is the flow down an inclined plane, which has been popular in the study of lubrication mechanisms, the mixing of fluid layers, and waves down an incline, [24]. Importance of this flow configuration in the current work is that it provides for a vehicle to study the effects of flow direction on the BJ slip parameter. This might be an important methodology from a cost-effective vantage point as experiments to approximate slip parameters could be costly, and some elegant theoretical methods of estimation are complicated and potentially time-consuming, [25]. With the availability of more data on the slip parameter for Darcy layers, experiments and/or techniques presented in [25] can estimate the most accurate value(s) for flow down an incline. Our proposed relations can then be used to estimate the corresponding slip parameter(s) for the Forchheimer layer.

We believe that this work provides some comparative data that could be of importance in further studies of flow of fluids with pressure-dependent viscosities through and over porous layers, among other more sophisticated flow.

2. Previous Results of Flow over Porous Layers

In their original work, Beavers and Joseph, [5], considered flow induced by a pressure gradient through a horizontal, straight channel of width h over a Darcy porous layer of semi-infinite extent, shown in **Fig. 1**. The Darcy velocity, u_D, is constant in the porous layer and the velocity profile in the channel is parabolic in the absence of the porous layer. The slip velocity, u_s, is envisaged to result due to the influence of the "slip" condition at the interface, as illustrated in **Fig. 1**.

Theoretical analysis and solutions to the governing equations of flow through a Navier-Stokes channel over a Darcy layer, and results of the work accomplished in [20] and [21], are summarized in what follows.

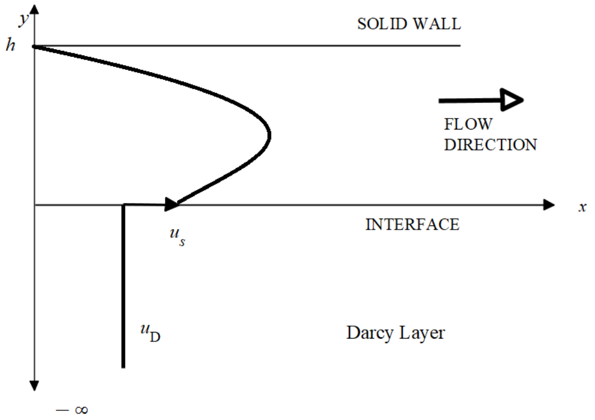

Fig. 1. Schematic Sketch for Beavers and Joseph Condition

Result 1: In the study of Navier-Stokes flow through a horizontal channel bounded by two solid, impermeable parallel plates at $y = 0$ and $y = h$, where no-slip conditions apply, governing equations reduce to

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} \qquad (1)$$

where $\frac{dp}{dx} < 0$ is the driving pressure gradient, $u(y)$ is the velocity in the channel, and μ is the viscosity of the fluid.

Solution to equation (1) satisfying $u(0) = u(h) = 0$ gives the following velocity profile across the channel:

$$u = \frac{1}{2\mu}\frac{dp}{dx}[y^2 - hy] \qquad (2)$$

Result 2: In the study of Navier-Stokes flow through a channel underlain by a semi-infinite Darcy layer where the channel is bounded by solid walls and described by $\{(x, y)|0 \leq y \leq h; -\infty < x < +\infty\}$, while the semi-infinite Darcy layer is described by $\{(x, y)|-\infty < y \leq 0; -\infty < x < +\infty\}$, flow in the channel is governed by equation (1) and the flow in the Darcy layer is governed by Darcy's equation. Darcy's equation is written in the form:

$$u_d = -\frac{k}{\mu}\frac{dp}{dx} \quad (3)$$

wherein $\frac{dp}{dx} < 0$ is the common driving pressure gradient, $u(y)$ is the velocity in the channel, u_d is the seepage velocity in the porous layer, and k is the constant permeability in the porous layer.

Equations (1) and (3) have been solved subject to the following conditions:

a) No-slip condition on the solid wall, $y = h$, namely

$$u(h) = 0 \quad (4)$$

b) Beavers and Joseph condition at the interface, namely

$$\frac{du}{dy} = \frac{\alpha_1}{\sqrt{k}}(u_{i1} - u_d)\ at\ y = 0 \quad (5)$$

where $u_{i1} = u(0^+)$ is the fluid velocity at the interface (referred to as the interfacial velocity), and α_1 is referred to as the slip conefficient.

Solution to system (1), (3), (4) and (5) yields the following velocity profile across the channel:

$$u = \frac{1}{2\mu}\frac{dp}{dx}[y^2 - h^2] + \frac{\alpha_1}{\sqrt{k}}(u_{i1} - u_d)[y - h] \quad (6)$$

where u_d is given by (3), and the velocity at the interface, $y = 0$, is determined as:

$$u_{i1} = -\frac{k\sigma}{2\mu}\left[\frac{\sigma + 2\alpha_1}{1 + \alpha_1\sigma}\right]\frac{dp}{dx} \quad (7)$$

wherein

$$\sigma = \frac{h}{\sqrt{k}} \quad (8)$$

Result 3: In their recent work, Abu Zaytoon and Hamdan, [20], modified the BJ condition and adjusted the values of its slip parameter to handle situations in which the Darcy layer is replaced by a Forchheimer layer. Corresponding forms of the BJ condition have also been derived for flow through composite porous layers, [21]. When the porous layer is of the type where Forchheimer's equation is valid, Abu Zaytoon & Hamdan, [20], provided the following problem formulation and solution. Equation (3) is replaced by

$$\frac{\mu}{k}u_{f1} + \frac{dp}{dx} + \frac{\rho C_f}{\sqrt{k}}{u_{f1}}^2 = 0 \quad (9)$$

where u_f is the tangential velocity component in the Forchheimer porous layer, ρ is the fluid density, and C_f is the Forchheimer drag coefficient. Equations (1) and (7) have been solved subject to conditions (3) and the following modified form of condition (4):

$$\frac{du}{dy} = \frac{\beta_1}{\sqrt{k}}(u_{b1} - u_{f1})\ at\ y = 0 \quad (10)$$

where u_{b1} is the velocity at the interface between the channel and the Forchheimer porous layer, and β_1 is the slip parameter associated with Forchheimer's porous interface. Solutions to (1) and (8), subject to conditions (4) and (9) are as follows.

Velocity distribution across the channel is given by:

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}[y^2 - h^2]$$

$$+\frac{\beta_1}{\sqrt{k}}(u_{b1} - u_{f1})[y - h];\ 0 < y < h \quad (11)$$

and the velocity distribution in the Forchheimer porous layer is given by:

$$u_{f1} = -\frac{\omega_1 p_x}{\mu};\ y < 0 \quad (12)$$

wherein

$$\omega_1 = \frac{\mu}{2\rho C_f\sqrt{k}p_x}\left[\mu - \sqrt{\mu^2 - 4\rho k\sqrt{k}\, C_f p_x}\right]$$

(13)

Velocity at the interface between the channel and porous layer has been calculated as

$$u_{b1} = -\frac{k}{2\mu}\left[\frac{\sigma^2+2\beta_1\sigma\omega_1}{1+\beta_1\sigma}\right]p_x \quad (14)$$

The following relationship between slip parameters α_1 and β_1 has been derived, [20], when the interfacial velocities are the same, namely $u_{i1} = u_{b1}$:

$$\beta_1 = \frac{\alpha_1[\sigma^2-2]}{[\sigma^2+2\alpha_1\sigma(1-\omega_1)-2\omega_1]} \quad (15)$$

3. Current Problem Formulation and Solution

Previous work and results are valid for unidirectional flow in a horizontal channel, over a porous layer, with the flow being driven by a common pressure gradient, p_x. The objective of the current work is to extend and modify the BJ condition to the case when flow is gravity-driven. To this end, we consider flow through an inclined channel over a semi-infinite porous layer and derive expressions for the interfacial velocity and the pressure distribution. Two types of porous media are considered. The first is one where Darcy's equation is valid and the second is where the Forchheimer equation is valid. This flow configuration represents a model that is used in the important problem of flow of a fluid with pressure-dependent viscosity, where the pressure distribution is a function of the normal physical variable.

We consider the steady flow of a viscous, incompressible fluid through a channel of depth h underlain by a semi-infinite porous layer inclined at an angle ϑ to the horizontal, as shown in **Fig. 2.**

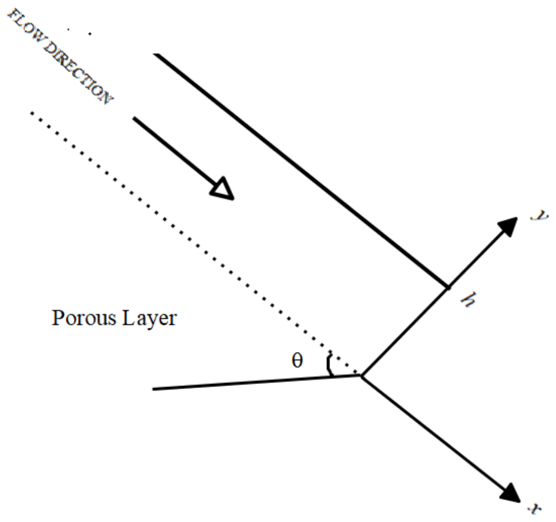

Fig. 2. Representative Sketch for Flow Down an Incline

The channel is bounded at $y = h$ by a solid, impermeable wall, on which the no-slip condition is imposed. The pressure gradient is to zero in the x-direction, the flow is assumed to be driven down the incline by gravity. Flow in the free-space channel if governed by the continuity and Navier-Stokes equations, which reduce to the following for the configuration at hand:

$$\frac{d^2u}{dy^2} = -\frac{\rho g}{\mu} sin\vartheta \quad (16)$$

$$-\frac{dp}{dy} - \rho g cos\vartheta = 0 \quad (17)$$

where $u = u(y)$ is the tangential flow velocity down the incline, $p = p(y)$ is the fluid pressure and g is the gravitational acceleration. We will provide the following three case solutions in the analysis below, in **Subsections 3.1, 3.2,** and **3.3**, respectively.

Case 1: Navier-Stokes flow through an inclined channel spanning the flow domain $\{(x, y)|0 < y < h; -\infty < x < +\infty\}$.

Case 2: Navier-Stokes flow through an inclined channel of height h over a semi-infinite Darcy porous layer with an interface located at $y = 0$.

Case 3: Navier-Stokes flow through an inclined channel of height h over a semi-infinite Forchheimer porous layer with an interface located at $y = 0$.

3.1. Navier-Stokes Flow in an Inclined Channel

Flow in the channel is governed by equations (16) and (17), subject to the following conditions on velocity and pressure, respectively

$$u(0) = u(h) = 0 \quad (18)$$

$$p(h) = p_0 \quad (19)$$

where p_0 is a constant, specified pressure that could be chosen as atmospheric pressure, [23]. Solutions to equations (16) and (17), satisfying conditions (18) and (19), are given, respectively, by:

$$u = \frac{\rho g}{2\mu} sin\vartheta[hy - y^2] \quad (20)$$

$$p(y) = p_0 + \rho g cos\vartheta\,(h - y);\ 0 \leq y \leq h. \quad (21)$$

Solution (20) gives the velocity distribution across the channel and illustrates its dependence of the angle of inclination, ϑ. Solution (21) gives the pressure distribution across the channel and illustrates its dependence on the angle of inclination, ϑ, and on the pressure distribution at the upper and lower channel boundaries, with the pressure being p_0 at the upper channel wall, and $p_0 + \rho g h cos\vartheta$ at the lower channel wall.

3.2. Navier-Stokes Flow over an Inclined Darcy Layer

In this case, flow in the channel is governed by equations (16) and (17), and the flow in the porous layer is governed by the equation of continuity and Darcy's equation, which reduce to the following forms for the configuration at hand:

$$\rho g sin\vartheta - \frac{\mu}{k} u_d = 0 \quad (22)$$

$$-\frac{dp_d}{dy} - \rho g cos\vartheta = 0 \quad (23)$$

where u_d is the tangential Darcy seepage velocity component, and $p_d = p_d(y)$ is the interstitial pressure. It is assumed that viscosities of the fluids in the channel and in the Darcy layer are equal and that the fluid densities are equal as well in order to avoid the occurrence of a free surface, or a curvilinear interface, between the fluid layers. The interface between the two domains is thus assumed to be sharp and is located at $y = 0$. At the interface of the two domains the following conditions hold:

$$\frac{du}{dy}(0) = \frac{\alpha_2}{\sqrt{k}}(u_{i2} - u_d) \quad (24)$$

$$p_d(0) = p(0) = p_0 + \rho g h cos \quad (25)$$

where $u_{i2} = u(0^+)$ is the fluid velocity at the interface, and α_2 is the slip parameter associated with the flow over an inclined Darcy layer. On the upper solid wall bounding the channel, velocity vanishes, namely $u(h) = 0$. Condition (24) is the well-known Bevers and Joseph slip condition, [5].

Darcy velocity profile is obtained algebraically from equation (22) as:

$$u_d = \frac{\rho g k}{\mu} sin\vartheta \quad (26)$$

Velocity distribution in the channel is obtained by solving equation (1) for $u(y)$. General solution to (1) is given by:

$$u = -\frac{\rho g}{2\mu} sin\vartheta\ y^2 + c_1 y + c_2 \quad (27)$$

where c_1 and c_2 are arbitrary constants that can be determined using conditions (24) together with $u(h) = 0$. Values of these constants c_1 and c_2 are thus given by:

$$c_1 = \frac{\alpha_2}{\sqrt{k}}(u_{i2} - u_d) \tag{28}$$

$$c_2 = \frac{\rho g h^2}{2\mu} sin\vartheta \; - \frac{h\alpha_2}{\sqrt{k}}(u_{i2} - u_d) \tag{29}$$

Using (28) and (29) in (27), we obtain

$$u = \frac{\rho g}{2\mu} sin\vartheta \; (h^2 - y^2) + \frac{\alpha_2}{\sqrt{k}}(u_{i2} - u_d)(y - h) \tag{30}$$

Using $u_{i2} = u(0^+)$ in (29), and solving for u_{i2}, we obtain the following expression for the velocity at the interface

$$u_{i2} = \rho g sin\vartheta \frac{k}{2\mu}\left[\frac{\sigma^2 + 2\alpha_2\sigma}{(1+\alpha_2\sigma)}\right] \tag{31}$$

Using (26) and (31) in (30), we obtain the following velocity distribution in the channel:

$$u(y) = \frac{\rho g sin\vartheta}{2\mu}\left[(h^2 - y^2) + \frac{\alpha_2(\sigma^2 - 2)}{(1+\alpha_2\sigma)}(y - h)\right] \tag{32}$$

Pressure distribution in the channel is as given by (21), and pressure distribution in the Darcy porous layer, satisfying condition (25), takes the form

$$p_d(y) = p_0 + \rho g cos\vartheta \; (h - y); \;\; y \leq 0 \tag{33}$$

3.3. Navier-Stokes Flow over an Inclined Forchheimer Layer

In this case, flow in the channel is governed by equations (16) and (17), and the flow in the porous layer is governed by the equation of continuity and Forchheimer's equation, which reduce to the following forms for the configuration at hand:

$$-\frac{\mu}{k}u_f - \frac{\rho C_f}{\sqrt{k}} u_f |u_f| + \rho g sin\vartheta = 0 \tag{34}$$

$$-\frac{dp_f}{dy} - \rho g cos\vartheta = 0 \tag{35}$$

where u_f and p_f are the tangential velocity and pressure in the Forchheimer layer, respectively.

Pressure distribution in the channel is as given by (21), and pressure distribution in the Forchheimer porous layer, satisfying the condition

$$p_f(0) = p(0) = p_0 + \rho g h cos\vartheta \tag{36}$$

takes the form

$$p_f(y) = p_0 + \rho g cos\vartheta \; (h - y); \;\; y \leq 0 \tag{37}$$

Algebraic solution to (34) renders the following constant Forchheimer velocity profile across the layer, where the positive root is chosen in order to obtain a positive velocity:

$$u_f = \frac{-1}{2\rho C_f \sqrt{k}}\left[\mu - \sqrt{\mu^2 + 4C_f k\sqrt{k}\rho^2 g sin\vartheta}\right] \tag{38}$$

Velocity distribution in the channel is obtained by solving equation (16) for $u(y)$ subject to the no-slip condition, $u(h) = 0$, and the following modified interfacial condition for the Forchheimer layer:

$$\frac{du}{dy}(0) = \frac{\beta_2}{\sqrt{k}}(u_{b2} - u_f) \tag{39}$$

where β_2 is the slip parameter associated with the flow over a Forchheimer porous layer, and u_{b2} is the corresponding interfacial velocity.

Velocity distribution in the channel, equation (27), thus takes the form

$$u = \frac{\rho g}{2\mu} sin\vartheta (h^2 - \; y^2) + \frac{\beta_2}{\sqrt{k}}\left[u_{b2} - u_f\right](y - h) \tag{40}$$

where u_f is given by (38) and velocity at the interface is given by

$$u_{b2} = \frac{\rho g sin\vartheta}{2\mu}\left[\frac{h^2}{(1+\beta_2\sigma)} + \frac{2k\beta_2\sigma\omega_2}{1+\beta_2\sigma}\right] \quad (41)$$

wherein

$$\omega_2 = \frac{\mu}{\rho g k sin\vartheta} u_f \quad (42)$$

4. Expressions for Slip Velocity, Shear Stress and Volumetric Flow Rate

In this section we provide expressions for the slip velocities when the flow is over a Darcy and a Forchheimer porous layers, in addition of shear stresses and volumetric flow rates for the three cases of flow through the channel, discussed in **Section 3**, above. Expressions for the the amount of increase in flow rate due to the presence of a porous boundary, relative to a solid boundary, are also obtained.

All expressions derived are exact, with the only source of uncertainty being the slip parameters' estimates, which are carried through the expressions by virtue of the interfacial velocities containing the slip parameters.

4.1. Expressions for slip velocities

In the flow through a horizontal channel over a Darcy porous layer, an expression for the slip velocity is given by:

$$U_{sd1} = u_{i1} - u_d = \frac{k}{\mu}\frac{dp}{dx}\left[1 - \frac{\sigma}{2}\left(\frac{\sigma+2\alpha_1}{1+\alpha_1\sigma}\right)\right] \quad (43)$$

The corresponding expression for slip velocity when the channel is inclined is given by:

$$U_{sd2=}u_{i2} - u_d = \frac{\rho g k sin\vartheta(\sigma^2-2)}{2\mu(1+\alpha_2\sigma)} \quad (44)$$

When the flow is through a horizontal channel over a Forchheimer porous layer, the slip velocity is given by:

$$U_{sf1} = u_{b1} - u_f = \left\{\frac{\omega_1}{\mu} - \frac{k}{2\mu}\left[\frac{\sigma^2+2\beta_1\sigma\omega_1}{1+\beta_1\sigma}\right]\right\} p_x \quad (45)$$

The corresponding expression for slip velocity when the channel is inclined is given by:

$$U_{sf2=}u_{b2} - u_{f2} = \frac{\rho g sin\vartheta}{2\mu}\left[\frac{h^2-2k\omega_2}{(1+\beta_2\sigma)}\right] \quad (46)$$

4.2. Expressions for volumetric flow rate, shear stress and

In their original work, Beavers and Joseph, [5], provided an expression for the relative increase in mass flux in the flow over a porous layer, as compared to the mass flow rate in a channel. In this section, we provide expressions for the volumetric flow rates $\int_0^h udy$ and their absolute changes for all flow configurations considered. Volumetric flow rates and shear stresses across the channel are obtained as follows.

For flow through a horizontal channel with solid, impermeable boundaries, volumetric flow rate, Q_{n1}, and shear stress, τ_{n1}, are obtained, respectively, from equation (2) as:

$$Q_{n1} = \int_0^h udy = -\frac{h^3}{12\mu}\frac{dp}{dx} \quad (47)$$

$$\tau_{n1} = \mu\frac{du}{dy} = \left(y - \frac{h}{2}\right)\frac{dp}{dx} \quad (48)$$

For flow through an inclined channel bounded by solid, impermeable walls, volumertic flowrate, Q_{n2}, and shear stress, τ_{n2}, across the channel are obtained, respectively, from (20) as

$$Q_{n2} = \int_0^h udy = \frac{\rho g h^3}{12\mu} sin\vartheta \quad (49)$$

$$\tau_{n2} = \mu\frac{du}{dy} = \frac{\rho g}{2} sin\vartheta[h - 2y] \quad (50)$$

For flow through a horizontal channel over a Darcy layer, volumetric flow rate, Q_{d1}, and shear stress, τ_{d1}, across the channel are obtained, respectively, from (6) as follows:

$$Q_{d1} = \int_0^h u dy = -\frac{h^3}{3\mu}\frac{dp}{dx}$$

$$-\frac{\alpha_1\sqrt{k}}{\mu}\frac{dp}{dx}\left[1 - \frac{\sigma}{2}\left(\frac{\sigma+2\alpha_1}{1+\alpha_1\sigma}\right)\right] \quad (51)$$

$$\tau_{d1} = \mu\frac{du}{dy} =$$

$$\left\{y + \alpha_1\sqrt{k}\left[1 - \frac{\sigma}{2}\left(\frac{\sigma+2\alpha_1}{1+\alpha_1\sigma}\right)\right]\right\}\frac{dp}{dx} \quad (52)$$

For flow through a horizontal channel over a Forchheimer layer, volumetric flow rate, Q_{f1}, and shear stress, τ_{f1}, across the channel are obtained, respectively, from (11) as follows:

$$Q_{f1} = -\frac{h^3}{3\mu}\frac{dp}{dx} - \frac{\beta_1 h^2}{2\sqrt{k}}\left\{\frac{\omega_1}{\mu} - \frac{k}{2\mu}\left[\frac{\sigma^2+2\beta_1\sigma\omega_1}{1+\beta_1\sigma}\right]\right\}p_x$$

(53)

$$\tau_{f1} = \mu\frac{du}{dy} = y\frac{dp}{dx}$$

$$+\frac{\beta_1}{\sqrt{k}}\left\{\omega_1 - \frac{k}{2}\left[\frac{\sigma^2+2\beta_1\sigma\omega_1}{1+\beta_1\sigma}\right]\right\}p_x \quad (54)$$

For flow through an inclined channel over a Darcy layer, volumetric flow rate, Q_{d2}, and shear stress, τ_{d2}, across the channel are obtained, respectively, from (32) as follows:

$$Q_{d2} = \int_0^h u dy = \frac{\rho g sin\vartheta}{2\mu}[\frac{2}{3}h^3 - \frac{h^2}{2}\frac{\alpha_2(\sigma^2-2)}{(1+\alpha_2\sigma)} \quad (55)$$

$$\tau_{d2} = \mu\frac{du}{dy} = \frac{\rho g sin\vartheta}{2}\left[\frac{\alpha_2(\sigma^2-2)}{(1+\alpha_2\sigma)} - 2y\right)] \quad (56)$$

For flow through an inclined channel over a Forchheimer layer, volumetric flow rate, Q_{f2}, and shear stress, τ_{f2}, across the channel are obtained, respectively, from (40) as follows:

$$Q_{f2} = \int_0^h u dy = \frac{\rho g h^3}{3\mu} sin\vartheta$$

$$-\frac{\rho g sin\vartheta h\sigma\beta_2}{4\mu}\left[\frac{h^2-2k\omega_2}{(1+\beta_2\sigma)}\right] \quad (57)$$

$$\tau_{f2} = \mu\frac{du}{dy} = -y\frac{\rho g}{2}sin\vartheta + \frac{\rho g sin\vartheta\beta_2}{2\sqrt{k}}\left[\frac{h^2-2k\omega_2}{(1+\beta_2\sigma)}\right]$$

(58)

4.3. Expressions for volumetric flow rate increase due to a permeable boundary

The following change (amount of increase) in volumetric flow rate when the flow is through a horizontal channel over a Darcy porous layer, relative to volumetric flow rate through a horizontal channel bounded by solid wall, is obtained from equations (47) and (51) as:

$$Q_{d1} - Q_{n1} = -\frac{h^3}{4\mu}\frac{dp}{dx}$$

$$-\frac{\alpha_1\sqrt{k}}{\mu}\left[1 - \frac{\sigma}{2}\left(\frac{\sigma+2\alpha_1}{1+\alpha_1\sigma}\right)\right]\frac{dp}{dx} \quad (59)$$

and for an inclined channel, the increase in volumetric flow rate is obtained from (49) and (55) as:

$$Q_{d2} - Q_{n2} = \frac{\rho g sin\vartheta h^2}{4\mu}\left[h - \frac{\alpha_2(\sigma^2-2)}{(1+\alpha_2\sigma)}\right] \quad (60)$$

The corresponding increase in volumetric flow rate when the Darcy layer is replaced by a Forchheimer layer is obtained from (47) and (53) for a horizontal channel as:

$$Q_{f1} - Q_{n1} = -\frac{h^3}{4\mu}\frac{dp}{dx}$$

$$-\frac{\beta_1 h^2}{2\sqrt{k}}\left\{\frac{\omega_1}{\mu} - \frac{k}{2\mu}\left[\frac{\sigma^2+2\beta_1\sigma\omega_1}{1+\beta_1\sigma}\right]\right\}\frac{dp}{dx} \quad (61)$$

and from (48) and (56), for an inclined channel, as:

$$Q_{f2} - Q_{n2} = \frac{\rho g h sin\vartheta}{4\mu}[h^2 - \sigma\beta_2\left[\frac{h^2-2k\omega_2}{(1+\beta_2\sigma)}\right] \quad (62)$$

5. Results and Analysis

5.1. Relationship between angle of inclination and pressure gradient

In the flow through a horizontal channel over a Darcy layer, equation (3) gives Darcy velocity as $u_d = -\frac{k}{\mu}\frac{dp}{dx}$. When the flow is through an inclined channel over a Darcy layer, equation (26) gives Darcy velocity as $u_d = \frac{\rho g k}{\mu} sin\vartheta$. If we assume that the Darcy velocities are equal, we must have $\rho g sin\vartheta = -\frac{dp}{dx}$.

For a given fluid density and with the knowledge of g, this condition gives us the angle of inclination that produces the same Darcy velocity for a give pressure gradient as

$$\vartheta = sin^{-1}\left(-\frac{\frac{dp}{dx}}{\rho g}\right) \quad (63)$$

Equation (63) is also valid for the case of flow over a Forchheimer porous layer, as can be seen by comparing equations (9) and (34). Furthermore, for a given angle of inclination, we can choose ρg that correspond to a given pressure gradient, as shown in **Table 1.**

	$\vartheta = 30^\circ$	$\vartheta = 60^\circ$
$\frac{dp}{dx}$	ρg	ρg
-1	2	$2/\sqrt{3}$
-2	4	$4/\sqrt{3}$

Table 1. Values of ρg for different pressure gradients and angles of inclination

In the computations to follow, we will use the following ranges of parameters:

$C_f = 0.55, h = 1, \frac{\mu}{\rho} = \frac{0.0005097}{0.05097} = 0.01, g = 9.81.$

5.2. Pressure at the interface

Equations (33) and (37) give the pressure distributions in the Darcy and Forchheimer porous layers, respectively. For the same fluid density and gravitational acceleration, both distributions depend in the same way on the angle of inclination, ϑ, on the channel width, h, and on the pressure p_0 at the upper channel wall. At the interface between the channel and porous layers, equations (25) and (36) give the same value of pressure, namely $p_0 + \rho g h cos\vartheta$, regardless of the layer being a Darcy or a Forchheimer porous layer.

For different angles of inclination and a unit channel width, **Table 2** gives pressure at the interface and shows a decrease in pressure with increasing angle of inclination, for a given p_0. For a given angle of inclination, pressure at the interface increases with increasing p_0.

	$\vartheta = 30^\circ$	$\vartheta = 60^\circ$
p_0	$p(0)$	$p(0)$
1	$1 + \sqrt{3}$	2
2	$2 + \sqrt{3}$	3
5	$5 + \sqrt{3}$	6

Table 2. Pressure at the interface for different p_0 and ϑ.

5.3. Relationship between slip parameters when flow is down an incline.

Equations (31) and (41) are expressions for the interfacial velocities, u_{i2} and u_{f2}, in the case of flow over a Darcy layer and over a Forchheimer layer, respectively. In order to relate the slip parameters appearing in the BJ condition, equations (24) and (39), for the respective layers, we assume equality of the interfacial velocities u_{i2} and u_{f2} and derive the following relationship between slip parameters α_2 and β_2:

$$\beta_2 = \frac{\alpha_2[\sigma^2-2]}{\sigma^2+2\alpha_2\sigma[1-\omega_2]-2\omega_2} \tag{64}$$

where $\omega_2 = \frac{\mu}{\rho g k sin\vartheta} u_f$, as given by (42).

This is the same form of relationship between β_1 and α_1 for the case of flow through a horizontal channel over a Darcy and a Forchheimer layer, given by equation (15), with the difference being in the values of ω_1 and ω_2, given by (14) and (43), respectively.

Values of the constant Darcy's and Forchheimer's velocities in the porous layers, and values of ω_2, are given in **Table 3** to illustrate their dependence on permeability and angle of inclination. For a given permeability, **Table 3** shows that increasing the angle of inclination is accompanied with an expected increase, due to greater gravity influence, in both the Darcy and Forchheimer velocities.

It also shows that increasing permeability, for a given angle of inclination, results in an increase in the Darcy and Forchheimer velocities, as one would expect for flow through porous layers. However, in all cases, and for the same parameters, Darcy velocity is greater in value than the Forchheimer velocity. This might be due to the presence of a quadratic inertial term in the Forchheimer equation that tends to slow down the flow.

k	$\vartheta = 30^\circ$	$\vartheta = 60^\circ$
0.001	$u_d = 0.4905$ $u_f = 0.316393$ $\omega_2 = 0.6450418$	$u_d = 0.849571$ $u_f = 0.416595$ $\omega_2 = 0.4903595$
0.00001	$u_d = 0.004905$ $u_f = 0.004901$ $\omega_2 = 0.9991845$	

Table 3. Values of Darcy velocity, Forchheimer velocity and ω_2 for different k and ϑ

In the experiment of Beavers and Joseph, [5], values used for the slip parameter in the case of flow through a horizontal channel over a Darcy layer were 0.78, 1.45, and 4.0 for Foametal having average pore sizes of 0.016, 0.034, and 0.045 inches, respectively, and 0.1 for Aloxite with average pore size of 0.013 or 0.027 inches. This values will be used in this work for illustration.

	$\omega_2 = 0.6450418$, $\vartheta = 30^\circ$, $k = 0.001$	$\omega_2 = 0.4903595$, $\vartheta = 60^\circ$, $k = 0.001$	$\omega_2 = 0.9991845$, $\vartheta = 30^\circ$, $k = 0.00001$
α_2	β_2	β_2	β_2
0.1	0.099704795	0.099576724	0.0999999
0.78	0.7660148198	0.760076085	0.7799969
1.45	1.4032325215	1.383783069	1.44998913
4.0	3.6674052	3.539166284	3.999917412

Table 4. Values of β_2 corresponding to α_2 for different k and ϑ

Using equation (64), **Table 4** provides the corresponding values for β_2 for various values of β_2, ϑ, and k. In all cases considered, values of β_2 are less than the corresponding values of α_2. For a given permeability, a reduction in the angle of inclination results in an increase in the values of β_2, thus indicating that for the case of flow due to gravity over a Forchheimer porous layer, the slip parameter is dependent on the angle of inclination. For a given angle of inclination, a decrease in the permeability of the Forchheimer layer results in an increase in the value of β_2 until, ultimately, $\beta_2 = \alpha_2$.

5.4. Velocities at the interface

In deriving relationship (42) between β_2 and α_2, we assumed that the velocities at the interface u_{i2} and u_{b2} are equal. **Table 5** gives a listing of the equal quantities $\frac{\mu u_{i2}}{\rho g} = \frac{\mu u_{b2}}{\rho g}$ for different angles of inclination, different permeabilities and the selected values of α_2. Calculations are carried out using equations (31) and (41), with the values of β_2 listed in **Table 4**. We note that **Table 5** lists the velocities at the interface multiplied by the factor $\frac{\mu}{\rho g}$ due to the fact that all quantities considered in this work are dimensional and we are trying whenever possible to avoid using specific values of $\frac{\mu}{\rho g}$. Based on **Table 5**, the following observations are established. With increasing α_2, hence increasing β_2, values of velocities at the interface increase for all angles of inclination and all permeabilities. The effect of increasing angle of inclination, for a given permeability and given slip parameters, is an increase in the velocities at the interface. Decreasing permeability, however, for a given angle of inclination and slip parameters, results in decreasing the velocities at

	$\vartheta = 30^\circ$ $k = 0.001$	$\vartheta = 60^\circ$ $k = 0.001$	$\vartheta = 30^\circ$ $k = 0.00001$
α_2	$\frac{\mu \boldsymbol{u_{i2}}}{\rho g}$ or $\frac{\mu \boldsymbol{u_{b2}}}{\rho g}$	$\frac{\mu \boldsymbol{u_{i2}}}{\rho g}$ or $\frac{\mu \boldsymbol{u_{b2}}}{\rho g}$	$\frac{\mu \boldsymbol{u_{i2}}}{\rho g}$ or $\frac{\mu \boldsymbol{u_{b2}}}{\rho g}$
0.1	0.191137997	0.331051	0.007668204
0.78	0.01022112	0.01770298	0.001014438
1.45	0.005825163	0.01008918	0.000550220
4.0	0.002456999	0.00085921	0.000202482

Table 5. Values of product of $\frac{\mu}{\rho g}$ and velocity at the interface $\boldsymbol{u_{i2}}$ or $\boldsymbol{u_{b2}}$

5.5. Velocity distribution in the free-space channel

Fig. 3 illustrates typical effects of slip parameters on the velocity profile across the free-space channel. In the absence of a porous interface, it is clear from equation (20) that the velocity profile is parabolic, with $u(0) = u(h = 1) = 0$, and a maximum velocity reached in the middle of channel at $y = 0.5$.

When a porous interface is present between the channel and the Forchheimer layer, **Fig. 3** illustrates the effect of the porous boundary on $U = \frac{2\mu}{\rho g sin\vartheta}\, u(y)$. As the velocity at the interface increases with decreasing β_2, the amount of slip increases and causes a loss in the originally parabolic channel velocity profile. This behaviour is shown in **Fig. 3** for permeability $k = 0.001$.

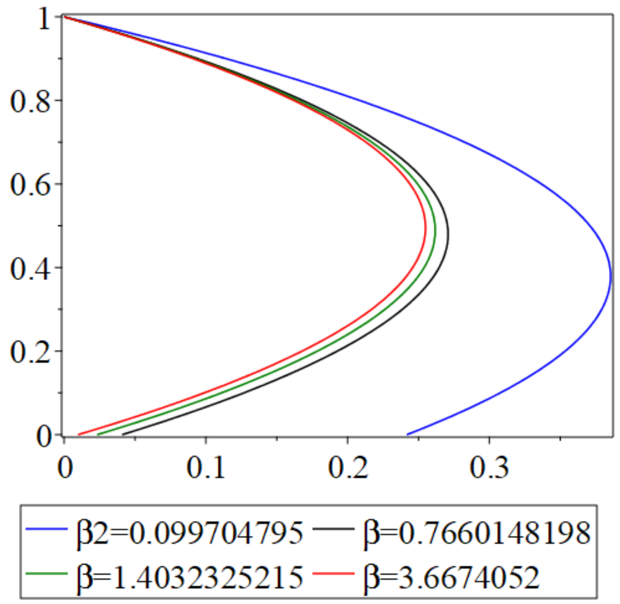

Fig. 3. Channel velocity profile $u(y)$ for k=0.001, $\omega_2 = 0.6450418$ and different values of β_2

6. Conclusion

In this work we considered flow through an inclined channel underlain by a Darcy and a Forchheimer porous layers. Beavers and Joseph condition was applied both cases in order to estimate the values of slip parameter when a Forchheimer layer is used. Solutions were obtained for the governing equations and an expression relating the slip parameter in a Forchheimer layer to that in a Darcy layer under the assumption of equal interfacial velocities. Expressions were also derived for the pressure distributions, slip velocities, and the increase in volumetric flow rates when the no-slip condition is replaced by a slip hypothesis. The main conclusions of this work are as follows:

1- In the flow down an inclined channel as compared to flow under a constant pressure gradient, the inclination angle that produces equal velocity to that of flow through a porous layer under a pressure gradient is given by equation (63).
2- Slip parameters, α_2 and β_2, in the flow over a porous layer down an incline are related by equation (64).
3- As permeability decreases, β_2 approaches α_2.
4- For a given α_2, the value of β_2 depends on medium and flow parameters, and on permeability and angle of inclination.

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References:

[1] D.A. Nield and A. Bejan, Convection in Porous Media, 3rd ed. Springer, 2006.
[2] Rudraiah, N., Flow Past Porous Layers and their Stability, in *Encyclopedia of Fluid Mechanics, Slurry FlowTechnology,* (N. P. Cheremisinoff, Ed., Houston, Texas: Gulf Publishing). Vol. 8,1986 , pp. 567-647.
[3] Sahraoui, M. and Kaviany, M., Slip and No-slip Velocity Boundary Conditions at Interface of Porous, Plain Media, *Int. J. Heat and Mass Transfer*, Vol. 35, 1992, pp. 927-944.
[4] Joseph, D.D. and Tao, L.N., Lubrication of a Porous bearing-Stokes Solution, *Journal of Applied Mechanics*, Vol. 33, 1966, pp. 753–760.
[5] Beavers, G.S. and Joseph, D.D., Boundary Conditions at a Naturally Permeable Wall, *J. Fluid Mechanics*, Vol. 30,1967, pp. 197-207.
[6] Alazmi, B. and Vafai, K., Analysis of Fluid Flow and Heat Transfer Interfacial Conditions Between a Porous Medium and a Fluid Layer, *Int. J. Heat and Mass Transfer,* Vol. 44, 2001, pp. 1735-1749.
[7] Nield, D.A., The Limitations of the Brinkman-Forchheimer Equation in

Modeling Flow in a Saturated Porous Medium and at an Interface, *Int. J. Heat and Fluid Flow,* Vol. 12, No. 3, 1991, pp. 269-272.

[8] Jamet, D. and Chandesris, M., Boundary Conditions at a Planar Fluid-Porous Interface for a Poiseuille Flow, *Int. J. Heat and Mass Transfer*, Vol. 49, 2006, pp. 2137-2150.

[9] Jamet, D. and Chandesris, M., Boundary Conditions at a Fluid-Porous Interface: An *a priori* Estimation of the Stress Jump Coeff-icients, *Int. J. Heat and Mass Transfer*, Vol. 50, 2007, pp. 3422-3436.

[10] Jäger, W. and Mikeli´c, A., On the Interface Boundary Condition of Beavers, Joseph, and Saffman, *SIAM Journal of Applied Mathematics*, Vol. 60, 2000, pp. 1111-1127.

[11] Nield, D.A., The Beavers–Joseph Boundary Condition and Related Matters: A Historical and Critical Note, *Transport in Porous Media*, Vol. *78,* 2009, pp. 537-540.

[12] Neale, G. and Nader, W., Practical Significance of Brinkman Extension of Darcy's Law: Coupled Parallel Flows within a Channel and a Bounding Porous Medium, *Canadian Journal of Chemical Engineering*, Vol. *52,* 1974, pp. 475-478.

[13] Abu Zaytoon, M.S., Alderson, T.A. and Hamdan, M.H., Flow through Variable Permeability Composite Porous Layers, *WSEAS Transactions on Fluid Mechanics*, Vol. 12, 2017, pp. 141-149.

[14] Saffman, P.G., On the Boundary Condi-tion at the Interface of a Porous Medium, *Studies in Applied Mathematics*, Vol. 1, 1971, pp. 93-101.

[15] Nield, D.A. and Kuznetsov, A.V., The Effect of a Transition Layer between a Fluid and a Porous Medium: Shear Flow in a Channel, *Transport in Porous Media*, Vol. 78, 2009, pp. 477-487.

[16] M. Ehrhardt, Report, An Introduction to Fluid-Porous Interface Coupling, Bergische Universitat Wuppertal,Pre-print BUW-AMNA-OPAP 10/15, Berlin, 2010. http://www.math.uni-wuppertl.de,

[17] Vafai, K. and Thiyagaraja, R., Analysis of Flow and Heat Transfer at the Interface Region of a Porous Medium, *Int. J. Heat and Mass Transfer*. Vol. 30, No. 7, 1987, pp. 1391-1405.

[18] Lyubimova, T. P., Lyubimov, D. V., Baydina, D. T., Kolchanova E. A. and Tsiberkin, K. B., Instability of Plane-Parallel Flow of Incompressible Liquid over a Saturated Porous Medium, *Physical Review E*, Vol. 94, 2016, pp. 1-12.

[19] Ochoa-Tapia, J.A. and Whitaker, S., Momentum Transfer at the Boundary between a Porous Medium and a Homogeneous Fluid: I) Theoretical Development, *Int. J. Heat and Mass Transfer*, Vol. 3, No. 14, 1995, pp. 2635-2646.

[20] Abu Zaytoon, M.S. and Hamdan, M.H., A Note on the Beavers and Joseph Condition for Flow over a Forchheimer Porous Layer, *Int. J. Research in Engineering and Science*, Vol. 5 # 3, 2017, pp. 13-20.

[21] Roberto Silva-Zea, M.H. Hamdan, Romel Erazo-Bone, Fidel Chuchuca-Aguilar, and Kenny Escobar-Segovia (2020) Modified Beavers and Joseph Condition in the Study of Flow through Composite Porous Layers, *Int. J. Research in Engineering and Science*, Vol. 8# 9, 2020, pp. 32-27.

[22] Nakshatrala, K.B. and Rajagopal, K.R., A numerical study of fluids with pressure-dependent viscosity flowing through a rigid porous medium. *Int. J. Numer. Meth. Fluids*, *67*, 2011, pp. 342-368.

[23] Kannan, K. and Rajagopal, K.R., Flow through porous media due to high pressure gradients. *J. Applied Mathematics and Computation*, Vol. 199, 2008, pp. 748-759.

[24] Pozrikidis, C., Multifilm flow down an inclined plane: Simulations based on the lubrication approximation and normal-mode decomposition of linear waves. In *Fluid Dynamics at Interfaces*, 1999, pp. 112-128, Wei Shyy and Ranga Narayanan eds. Cambridge University Press.

[25] Mierzwiczak, M., Fraska, A. and Grabski, J.K., Determination of the slip constant in the Beavers-Joseph experiment for laminar flow through porous media using a meshless method, *Mathematical Problems in Engineering*, 2019, Article ID 1494215, Hindawi.

Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

All three authors participated in pertinent literature review to identify missing state-of-the-art knowledge in this area of research.
Roberto Silva-Zea outlined the step to take in conducting this research and what quantities to calculate.
All three authors independently obtained the solutions to governing equations.
M.S. Abu Zaytoon provided calculations and graphing of results using *Maple*.
Roberto Silva-Zea analysed the results.
M.H. Hamdan collected information from co-authors and wrote the manuscript.

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